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Chapter 1

Introduction

The AdS-CFT duality is a string theory -based, conjectured[1] relation between string theory in Anti de Sitter space and conformal field theory in Minkowski space. On a specific limit the duality simplifies to a relation between ordinary general relativity and conformal hydrodynamics. The purpose of this thesis is to give an introduction to this subject. Although many introductions are already available, this aims to differ from them in various aspects. Firstly, the objective has been to collect all information needed for the process of deriving conformal hydrodynamics into one work and to represent it in a chronological way, i.e. explaining subjects along the way instead of providing first some knowledge whose purpose is revealed only after several chapters. Secondly, we have tried to fill in the gaps in the reviews written by some of the most brilliant researchers of this area, gaps that might seem trivial to those who have explored the subject for years, but puzzling to newbies. Finally, the emphasis has been on a straightforward point of view, meaning that we have chosen to explain things using the approach, which is easiest to find or to understand[2] instead of the mathematically most beautiful or most generalizable approach, if the generalizations and mathematical formulations are not needed for what is represented later in this thesis.

Of course, within the scope of this thesis, some things had to be left out. We expect the reader to be familiar with general relativity, specifically with black holes. Also we expect that the concept of quantum field theory and its path integral formulation is known to the reader. As a smaller subject, conformal symmetry is expected to be known. Otherwise this thesis is supposed to be as self-explanatory as possible. This is not to say that all subjects related to the AdS-CFT -duality are explained here. It would be impossible to fully describe string theory and supersymmetry here, as they are subjects to whole books. Nevertheless, as the thesis is aimed for younger students, it would have been unwise to have these subjects as prerequisites here. Therefore, the decision was made to only give short, qualitative descriptions for these subjects and to merely list the results needed from them in this thesis. This was possible, as during the main process of deriving hydrodynamics from gravity one does not have to be immediately aware of the

[1] Conjectured in a mathematical sense, the duality has passed all the numerous tests made so far.

[2] What is the most straightforward way is naturally always contestable.

deeper foundations of the machinery although it will be useful when trying to apply it further. For readers not familiar with but interested in these topics, references to various books and introductory notes are given.

We begin by reviewing some concepts of hydrodynamics and providing a way to derive relativistic hydrodynamics up to some constants. Then we go through the same procedure for conformal fluids. The rest of the thesis is about a process which fixes the constants for the conformal fluids. In chapter 3, we will motivate the conjectured AdS-CFT -duality, based on string theory, and then show how it simplifies at a certain limit to a practical machinery between gravity and conformal hydrodynamics. In chapter 4, the machinery is used in order to sketch a procedure giving conformal hydrodynamics as a gradient expansion. Then it is used to compute the stress-energy tensor for conformal fluids explicitly up to first order. Here some mechanical calculations made using Maple are omitted, as it would probably not be illuminating. Finally, some further results are noted.

Throughout this thesis we use the mostly-plus metric, $\eta^{\alpha\beta} = diag(-1, 1, \ldots, 1)$. Also the Einstein sum convention is used.

Chapter 2

Hydrodynamics

Originally, hydrodynamics meant a dynamic statistical description of liquids. However, it can be seen in a wider perspective as an effective theory for any interacting many-particle model, or more generally, for any interacting field theory, under certain conditions. These conditions are called *the hydrodynamic limit* of the underlying theory. In the following, we will specify what this hydrodynamic limit means, and then formulate hydrodynamic equations for any relativistic theory. After this we will consider the special case of conformal hydrodynamics, i.e. the hydrodynamic limit of any conformally symmetric field theory, which is the most important result of this chapter needed later in this thesis.

2.1 Hydrodynamics as an effective theory

In hydrodynamic description, one wants to specify the state of the system using only thermodynamic variables (T, p, ρ, $\vec{v}, \ldots$), which are continuous variables in space and time. Then one needs some dynamic equations (involving some constants typical for the system examined) defining the time development of the system. Clearly, this is not always possible. Before moving to hydrodynamic evolution, consider first the conditions under which the state of a system can be exactly described using only the thermodynamical variables. In order to have well-defined temperature, the system has to be in thermodynamic equilibrium. Also, as the thermodynamic variables are continuous in space, there may be no structure, as point-like particles, even when considering very small scales. On the other hand, the matter has to fill the whole space considered in order to have thermodynamic variables defined everywhere. These conditions can be summarized as

1. Global thermodynamic equilibrium

2. Continuous matter filling all space at all scales

It is clear that no realistic situation satisfies these conditions. However, usually when considering a many-particle situation, one is not interested in the physics of arbitrary small scales. Therefore, if the aforementioned conditions are satisfied above some length scale l, it should be possible to obtain hydrodynamic description which similarly is valid (i.e. a good approximation) above this scale.

Every interacting many-particle or field theory has a built-in length scale, the *mean free path* l_{mfp}, which is the average distance a particle travels between successive collisions or, more generally, interactions. Intuitively, if we examine the system only at scales $l \gg l_{mfp}$, we can assume that enough collisions take place to relax the system into local thermodynamic equilibrium, so that the hydrodynamic description is valid at all scales examined. We can approach the situation more formally: first assume the whole system is in global thermodynamic equilibrium. Then the hydrodynamic description is trivially valid at all scales above any small-scale inhomogeneities, which violate the second assumption above. Then, let some fluctuations take place around this global equilibrium. If the fluctuation wavelength $\lambda \gg l_{mfp}$, we have local thermodynamic equilibrium everywhere. Of course, the fluctuations have to be long also in time, so that for fluctuation frequency ω we have $\omega t_{mft} \ll 1$, where t_{mft} is the mean time between successive interactions. So we can formulate practical conditions, under which we can have a valid hydrodynamic effective description for a field theory or classical many-body system [1]:

1. For all fluctuations around global thermodynamic equilibrium, $kl_{mfp} \ll 1$ and $\omega t_{mft} \ll 1$.
2. We examine the situation only at scales $l > l_{mfp}$ and $\Delta t > t_{mft}$.
3. There are no inhomogeneities at scales $l > l_0$.

The last condition is rarely restrictive, as the scale of inhomogeneities (for example, the size of a particle or localized wavefunction) is usually much smaller than l_{mfp}. If not, the particles are either very big or sparsely scattered, so that one intuitively does not even consider applying the hydrodynamic approach.

2.2 Relativistic hydrodynamics

In the following, we consider 3+1 -dimensional relativistic hydrodynamics. The stress-energy tensor $T^{\alpha\beta}$ contains all information about the energy and momentum density of the system. These are conserved because of Poincare symmetry and these conservation laws can be summarized as [2]

$$\nabla_\alpha T^{\alpha\beta} = 0 \tag{2.1}$$

This equation contains four scalar equations. The stress-energy tensor can be calculated from the action of the theory as [2]

$$T^{\alpha\beta} = 2\frac{\delta \mathcal{L}_m}{\delta g_{\alpha\beta}} + g^{\alpha\beta}\mathcal{L}_m, \tag{2.2}$$

where $\mathcal{L}_m$ is the matter part of the action, i.e. one gets the total action by adding $\mathcal{L}_m$ to the Einstein-Hilbert action. Note that the stress-energy tensor is symmetric.

However, as we are looking for an effective, hydrodynamic theory, we do not want to compute the stress-energy tensor from the exact, underlying action, which in general can be unknown. As the stress-energy tensor contains thermodynamic information, we should be able to express it using thermodynamic variables. The relevant variables are the energy density ρ (note that this consists of both the rest mass energy and kinetic energy), the densities of conserved charges q_I, fluid velocity u_α, pressure p, temperature T and chemical potentials μ_I. Of these, p, T, μ_I are intrinsic and are defined by a system-dependent equation of state in terms of the other variables. [3],[4] So we have $5+I$ scalar variables left in total, where I is the number of conserved charges. As already mentioned, equation (2.1) gives only four scalar equations, so we need more equations in order to formulate the effective hydrodynamic theory. First, for the conserved charges we trivially have

$$\nabla_\mu J_I^\mu = 0, \tag{2.3}$$

where J_I^μ is the current corresponding to conserved charge q_I. This gives I scalar equations more. One more scalar variable is removed by noting that $u_\mu u^\mu = -1$. Thus the effective hydrodynamic theory can be formulated as

$$\begin{aligned} \nabla_\alpha T^{\alpha\beta} &= 0 \\ \nabla_\mu J_I^\mu &= 0 \end{aligned} \tag{2.4}$$

All that remains is to define $T^{\alpha\beta}$and J_I^μ by using the above-listed variables of this effective theory.

2.2.1 Ideal fluid

We begin with the stress-energy tensor and conserved currents of an ideal fluid. Ideal fluid consists of identical, noninteracting particles, the velocities of which are isotropically distributed. In the local rest frame we can then deduce the form of the stress-energy tensor directly from the standard interpretation of its components [2]:

T^{00} is the energy density, in the rest frame it is just ρ

T^{0i} is the i:th component of energy flux, so for ideal fluid in its rest frame it has to be zero

T^{ik} is the k:th component of the flux of the i:th component of momentum. As in the rest frame the situation is isotropic, we must have $T^{ik} = C\delta_{i,k}$. The T^{ii}-components are easily interpreted: the change of momentum across a surface is equal to the force across the surface, so these diagonal components are just the pressure p.

Thus, noting that in the rest frame must be $u^\mu = (-1, 0, 0, 0)$, we can write the stress-energy tensor for ideal fluid in its rest frame as

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + p(g^{\alpha\beta} + u^\alpha u^\beta). \tag{2.5}$$

As this is already in covariant form, (2.5) is valid in any frame. The conserved currents are trivially

$$J_I^\mu = q_I u^\mu. \tag{2.6}$$

2.2.2 Gradient expansion

Ideal fluid is a simple but not a very realistic model. It does not capture any dissipative effects, which we know through our everyday experience to be present in most realistic fluids. The standard procedure to create effective theories for QFT's is to list all terms allowed by the symmetries of the exact theory and add them up, multiplied by phenomenological constants to form the effective theory Lagrangian. Correspondingly, we can now list all terms allowed by the fact that the stress-energy tensor is symmetric and then add them up to form the stress-energy tensor and conserved currents. As noted before, the condition for having a valid hydrodynamic description was that the wavelength of the fluctuations around global thermal equilibrium is long. Therefore, it is sensible to write the stress-energy tensor and the conserved currents in a gradient expansion: the more derivatives in a term, the smaller it has to be, as short-length fluctuations do not exist [3]. It means that we list the terms up to some order of derivatives. In principle, we have to consider the derivatives of u^μ, p and ρ, but when considering the n-derivative terms, the $n-1$ -order equations (2.4) relate the n:th derivatives of p and ρ to the n:th derivatives of u^μ, and we can use these equations to eliminate some of the derivatives.

We can further constrain the number of allowed terms in the expansion. Because in general relativity, mass and energy cannot be distinguished, we have an ambiguity defining the velocity field: it is not unambiguous, whether it is the velocity of the conserved charges, the velocity of energy or something else. Here we choose the *Landau frame*, following [3]. It means, that we define the velocity field to describe the energy flow, i.e. we define u^μ so that in the local rest frame, the components of the stress-energy tensor longitudinal to the velocity give the local energy density. This is exactly what we did already when forming the stress-energy tensor for ideal fluid. We see that it can be written as follows:

$$\begin{aligned} T^{\alpha\beta} &= \rho u^\alpha u^\beta + p P^{\alpha\beta}, \qquad P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \Rightarrow \\ P^{\alpha\beta} u_\beta &= u^\alpha + u^\alpha(-1) = 0 \text{ and } P^{\alpha\beta} P_{\beta\gamma} = g^\alpha_\gamma + 2u^\alpha u_\gamma - u^\alpha u_\gamma = P^{\alpha\beta} g_{\beta\gamma} \end{aligned} \tag{2.7}$$

So $P^{\alpha\beta}$ is a projection operator to the subspace orthogonal to the velocity. The component longitudinal to the velocity is the local energy density and we are indeed in the Landau frame. An equivalent definition for the Landau frame is that the dissipative part of the stress-energy tensor is orthogonal to the velocity field. If it were not, the component of the stress-energy tensor, which is longitudinal to the velocity, would contain dissipation, i.e. energy flux, which would be in contradiction to the original definition of Landau frame. Thus we have three constraints for the terms to be included in the gradient expansion:

1. The stress-energy tensor is symmetric

2. We are in the Landau frame, so $(T^{\alpha\beta} - \rho u^\alpha u^\beta)u_\beta = 0$

3. The lower order hydrodynamic equations constrain the number of independent terms

Next we will construct the gradient expansion to the first-order. As there can be only terms orthogonal to the velocity field in the expansion, it is useful to list separately the scalar, orthogonal and longitudinal parts of the constraint equations from the expansion to the zeroth order:

$$\begin{aligned} u_\beta \nabla_\alpha T^{\alpha\beta}_{(0)} &= 0 \Longleftrightarrow \\ u_\beta u^\beta u^\alpha \nabla_\alpha \rho + \rho u_\beta u^\beta \nabla_\alpha u^\alpha + \rho u_\beta u^\alpha \nabla_\alpha u^\beta & \\ + u_\beta P^{\alpha\beta} \nabla_\alpha p + p u_\beta u^\beta \nabla_\alpha u^\alpha + p u_\beta u^\alpha \nabla_\alpha u^\beta &= 0 \Longleftrightarrow \\ -u^\alpha \nabla_\alpha \rho - (p+\rho)\nabla_\alpha u^\alpha &= 0 \end{aligned} \tag{2.8}$$

$$\begin{aligned} P_{\beta\mu} \nabla_\alpha T^{\alpha\beta}_{(0)} &= 0 \Longleftrightarrow \\ P_{\beta\mu} u^\alpha u^\beta \nabla_\alpha \rho + \rho P_{\beta\mu} u^\alpha \nabla_\alpha u^\beta + \rho P_{\beta\mu} u^\beta \nabla_\alpha u^\beta + & \\ P_{\beta\mu} P^{\alpha\beta} \nabla_\alpha p + p P_{\beta\mu} u^\alpha \nabla_\alpha u^\beta + p P_{\beta\mu} u^\beta \nabla_\alpha u^\beta &= 0 \Longleftrightarrow \\ (\rho + P) P_{\beta\mu} a^\beta + P^\alpha_\mu \nabla_\alpha p &= 0 \end{aligned} \tag{2.9}$$

$$\nabla_\mu J^\mu_I = u^\mu \nabla_\mu q_I + q_I \nabla_\mu u^\mu = 0 \tag{2.10}$$

where we have used $a^\beta = u^\gamma \nabla_\gamma u^\beta$, the acceleration.

We can eliminate all other derivatives except velocity gradients and the acceleration term $(\rho + P)P_{\beta\mu}a^\beta$ using the constraint equations (2.8)-(2.10). However, as the acceleration is orthogonal to the velocity, it can not be part of any two-tensor in the Landau frame containing only one derivative. Therefore, for the stress-energy tensor, we need to examine only different combinations of velocity gradients. Decomposing $\nabla^\alpha u^\beta$ gives

$$\begin{aligned} \nabla^\alpha u^\beta &= -u^\alpha u^\gamma \nabla_\gamma u^\beta + \frac{1}{2}(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha) + \frac{1}{2}(u^\alpha u^\gamma \nabla_\gamma u^\beta + u^\beta u^\gamma \nabla_\gamma u^\alpha) \\ &\quad - \frac{1}{d-1}\nabla_\gamma u^\gamma P^{\alpha\beta} + \frac{1}{2}(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha) \\ &\quad + \frac{1}{2}(u^\alpha u^\gamma \nabla_\gamma u^\beta - u^\beta u^\gamma \nabla_\gamma u^\alpha) + \frac{1}{d-1}\nabla_\gamma u^\gamma P^{\alpha\beta} \\ &= -u^\alpha a^\beta + \sigma^{\alpha\beta} + \omega^{\alpha\beta} + \frac{1}{d-1}\theta P^{\alpha\beta}, \end{aligned} \tag{2.11}$$

where we have defined

$$\sigma^{\alpha\beta} = \frac{1}{2}(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha) + \frac{1}{2}(u^\alpha u^\gamma \nabla_\gamma u^\beta + u^\beta u^\gamma \nabla_\gamma u^\alpha) - \frac{1}{d-1}\nabla_\gamma u^\gamma P^{\alpha\beta} \tag{2.12}$$

$$\omega^{\alpha\beta} = \frac{1}{2}(\nabla^\alpha u^\beta - \nabla^\beta u^\alpha) + \frac{1}{2}(u^\alpha u^\gamma \nabla_\gamma u^\beta - u^\beta u^\gamma \nabla_\gamma u^\alpha) \tag{2.13}$$

$$\theta = \nabla_\gamma u^\gamma \tag{2.14}$$

Clearly, $\omega^{\alpha\beta}$ is antisymmetric and thus cannot be part of the gradient expansion. Also, $\sigma^{\alpha\beta}$ is clearly symmetric and traceless:

$$\begin{aligned}\sigma^{\alpha}_{\ \alpha} &= \nabla^\alpha u_\alpha + u^\alpha u^\gamma \nabla_\gamma u_\alpha - \frac{1}{d-1}\nabla_\gamma u^\gamma P^{\alpha}_{\ \alpha} \\ &= \nabla^\alpha u_\alpha + u^\alpha u^\gamma \nabla_\gamma u_\alpha - \nabla_\gamma u^\gamma = u^\alpha a_\alpha = 0 \end{aligned} \tag{2.15}$$

and orthogonal to the velocity:

$$\begin{aligned}\sigma^{\alpha\beta}u_\beta &= \frac{1}{2}u_\beta(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha) + \frac{1}{2}u_\beta(u^\alpha a^\beta + u^\beta a^\alpha) - \frac{1}{d-1}\theta P^{\alpha\beta}u_\beta \\ &= \frac{1}{2}u_\beta \nabla^\alpha u^\beta + \frac{1}{2}u^\beta \nabla_\beta u^\alpha - \frac{1}{2}a^\alpha = \frac{1}{2}u_\beta \nabla^\alpha u^\beta = -\frac{1}{2}u_\beta \nabla^\alpha u^\beta = 0, \end{aligned} \tag{2.16}$$

where the second last equation comes by partial integration. Therefore, this will be included in the gradient expansion. Another term included is $\theta P^{\alpha\beta}$, as it is symmetric and orthogonal to the velocity. The $u^\alpha a^\beta$ -term will not be included, because we are working in the Landau frame and one easily checks that this term is orthogonal to $P^{\alpha\beta}$:

$$u^\alpha a^\beta P_{\alpha\beta} = u^\alpha a^\beta(g_{\alpha\beta} + u_\alpha u_\beta) = -1 + (-1)^2 = 0. \tag{2.17}$$

Thus we have the relativistic stress-energy tensor to the first order in gradient expansion:

$$T^{\alpha\beta} = \rho u^\alpha u^\beta + pP^{\alpha\beta} - 2\eta\sigma^{\alpha\beta} - \varsigma\theta P^{\alpha\beta}. \tag{2.18}$$

The coefficients have been chosen so that when considering the non-relativistic limit, the usual classical hydrodynamic equations are recovered. [3] We can interpret the meaning of different terms already before any calculation, by analogy to the classical situation. $\omega^{\alpha\beta}$ is “cross-product -like” term, and must be the vorticity, as it is antisymmetric and does not cause dissipation. $\sigma^{\alpha\beta}$ is the traceless symmetric part of the stress-energy tensor, so it represents shear stress, and therefore η must be the shear viscosity. As θ is a “divergence-like” term and contains dissipative effects occurring in the flux direction (diagonal terms), it must represent the viscosity of the fluid, so ς is the bulk viscosity.

We have not yet calculated the first-order corrections to the conserved currents, let us do that next. Here, we can again eliminate some terms using the constraint equations (2.8)-(2.10). However, as we are now searching for vectors in Landau frame, not two-tensors, the acceleration term $(\rho+P)P_{\beta\mu}a^\beta$ is now contributing. Instead, no other velocity gradient terms contributes: we should contract $\sigma^{\alpha\beta}$ and $\omega^{\alpha\beta}$ with u^μ in order to get vectors, but these are both orthogonal to u^μ, and the only vector we get from θ is θu^μ, which can not be used in Landau frame. We choose to eliminate acceleration with the constraint equations, so we are left with

$$J^\mu_I = q_I u^\mu - \chi_{IJ}P^{\mu\gamma}\nabla_\gamma q_J - \tilde{\gamma_{I1}}P^{\mu\gamma}\nabla_\gamma \rho - \tilde{\gamma_{I2}}P^{\mu\gamma}\nabla_\gamma p, \tag{2.19}$$

which can be written using the chemical potentials and temperature ($\mu_I = q_I T$):

$$J^\mu_I = q_I u^\mu - \chi_{IJ}P^{\mu\gamma}\nabla_\gamma \frac{\mu_J}{T} - \gamma_I P^{\mu\gamma}\nabla_\gamma T. \tag{2.20}$$

We now have the stress-energy tensor (2.18) and conserved currents (2.20) to the first order in gradient expansion, and we can get the corresponding hydrodynamic equations by inserting them into (2.1) and (2.3), respectively. Whereas the form of the conservation equations, stress-energy tensor and conserved currents is the same, the constants, i.e. the different transport coefficients that appeared, have to be computed separately for fluids of a different kind. Depending on the specific situation, one could try to compute these constants based on the microscopic properties of the fluid, or determine them phenomenologically. However, it might be a hard task. While the gradient expansion can be continued quite mechanically to arbitrary order, it will bring forth more and more constants, and so the challenge of defining them all will grow rapidly in complexity. Thus, it is remarkable that it is possible to define these constants unambiguously and to all orders for conformal fluids, with a process described later in this thesis.

2.3 Conformal hydrodynamics

Conformal fluids are a special case of relativistic fluids and the theory describing them is called conformal hydrodynamics. As hydrodynamics can be seen as an effective theory for field theories, conformal hydrodynamics is respectively an effective theory for conformal field theories (CFTs). In order to be conformal, the fluid (i.e. its stress-energy tensor, conserved currents, and the equations describing its behaviour) must be conformally invariant. The conformal invariance of a tensor means that under a conformal transformation (φ is a smooth scalar function of the coordinates)

$$\widetilde{g}_{\alpha\beta} = e^{-2\varphi} g_{\alpha\beta} \tag{2.21}$$

the tensor transforms as

$$\widetilde{T} = e^{\omega\varphi} T \tag{2.22}$$

where ω is the conformal weight of the tensor and is clearly dependent on the raising and lowering of the tensor indices.

We want to explore how the equation (2.1) behaves under conformal transformation. For this, we have to know what $\widetilde{\nabla}_\alpha$ is. Using some basic differential geometry results [5], we know that (as with the Christoffel connection)

$$C^\alpha_{\beta\gamma} = \frac{1}{2}\widetilde{g}^{\alpha\delta}(\nabla_\beta \widetilde{g}_{\gamma\delta} + \nabla_\gamma \widetilde{g}_{\beta\delta} - \nabla_\delta \widetilde{g}_{\beta\gamma}) \text{ , where} \tag{2.23}$$

$$C^\alpha_{\beta\gamma} T_\alpha = \nabla_\beta T_\gamma - \widetilde{\nabla}_\beta T_\gamma . \tag{2.24}$$

By using $\nabla_\alpha g_{\beta\gamma} = 0$ and (2.21), we get

$$\begin{aligned} C^\alpha_{\beta\gamma} &= \frac{1}{2} e^{2\varphi} g^{\alpha\delta}(-2e^{-2\varphi} g_{\gamma\delta}\nabla_\beta\varphi - 2e^{-2\varphi} g_{\beta\delta}\nabla_\gamma\varphi + 2e^{-2\varphi} g_{\beta\gamma}\nabla_\delta\varphi) \\ &= g^{\alpha\delta} g_{\beta\gamma}\nabla_\delta\varphi - \delta^\alpha_\gamma \nabla_\beta\varphi - \delta^\alpha_\beta \nabla_\gamma\varphi \end{aligned} \tag{2.25}$$

So for (2.1) we get

$$\begin{aligned}\widetilde{\nabla}_\alpha \widetilde{T}^{\alpha\beta} &= \nabla_\alpha(e^{n\varphi}T^{\alpha\beta}) + C^\alpha_{\gamma\alpha}e^{n\varphi}T^{\gamma\beta} + C^\beta_{\gamma\alpha}e^{n\varphi}T^{\alpha\gamma} \\ &= e^{n\varphi}\nabla_\alpha T^{\alpha\beta} + (g^{\alpha\delta}g_{\alpha\gamma}\nabla_\delta\varphi - \delta^\alpha_\gamma\nabla_\alpha\varphi - \delta^\alpha_\alpha\nabla_\gamma\varphi)e^{n\varphi}T^{\gamma\beta} \\ &\quad +(g^{\beta\delta}g_{\alpha\gamma}\nabla_\delta\varphi - \delta^\beta_\gamma\nabla_\alpha\varphi - \delta^\beta_\alpha\nabla_\gamma\varphi)e^{n\varphi}T^{\alpha\gamma} \\ &= e^{n\varphi}\nabla_\alpha T^{\alpha\beta} + (n-d-2)e^{n\varphi}T^{\alpha\beta}\nabla_\alpha\varphi + g^{\beta\delta}e^{n\varphi}T^\alpha_\alpha\nabla_\delta\varphi \qquad (2.26)\end{aligned}$$

Now by demanding this to be conformally invariant we get two constraints for $T^{\alpha\beta}$:

$$T^\alpha_\alpha = 0 \qquad (2.27)$$

$$\widetilde{T}^{\alpha\beta} = e^{(d+2)\varphi}T^{\alpha\beta} \qquad (2.28)$$

So the stress-energy tensor must be traceless and its conformal weight is $d+2$.

2.3.1 Ideal conformal fluid

Previously, we got for an ideal fluid the stress-energy tensor $T^{\alpha\beta} = \rho u^\alpha u^\beta + p(g^{\alpha\beta} + u^\alpha u^\beta)$. Using the constraints (2.27,2.28), we can see how the stress-energy tensor for an ideal conformal fluid looks like. From the tracelessness, we get

$$\begin{aligned}T^\alpha_\alpha &= -\rho + p(d-1) = 0 \Longleftrightarrow \\ \rho &= (d-1)p \qquad (2.29)\end{aligned}$$

Also by using (2.21) and the normalization of the velocity field $u_\mu u^\mu = -1$, we get

$$\begin{aligned}\widetilde{u}_\mu \widetilde{g}^{\alpha\mu}\widetilde{u}_\alpha &= 1 \Longleftrightarrow \\ \widetilde{u}_\mu\widetilde{u}_\alpha e^{2\varphi}g^{\alpha\mu} &= 1 \Rightarrow \\ \widetilde{u}_\mu &= e^{-\varphi}u_\mu \qquad (2.30)\end{aligned}$$

Now we get from the second constraint (2.28)

$$\begin{aligned}\widetilde{T}^{\alpha\beta} &= e^{(d+2)\varphi}T^{\alpha\beta} = \widetilde{p}e^{2\varphi}(g^{\alpha\beta} + du^\alpha u^\beta) \Rightarrow \\ \widetilde{p} &= e^{d\varphi}p \qquad (2.31)\end{aligned}$$

Using $p \propto \rho \propto T^d$, one can write the stress-energy tensor for an ideal conformal fluid simply as

$$T^{\alpha\beta} = CT^d(g^{\alpha\beta} + du^\alpha u^\beta), \qquad (2.32)$$

where C is a dimensionless constant depending on the fluid in question.

2.3.2 Gradient expansion for conformal fluids

Just as the conformal symmetry restricted the form of the stress-energy tensor for ideal conformal fluids, it also restricts the terms allowed in the gradient expansion. We have to check which terms of the gradient expansion for usual relativistic fluids comply with the restrictions given by conformal symmetry. In the first-order expansion (2.18), we have two terms to investigate: $\eta\sigma^{\alpha\beta}$ and $\varsigma\theta P^{\alpha\beta}$. Let us begin with the latter term. As shown above (2.27), conformal symmetry demands that the stress-energy tensor is traceless. Because $tr(\varsigma\theta P^{\alpha\beta}) = \varsigma\theta(d-1)$, must be $\varsigma = 0$, as in interesting situations $d \neq 1$.

For the first term, we have to compute how it behaves under conformal transformations, as it is traceless by construction, as seen in (2.15). Using again (2.24) and (2.25), we get

$$\begin{aligned}
\widetilde{\sigma}^{\alpha\beta} &= \frac{1}{2}(\widetilde{\nabla}^{\alpha}\widetilde{u}^{\beta} + \widetilde{\nabla}^{\beta}\widetilde{u}^{\alpha}) + \frac{1}{2}(\widetilde{u}^{\alpha}\widetilde{u}^{\gamma}\widetilde{\nabla}_{\gamma}\widetilde{u}^{\beta} + \widetilde{u}^{\beta}\widetilde{u}^{\gamma}\widetilde{\nabla}_{\gamma}\widetilde{u}^{\alpha}) - \frac{1}{d-1}\widetilde{\nabla}_{\gamma}\widetilde{u}^{\gamma}\widetilde{P}^{\alpha\beta} \\
&= \frac{1}{2}(\widetilde{g}^{\alpha\gamma} + e^{\varphi}u^{\alpha}e^{\varphi}u^{\gamma})(\nabla_{\gamma}(e^{\varphi}u^{\beta}) + C^{\beta}_{\delta\gamma}e^{\varphi}u^{\delta}) \\
&\quad + \frac{1}{2}(\widetilde{g}^{\beta\gamma} + e^{\varphi}u^{\beta}e^{\varphi}u^{\gamma})(\nabla_{\gamma}(e^{\varphi}u^{\alpha}) + C^{\alpha}_{\delta\gamma}e^{\varphi}u^{\delta}) \\
&\quad - \frac{1}{d-1}(\nabla_{\gamma}(e^{\varphi}u^{\gamma}) + C^{\gamma}_{\delta\gamma}e^{\varphi}u^{\delta})e^{2\varphi}P^{\alpha\beta} \\
&= e^{3\varphi}\sigma^{\alpha\beta} + \frac{1}{2}e^{3\varphi}(P^{\alpha\gamma}u^{\beta} + P^{\beta\gamma}u^{\alpha})\nabla_{\gamma}\varphi + \frac{1}{2}e^{3\varphi}(P^{\alpha\gamma}C^{\beta}_{\delta\gamma}u^{\delta} + P^{\beta\gamma}C^{\alpha}_{\delta\gamma}u^{\delta}) \\
&\quad - \frac{1}{d-1}e^{3\varphi}(u^{\gamma}\nabla_{\gamma}\varphi + C^{\gamma}_{\delta\gamma}u^{\delta})P^{\alpha\beta} \\
&= e^{3\varphi}\sigma^{\alpha\beta} + \frac{1}{2}e^{3\varphi}(P^{\alpha\gamma}u^{\beta} + P^{\beta\gamma}u^{\alpha})\nabla_{\gamma}\varphi \\
&\quad + \frac{1}{2}e^{3\varphi}P^{\alpha\gamma}(g^{\beta\mu}g_{\delta\gamma}\nabla_{\mu}\varphi - \delta^{\beta}_{\gamma}\nabla_{\delta}\varphi - \delta^{\beta}_{\delta}\nabla_{\gamma}\varphi)u^{\delta} \\
&\quad + \frac{1}{2}e^{3\varphi}P^{\beta\gamma}(g^{\alpha\mu}g_{\delta\gamma}\nabla_{\mu}\varphi - \delta^{\alpha}_{\gamma}\nabla_{\delta}\varphi - \delta^{\alpha}_{\delta}\nabla_{\gamma}\varphi)u^{\delta} + \nabla_{\delta}\varphi e^{3\varphi}u^{\delta}P^{\alpha\beta} \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\widetilde{\sigma}^{\alpha\beta} &= e^{3\varphi}\sigma^{\alpha\beta} + \frac{1}{2}e^{3\varphi}(P^{\alpha\gamma}u^{\beta} + P^{\beta\gamma}u^{\alpha})\nabla_{\gamma}\varphi + \frac{1}{2}e^{3\varphi}P^{\alpha\gamma}g^{\beta\mu}\nabla_{\mu}\varphi u_{\gamma} \\
&\quad - \frac{1}{2}e^{3\varphi}P^{\alpha\beta}\nabla_{\delta}\varphi u^{\delta} - \frac{1}{2}e^{3\varphi}P^{\alpha\gamma}\nabla_{\gamma}\varphi u^{\beta} + \frac{1}{2}e^{3\varphi}P^{\beta\gamma}g^{\alpha\mu}\nabla_{\mu}\varphi u_{\gamma} \\
&\quad - \frac{1}{2}e^{3\varphi}P^{\beta\alpha}\nabla_{\delta}\varphi u^{\delta} - \frac{1}{2}e^{3\varphi}P^{\beta\gamma}\nabla_{\gamma}\varphi u^{\alpha} + \nabla\varphi e^{3\varphi}u^{\delta}P^{\alpha\beta} \\
&= e^{3\varphi}\sigma^{\alpha\beta}
\end{aligned} \tag{2.33}$$

Thus this shear stress part is conformally invariant (must be $\widetilde{\eta} = e^{(d-1)\varphi}\eta)$, and the stress-energy tensor for conformal fluids up to first order in gradient expansion can now be written as

$$T^{\alpha\beta} = CT^{d}(g^{\alpha\beta} + du^{\alpha}u^{\beta}) - 2\eta\sigma^{\alpha\beta}. \tag{2.34}$$

The process above can be continued as long as wanted. One first lists the terms allowed for general relativistic fluid, and then examines them one by one from the constraints given by the conformal symmetry point of view. For example, with second order contributions the stress-energy tensor becomes [3]

$$\begin{aligned} T^{\alpha\beta} &= CT^d(g^{\alpha\beta} + du^\alpha u^\beta) - 2\eta\sigma^{\alpha\beta} + 2\tau_\pi \eta u^\mu \mathcal{D}_\mu \sigma^{\alpha\beta} + \kappa C^{\alpha\beta}_{\mu\nu} u^\mu u^\nu + \\ & 2\lambda_1(\sigma^{\mu\alpha}\sigma^\beta_\mu - \sigma^{\mu\beta}\sigma^\alpha_\mu) + \lambda_2(\sigma^{\mu\alpha}\omega^\beta_\mu - \sigma^{\mu\beta}\omega^\alpha_\mu) + \frac{\lambda_3}{2}(\omega^{\mu\alpha}\omega^\beta_\mu - \omega^{\mu\beta}\omega^\alpha_\mu), \end{aligned} \tag{2.35}$$

where τ_π, κ, $\lambda_{1,2,3}$ are new transport coefficients appearing at this order, $C^{\alpha\beta}_{\mu\nu}$ is the Weyl tensor [6] and $\mathcal{D}_\alpha$is the Weyl covariant derivative [7]. As mentioned already, for conformal fluids these coefficients can be determined, using a process built upon the AdS-CFT duality. In the next chapter, the duality is explained and in chapter 4 is shown how it can be used to determine the coefficients mentioned.

Chapter 3

The Duality

The AdS-CFT duality (called also AdS-CFT correspondence, gauge-gravity duality/ correspondence, Maldacena duality..., with some terms describing a larger idea than others) is a conjectured relationship between two seemingly completely different theories. The conjecture was first proposed by Juan Maldacena in late 1997 [8]. AdS stands for Anti-de-Sitter, referring to the one side of the duality: here the type IIB string theory on $AdS_5 \times S^5$ space. CFT refers to the other side: here $\mathcal{N} = 4$ (the number of supercharges) supersymmetric Yang-Mills theory on four-dimensional Minkowski space, thus a conformal field theory (CFT). Since its discovery, the duality has passed numerous tests and created a whole new branch of theoretical physics although it has not been proven mathematically. The duality gives mathematical machinery relating objects and phenomena in one theory to those of the other. This is useful, as it turns out that the duality maps some features hard to solve on the CFT side into the features of string theory that we can compute and vice versa. The AdS-CFT duality in general is a kind of an umbrella term for various specific dualities like the one mentioned above, but in this thesis we shall need only this one.

As the process used computing the coefficients for the stress-energy tensor of conformal fluids is based on this duality, we will go through the basic and relevant properties of it. First we note some results of string theory without calculations, in order to be able to present the D3-brane, an object of string theory around which the duality is built. Secondly, the duality is motivated (as it is only a conjecture, a derivation can not be given) and we note the exact formulation of the duality given by Edward Witten [9] and Gubser et al. [34]. Finally, we show how using a special limit of the duality, one can achieve a link between conformal hydrodynamics and Einstein gravity: the fluid-gravity correspondence. We encounter many topics so comprehensive that it would take books to fully describe all their features (string theory, supersymmetry, conformal field theory etc.). Therefore, only short general descriptions are included here, with references to more comprehensive sources for readers interested but unfamiliar with the topics. The main focus here lies on the idea of the duality and some specific features and calculations, which are the most important for the purpose of this thesis, the description of the process for the calculation of the stress-energy tensor for conformal fluids.

3.1 Results from string theory

String theory, first invented in the late 1960s in an unsuccessful attempt to model the strong interaction [10], is now one of the most promising programs to unify the particle-physics standard model and general relativity and has already given rise to many by-products, like the AdS-CFT duality in question here. The basic idea in string theory is to represent all particles as one-dimensional objects, strings, instead of zero-dimensional particles. This single assumption, with the demand of having a self-consistent theory, turns out to be very restrictive and it outlines a theory which seems to address many of the problems of the standard model, e.g. it removes the high-energy singularities. Also the theory is bound to contain general relativity and some kind of gauge theory. [11] Some of the main problems left are that string theory gives few testable predictions and that it has turned out to be hard to find a string theory solution, the low-energy limit of which would describe the standard model gauge theories.[1]

One can imagine two kinds of strings: a closed, rubber band -like string, and an open one. A closed one will just float in space, but the open one can have its endpoints attached to somewhere, or, more exactly, it has certain constraining boundary conditions for its endpoints in some dimensions. These boundaries, on which the open strings are attached, are called branes.

For a single string, the theory is rather straightforward. The vibrations of the string are quantized, and demanding the symmetries of the supersymmetric[2] string action to apply also at quantum level restricts the number of dimensions to be 9+1. With multiple strings, the situation becomes more complicated. In general, one can then work with only effective theories left at the low-energy and small-curvature limit: at this limit one can get some more complicated restrictions instead of the condition $D = 9 + 1$ through a series expansion, and these restrictions turn out to be classical equations of motion for some theories, which are then recognized as effective theories for the corresponding string theory setup. In general, the effective theories for closed strings are supergravity theories, and gauge theories for open strings attached to branes.

There are, however, some exact solutions known. Here we will need the solution [10] for type IIB string theory with a stack of D3-branes[3]. Basically the IIB string theory contains only closed strings, but the introduction of the branes brings open strings there too, as the excitation states of the branes. We will here need only the background metric of the solution:

$$ds^2 = \frac{1}{\sqrt{1+(\frac{R}{r})^4}}(-dt^2 + \sum_{i=1}^{3} dx_i^2) + \sqrt{1+(\frac{R}{r})^4}(dr^2 + r^2 d\Omega_5^2) \tag{3.1}$$

[1]For a comprehensive introduction to string theory, see e.g. [11] or [10]. For a qualitative introduction, see for example [12].

[2]For an introductory review of supersymmetry, see e.g. [29]

[3]D3-brane is a stable 3+1 -dimensional brane. For more information of different kinds of branes, see e.g. [21]

with

$$R^4 = l_s^4 4\pi g_s N, \tag{3.2}$$

where R is a curvature parameter, l_s is the string length, g_s is the closed string coupling constant and N is the number of adjacent branes. Open strings are bound to live on the branes, whereas closed strings are free to move in the surrounding bulk space, which is curved by the branes.

For the purposes of this thesis, we will need some more information about the coupling constants of open and closed strings. Rather easily one achieves the result [11]

$$g_{cs} = g_{os}^2. \tag{3.3}$$

Without knowing string theory, one can justify this informally by thinking of the worldsheet of a basic closed string interaction, where three closed strings meet each other (imagine a junction of three tubes). If one wants to change this worldsheet to describe an interaction of three open strings (imagine a junction of three sheets), one needs to make two cuts for the worldsheet - with one only two of the three tubes can be cut open. Also we note that by analyzing the string action and the dimensions of the coupling constants, one can deduce that closed strings are non-interacting at the low-energy limit. This is essentially the statement that gravity is infrared-free, as the low-energy theory of closed strings was supergravity. [15]

3.2 The Maldacena-duality

3.2.1 Motivation

Consider the metric of N adjacent D3-branes (3.1). The brane curvature parameter R divides the space in two areas: one with $r \ll R$ and other with $r \gg R$. In the first case, the metric is approximately (as $1 \ll \frac{R}{r}$)

$$ds^2 = (\frac{r}{R})^2(-dt^2 + \sum_{i=1}^{3} dx_i^2) + (\frac{R}{r})^2 dr^2 + R^2 d\Omega_5^2, \tag{3.4}$$

i.e. an $AdS_5 \times S^5$-metric. We call this the near-throat limit. The properties of the AdS_5-metric will be examined in the next section, when we have first seen why it is important.

On the second limit, the metric becomes (as $1 \gg \frac{R}{r}$)

$$ds^2 = -dt^2 + \sum_{i=1}^{3} dx_i^2 + dr^2 + r^2 d\Omega_5^2 = -dt^2 + \sum_{i=1}^{9} dx_i^2. \tag{3.5}$$

This is the well-known ten-dimensional Minkowski metric.

Next one can think of two different limits, or points of view, $R \ll l_s$ and $R \gg l_s$, with l_s being again the string length. The first limit, $R \ll l_s$, corresponds to a very stringy viewpoint. Here the stack of branes can well be thought as a four-dimensional object in an otherwise flat spacetime, as the area where the metric is of the form (3.4) is virtually non-existent. Therefore, the physics here consists of open strings with endpoints attached to the D3 branes and closed strings propagating in the Minkowskian bulk (see 3.1).

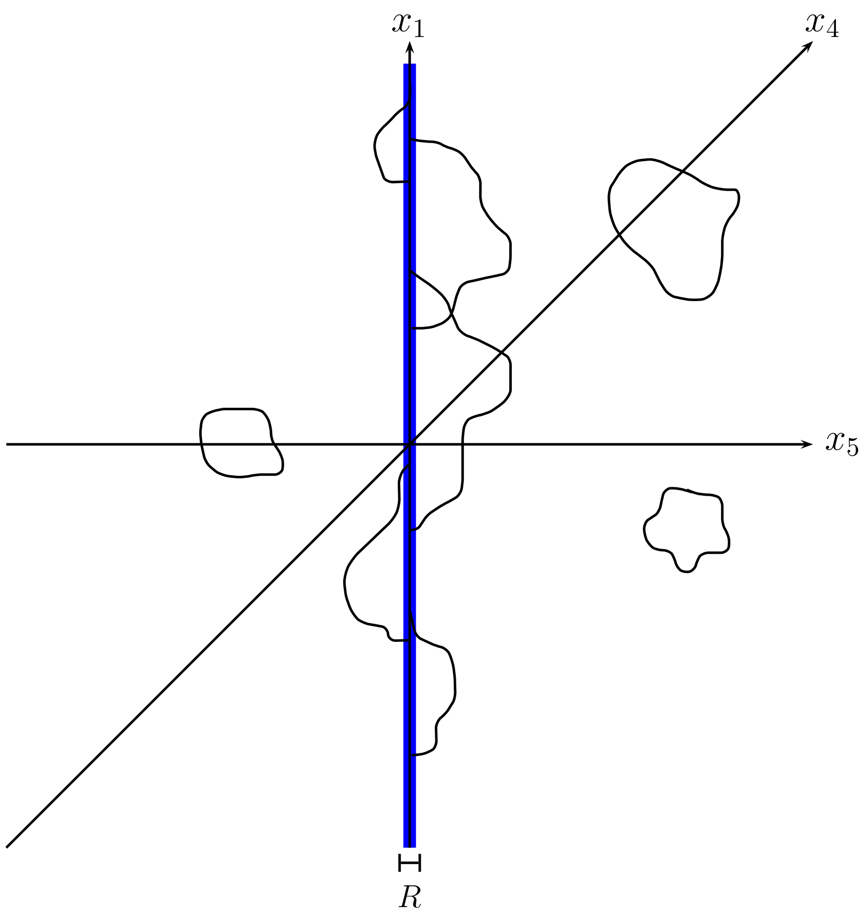

Figure 3.1: The solution of N D3-branes from the viewpoint of $R \ll l_s$. The situation is projected to directions x_1, x_4 and x_5. The branes are extended objects in directions x_1, x_2 and x_3 and are marked by blue in the picture. As $R \ll l_s$, the area where metric would be of the form (3.4) is too small to have any effect. Some open and closed strings are drawn in the picture.

In order to see the duality take shape, it is enough to consider merely the low-energy limit of the physics. In section 3.1, we noted that the low-energy effective theories for open strings attached to branes are gauge theories in general. In this case, the low-energy theory turns out to be the $\mathcal{N} = 4$ $U(N)$ SYM -theory. For the closed strings, the effective theory is simply IIB supergravity. Because the closed strings are non-interacting at the low-energy limit (as noted in section 3.1), they decouple from the open strings and we are left with $\mathcal{N} = 4$ $U(N)$ SYM -theory describing the physics on the branes, i.e. in

3+1 dimensions (with N still being the number of branes in the stack), and IIB SUGRA describing it in the bulk, i.e. in 9+1 dimensions.

The second viewpoint gives a different kind of interpretation for the situation. When $R \gg l_s$, both areas $1 \ll \frac{R}{r}$ and $1 \gg \frac{R}{r}$ are relevant to the physics. Now the physics consists of closed strings moving in curved spacetime (see 3.2). Locally, the low-energy limit is everywhere the IIB SUGRA, but now we have to also consider the global geometry when thinking about the low-energy limit of the system as a whole.

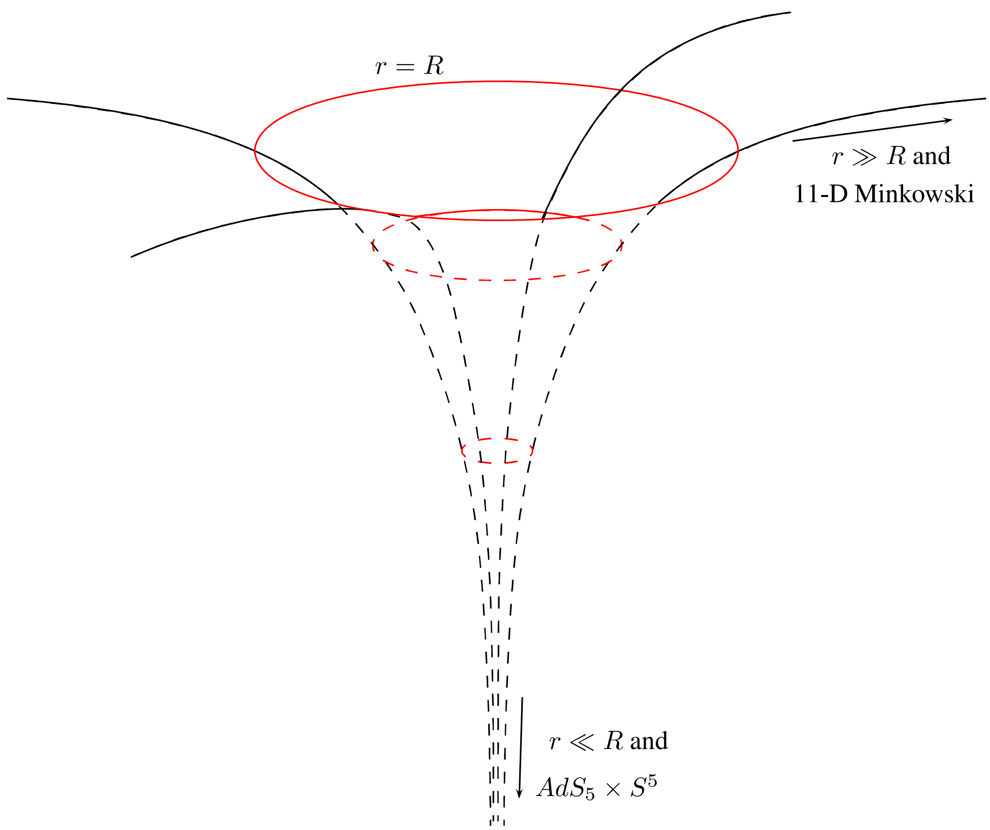

Figure 3.2: A schematic picture of the viewpoint $R \gg l_s$. The red lines are $r =$constant lines and the black ones are radial lines. Far away from the throat the whole metric approaches the ten-dimensional Minkowski metric (3.5) whereas in the throat it becomes (3.4), the $AdS_5 \times S^5$. The branes can be thought as lying deep down in the throat, at $r = 0$, creating the curved background where the closed strings move.

The space is now roughly divided into two parts (3.4) and (3.5). As the space is asymptotically Minkowskian, the physics of closed strings moving far from the throat at the low-energy limit is again described by the IIB SUGRA, as in the previous case. However, in the throat all string states are left when taking the low-energy limit: the further down in the throat a string is, the more curved the background space is, and this leaves more and more states left in the effective theory. This means that we need the whole IIB string theory for depicting the physics near the throat. On the other hand, the strings in the throat decouple from those far away from it, as the latter ones are non-interacting at the low-energy limit, therefore the string theory physics in the throat is independent of the IIB SUGRA far away from it. One can think of this also from the point of view of an observer in infinity: the deeper a string is in the throat, the more energy it has to have in order to be seen from infinity because of the redshift. From this viewpoint, the surface of the branes is so deep down in the throat, that the open string excitations cannot be seen at all and therefore they do not play a role in the low-energy limit description.

We now have two different pictures of the situation. However, there does not exist any strict border between them: one can start from the latter viewpoint and smoothly shrink the curvature parameter R until reaching the first viewpoint. In particular, one can think about cases where $R \approx l_s$. This brings forth the idea that these two different pictures should describe the very same underlying, real physics - and that there should exist some kind of connection between them, the duality. We can summarize the situation on the following table:

	The low-energy theory near the brane	The low-energy theory far away from the brane
$R \ll l_s$: a stringy point of view	$\mathcal{N} = 4\ U(N)$ SYM	IIB SUGRA in ten-dimensional Minkowski space
$R \gg l_s$: a supergravity point of view	IIB string theory on $AdS_5 \times S^5$-background	IIB SUGRA in ten-dimensional Minkowski space

Table 3.1: A summary of the above discussion. In both situations, the physics far from the branes is independent of that close to them. It is reasonable to assume that the IIB SUGRA in ten-dimensional Minkowski space and the IIB string theory in $AdS_5 \times S^5$ space describe the same physical situation.

It seems now somehow reasonable to predict that there must exist some machinery between IIB string theory in $AdS_5 \times S^5$-space and the $\mathcal{N} = 4\ U(N)$ super Yang-Mills -theory. This is the duality first postulated by Maldacena in [8]. Because the different low-energy theories were obtained by taking different limits, the duality is just a conjecture. [22] In order to make this more exact and plausible, we review some of the basic properties of AdS-spaces and consider the symmetries of these two theories, before giving the exact formulation of the duality by Witten.

3.2.2 The AdS-geometry and symmetry comparison

An $1+p$-*dimensional anti-de Sitter space* can be defined [13] as a surface

$$-\sum_{i=1}^{2} x_i^2 + \sum_{i=3}^{p+2} x_i^2 = -R^2 \tag{3.6}$$

embedded in a $2+p$-dimensional Minkowski space $\mathcal{M}_2^{2+p}$ with two timelike dimensions, i. e. a $2+p$-dimensional space with the metric

$$ds^2 = -\sum_{i=1}^{2} dx_i^2 + \sum_{i=3}^{p+2} dx_i^2. \tag{3.7}$$

We begin by showing that the metric (3.4) is indeed $AdS_5 \times S^5$ with the above definition. This can be seen through a coordinate transformation. First define a new set of coordinates in $\mathcal{M}_2^{2+p}$ as

$$\begin{aligned} r &= x_1 + x_3 \\ v &= x_1 - x_3 \\ y_i &= \frac{Rx_i}{x_1 + x_3} = \frac{Rx_i}{r} \quad \forall\, i > 3 \\ t &= \frac{Rx_2}{x_1 + x_3} = \frac{Rx_2}{r} \end{aligned} \tag{3.8}$$

Also define notation $y^2 = \sum_{i=4}^{p+2} y_i^2$. In these coordinates, the equation defining the AdS-space becomes

$$-rv - \frac{r^2}{R^2}t^2 + \frac{r^2}{R^2}y^2 = -R^2. \tag{3.9}$$

From this one can easily solve for v:

$$v = \frac{R^2}{r} + \frac{r}{R^2}(y^2 - t^2). \tag{3.10}$$

Inverting the transformation (3.8) and applying (3.10) we get

$$\begin{aligned} x_1 &= \frac{1}{2}(r + v) = \frac{1}{2}(r + \frac{R^2}{r} + \frac{r}{R^2}(y^2 - t^2)) \\ x_2 &= \frac{r}{R}t \\ x_3 &= \frac{1}{2}(r - v) = \frac{1}{2}(r - \frac{R^2}{r} - \frac{r}{R^2}(y^2 - t^2)) \\ x_i &= \frac{r}{R}y_i \quad \forall\, i > 3 \end{aligned}$$

Using this, we can write down the induced metric on the manifold (3.6), i.e. the metric of the AdS-space as

$$\begin{aligned}
ds^2 &= -\frac{1}{4}(dr - \frac{R^2}{r^2}dr + \frac{1}{R^2}(y^2 - t^2)dr + \frac{2r}{R^2}(y_i dy_i - tdt))^2 - \frac{1}{R^2}(tdr + rdt)^2 \\
&\quad + \frac{1}{4}(dr + \frac{R^2}{r^2}dr - \frac{1}{R^2}(y^2 - t^2)dr - \frac{2r}{R^2}(ydy - tdt))^2 + \frac{1}{R^2}\sum_{i=3}^{2+p}(rdy_i + y_i dr)^2 \\
&= \frac{R^2}{r^2}dr^2 - \frac{1}{R^2}(y^2 - t^2)dr^2 - \frac{2r}{R^2}(y_i dy_i - tdt)dr - \frac{t^2}{R^2}dr^2 - \frac{2tr}{R^2}drdt \\
&\quad - \frac{r^2}{R^2}dt^2 + \frac{r^2}{R^2}\sum_{i=3}^{2+p} dy_i^2 + \frac{2ry_i}{R^2}drdy_i + \frac{y^2}{R^2}dr^2 \\
&= \frac{R^2}{r^2}dr^2 - \frac{r^2}{R^2}dt^2 + \frac{r^2}{R^2}\sum_{i=3}^{2+p} dy_i^2, \qquad (3.11)
\end{aligned}$$

which is of the form (3.4) (without the S^5 part, of course).

Note that the singularity $r = 0$ of the metric (3.11) arose only because the coordinate transformation (3.8) was not defined for $r = 0$. Thus, this is merely a coordinate singularity from the viewpoint of the whole AdS_5-space - the new coordinates do not completely cover it. We are not interested in examining this any further, we have shown that the metric (3.8) is $AdS_5 \times S^5$.

Next we will consider the symmetries of the AdS-space. By construction (3.6), the AdS_{1+p}-space is clearly symmetric under $SO(2,p)$. The important observation is that this is isomorphic to the conformal group in p-dimensional Minkowski space. Begin by considering the generators of the latter [14] ($\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$):

$$\begin{array}{rrcl}
\text{Translation} & P_\mu & = & -i\partial_\mu \\
\text{Rotation} & L_{\mu\nu} & = & i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
\text{Dilation} & D & = & -ix^\mu\partial_\mu \\
\text{Special conformal transformation (SCT)} & K_\mu & = & -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)
\end{array} \qquad (3.12)$$

and the corresponding conformal algebra [14]:

$$\begin{array}{rcl}
[D,\, P_\mu] & = & iP_\mu \\
[D,\, K_\mu] & = & -iK_\mu \\
[K_\mu,\, P_\nu] & = & 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho,\, L_{\mu\nu}] & = & i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\
[P_\rho,\, L_{\mu\nu}] & = & i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu},\, L_{\rho\sigma}] & = & i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \\
[D,\, L_{\mu\nu}] = [K_\mu,\, K_\nu] = [P_\mu,\, P_\nu] & = & 0
\end{array} \qquad (3.13)$$

Then one can define a new basis for this algebra:

$$\begin{array}{rcl} J_{0,p+1} & = & D \\ J_{0,\mu} & = & \frac{1}{2}(P_\mu - K_\mu) \\ J_{p+1,\mu} & = & \frac{1}{2}(P_\mu + K_\mu) \\ J_{\mu\nu} & = & L_{\mu\nu} \end{array} \tag{3.14}$$

with $J_{ab} = -J_{ba},\ a,b = \{0,1,\ldots,p,p+1\}$. One can now check that for this basis the algebra comes into the form (with $\eta_{\mu\nu} = diag(-1,-1,1,\ldots 1)$)

$$[J_{ab},\ J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}), \tag{3.15}$$

which is the algebra of the generators of $SO(2,p)$. Thus the spatial symmetry of $\mathcal{N} = 4\ U(N)$ SYM is the same as that of the AdS_5-space.

We still have to check that other symmetries of these theories, IIB ST in $AdS_5 \times S^5$-space and the $\mathcal{N} = 4\ U(N)$ SYM in 4-dimensional Minkowski are equal. For this we need some knowledge of supersymmetric theories and their symmetries. Here we will just note the results needed and refer to sources for interested readers.

The whole symmetry of the $\mathcal{N} = 4\ U(N)$ SYM side consists of the covering group[4] $SU(2,2)$ for the conformal group $SO(2,4)$, of the internal symmetry group for the $\mathcal{N} = 4\ U(N)$ SYM, which is $SU(4)$, and of the fermionic generators for supersymmetry transformations. For a general $\mathcal{N} = 4$ supersymmetric theory this would mean $2 \times 2 \times 4 = 16$ supercharges (supersymmetry transformation generators), but the conformal symmetry brings with it the special supersymmetry transformations associated with the special conformal transformations[5]. Therefore, the complete symmetry group is the Lie supergroup $SU(2,2|4)$. [16] This is exactly the same as the total symmetry of the string theory side: there we have the spatial symmetry $SO(2,4) \times SO(6)$ (the symmetry groups of the AdS_5 and S^5, respectively), for which we again have to use the covering group, which is $SU(2,2) \times SU(4)$. In addition, as noted in section 3.1, there are 32 fermionic supercharges, which again combined with $SU(2,2) \times SU(4)$ give the Lie supergroup $SU(2,2|4)$. Thus the symmetries of these two seemingly different theories are indeed identical.

3.2.3 The boundary of AdS-space

Seen that the symmetries of the theories are equal, we want to examine further how to link the theories. In order to do this, one has to do something for the extra dimensions on the string theory side. The most natural way to get rid of them is to somehow integrate them out, so that we are left with a four-dimensional theory. The question is, what kind of 4-dimensional surface should we pick in the ten-dimensional space? Consider

[4]For more information of covering groups, see e.g. ch. 4.6.5 of [17]. We have to use them because we have supersymmetry and therefore spinor representation is relevant. [16]

[5]For a comprehensive discussion of $\mathcal{N} = 4\ U(N)$ SYM and its symmetries, see e.g. [18] The complete algebra can be found there on p. 48.

the metric near the branes (3.4). Here the branes can be thought as being located at $r = 0$, and the coordinates on the branes are t, x_1, x_2 and x_3. So we should choose a surface which leaves these coordinates free, resembles the four-dimensional Minkowski space and has the same transformation structure as $\mathcal{N} = 4$ $U(N)$ SYM when acted by $SO(2,4)$. It is natural to assume that we have to integrate out the spherical part: we already saw that the algebra of the symmetry group for the AdS part matches that for the four-dimensional Minkowski and those of the spherical part match the non-spatial symmetries of the $\mathcal{N} = 4$ $U(N)$ SYM. So we expect to find the right four-dimensional surface from the AdS part, setting the spherical coordinates to zero. Naively one could try setting $r = 0$, but this does not work as there is a coordinate singularity in $r = 0$, as mentioned. Thus something more has to be done in order to find the right surface, which can be identified with the branes.

Using the coordinates (3.8), we can reformulate the problem as finding two constraints for these coordinates which leave t, y_1, y_2 and y_3 free and which define a surface having the right topology and transformation properties. The definition for AdS space (3.6) gives already one constraint, so only one equation more is needed. Consider a scaling of the original coordinates on $\mathcal{M}_2^6$(3.7): $x_i = K\tilde{x}_i$ with $K \to \infty$. Now (3.6) becomes

$$-\sum_{i=1}^{2}\tilde{x}_i^2 + \sum_{i=3}^{6}\tilde{x}_i^2 = -(\frac{R}{K})^2 \to 0 \tag{3.16}$$

So we have reached a specific limit of the AdS space. However, one reaches this same limit also by considering a different scaling, $K' = bK$ with any nonzero $b \in \mathbb{R}$. This means that when considering this limit, one has to identify a solution $(x_1, \dots, x_6)$ of (3.16) with other solutions of the form $(bx_1, \dots, bx_6)$. Therefore, this limit defines a four-dimensional surface by

$$\begin{aligned} -\sum_{i=1}^{2}x_i^2 + \sum_{i=3}^{6}x_i^2 &= 0 \\ (x_1, \dots, x_6) &\sim (bx_1, \dots, bx_6) \end{aligned} \tag{3.17}$$

Note that the scaling $x_i = K\tilde{x}_i$ leaves the coordinates t, y_1, y_2 and y_3 unchanged, so when we identify different scalings, we do not identify points with different t, y_1, y_2 and y_3-coordinates. Therefore, there is a bijection between this limit and the branes, and so this is a good candidate for the four-dimensional surface we want. (3.17) is usually referred to as the *boundary of the AdS-space* although it is a boundary only in a conformal sense[15]. In these $(r,\ t,\ y_1,\ y_2,\ y_3)$-coordinates, this boundary is at $r = \infty$, which we can identify with the border separating the areas $r \ll R$ and $r \gg R$. [22]

We can now consider different ways to choose the representatives for each equivalence class. One way is to choose for any point $(x_1, \dots, x_6)$ a point $(bx_1, \dots, bx_6)$ from the same equivalence class so that $b^2\sum_{i=3}^{6}x_i^2 = 1$.[6] This means that one can choose one representative

[6] As we are considering the limit of big x_i, points with $\sum_{i=3}^{6}x_i^2 = 0$ clearly are not included - that would

from each equivalence class so that the surface (3.17) becomes

$$\sum_{i=1}^{2} x_i^2 = \sum_{i=3}^{6} x_i^2 = 1, \tag{3.18}$$

the topology of which is $S^1 \times S^3$, which is the same as the four-dimensional Minkowski space compactified with the point at infinity. So this choice clearly demonstrates the topology of the boundary being a suitable one.

Another useful way to choose is to change into coordinates

$$\begin{aligned} r &= x_1 + x_2 \\ v &= x_1 - x_2 \\ y_i &= x_i \end{aligned} \tag{3.19}$$

In which the definition of the boundary comes into the form

$$\begin{aligned} -rv + y^2 &= 0 \\ (r,\, v,\, \vec{y}) &\sim (br,\, bv,\, b\vec{y}) \end{aligned} \tag{3.20}$$

and choose $r = 1$. [7]This is possible if $r \neq 0$. Then $v = y^2$ and we can use just $\vec{y}$ to define the point on the boundary. If also $v \neq 0$, we can choose $v = 1$ and use another coordinate chart with $r = \tilde{y}^2$. These are clearly related through $\vec{y} = \frac{\vec{\tilde{y}}}{\tilde{y}^2}$, or, equivalently, through $\vec{\tilde{y}} = \frac{\vec{y}}{y^2}$. When $r = 0 = v$, we clearly have just $\vec{y}$ left to define the point. This choice is useful, as it allows us to see how the symmetry group $SO(2,4)$ of the AdS_5-space acts on the boundary: we just have to track what happens to the coordinates $(y_2, y_4, y_5, y_6) = \vec{y}$, as these are enough to specify a point at the boundary. It is sufficient to go through just the case $r \neq 0$, as the other two are identical.

Consider now the symmetry group $SO(2,4)$ of the AdS_5-space acting on the surface (3.20), following [16]. In these coordinates (3.19), an infinitesimal transformation $\Lambda \in SO(2,p)$ must be of the form $\Lambda = \mathbf{1}_{(2+p)\times(2+p)} + \Omega$, with

$$\Omega = \begin{pmatrix} a & 0 & 2\vec{\alpha}^T \\ 0 & -a & 2\vec{\beta}^T \\ \vec{\beta} & \vec{\alpha} & \omega \end{pmatrix}, \tag{3.21}$$

with $\vec{\alpha}, \vec{\beta}$ infinitesimal 4-vectors and ω an infinitesimal Lorentz-transformation (4×4 - matrix). This can be easily checked by demanding that the length of a vector is unchanged in the transformation. Now let $(r, v, \vec{y})$ be a point on the boundary so that $r = 1$. Then

$$(\mathbf{1} + \Omega) \begin{pmatrix} r \\ v \\ \vec{y} \end{pmatrix} = \begin{pmatrix} r + ar + 2\vec{\alpha}\cdot\vec{y} \\ v - av + 2\vec{\beta}\cdot\vec{y} \\ \vec{\beta} r + \vec{\alpha} v + (1+\omega)\vec{y} \end{pmatrix} \equiv \begin{pmatrix} r' \\ v' \\ \vec{y}' \end{pmatrix}. \tag{3.22}$$

---

require also $\sum_{i=1}^{2} x_i^2 = 0$.

[7]Here $\vec{y} = (y_2, y_4, y_5, y_6)$ and $y^2 = -y_2^2 + \sum_{i=4}^{6} y_i^2$

However, now for the transformed point $(r', v', \vec{y}')$ $r \neq 1$ in general, but we can always choose another representative for the equivalence class the transformed point belongs to so that $r = 1$. That means, remembering $v = y^2$,

$$\begin{pmatrix} r' \\ v' \\ \vec{y}' \end{pmatrix} \sim \begin{pmatrix} 1 \\ \frac{y^2 - ay^2 + 2\vec{\beta}\cdot\vec{y}}{1+a+2\vec{\alpha}\cdot\vec{y}} \\ \frac{\vec{\beta}+\vec{\alpha}y^2+(1+\omega)\vec{y}}{1+a+2\vec{\alpha}\cdot\vec{y}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{y^2 - ay^2 + 2\vec{\beta}\cdot\vec{y}}{1+a+2\vec{\alpha}\cdot\vec{y}} \\ \vec{\beta} + \vec{\alpha}y^2 + (1+\omega)\vec{y} - a\vec{y} - 2(\vec{\alpha}\cdot\vec{y})\vec{y} \end{pmatrix} \tag{3.23}$$

Therefore, the transformed coordinate on the boundary is $\tilde{y} = \vec{y} + \vec{\beta} + \omega\vec{y} - a\vec{y} + \vec{\alpha}y^2 - 2(\vec{\alpha}\cdot\vec{y})\vec{y}$. This transformation is actually a general infinitesimal conformal transformation: it consists of translations $(\vec{\beta})$, Lorentz transformations (ω), dilation (a) and special conformal transformation $(\vec{\alpha})$. So the symmetry group $SO(2,4)$ of the AdS_5-space acts on the boundary just like the conformal group acting on four-dimensional Minkowski space.

We have now shown that all AdS spaces have a specific surface (3.17), which can be thought of as the boundary of the AdS space, in a conformal sense (as it was defined by using the scaling equivalence) [15]. Specifically, for AdS_5, this boundary has the topology of the compactified four-dimensional Minkowski space. In addition the symmetry group $SO(2,4)$ of the AdS_5-space acts on the boundary just like the conformal group acting on four-dimensional Minkowski space. As there is a bijection between points on the boundary and points on the branes, we can identify the boundary and the branes, and, more importantly, the physics on the branes and on the boundary.

3.2.4 Formal definition of the duality

As we have now motivated the duality, we are ready to give an exact definition for it. The proposed relation, formulated by Witten in [9] and Gubser et al. in [34], is

$$Z_{CFT}[J] = \int e^{-S_{CFT} - \int J(x)\mathcal{O}(x)d^4x} \mathcal{D}\mathcal{O} = Z_{ST}\left[\varphi(x)|_{\partial AdS} = J(x)\right] \tag{3.24}$$

Basically, this tells just that the (euclidianized) partition function of the conformal field theory is identical to that of the string theory on the conformal boundary. It states that for any operator $\mathcal{O}$ of the CFT exists a source field J, that is also the value of a string theory field φ at the conformal boundary, and the theories are linked through this connection. It is to remember that this relation is still in a mathematical sense a very hazily defined conjecture. On the other hand, the proposition has so far survived all the numerous tests made without any redefinition or tuning. [19]

With the duality, one can now calculate any correlation function of the CFT using the string theory side:

$$\langle \mathcal{O}(x) \rangle = -\frac{1}{Z[0]} \left.\frac{\delta Z_{CFT}[J]}{\delta J}\right|_{J=0} = -\frac{1}{Z[0]} \left.\frac{\delta Z_{ST}\left[\varphi(x)|_{\partial AdS} = J(x)\right]}{\delta J}\right|_{J=0} \tag{3.25}$$

Thus the duality relates the correlator functions of the CFT to the field configurations of the string theory.

3.3 The emergence of the fluid-gravity correspondence

Next we will show how the duality can be used to derive conformal hydrodynamics. Clearly, we want to compute $\langle T^{\alpha\beta}\rangle$ using the duality. By definition, the stress-energy tensor is given by $T^{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}$, this being equivalent with (2.2). Thus, we should get the stress-energy tensor via the duality as ($g_{\alpha\beta}$ being the metric of the field theory and γ_{MN} that of the string theory side[8], and C_{S^5} being an irrelevant normalization constant taking care of the constant coefficient coming from the integration over the spherical part)

$$\langle T^{\alpha\beta}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta S_{Eff}}{\delta g_{\alpha\beta}} = -\frac{2}{\sqrt{-g}}\frac{1}{C_{s^5}}\frac{1}{Z\left[0\right]}\left.\frac{\delta Z_{ST}\left[\gamma_{MN}(x)|_{\partial AdS} = g_{\alpha\beta}\right]}{\delta g_{\alpha\beta}}\right|_{g_{\alpha\beta}=0} \tag{3.26}$$

This leads immediately into problems. Apparently, the full string theory generating functional in AdS-space would be needed, but that is not available - as discussed in section (3.1), in general, one can do only either a loop expansion in flat background (string field theory) or classical string theory in curved background with string length divided by the curvature scale as the expansion coefficient. The exact string theory solution (3.1), on which the duality was built, was an exception.

The solution is to note that we are interested in something related to conformal hydrodynamics, which is a special limit of the conformal field theory. In this section is first discussed what is left of the duality when going into this limit. Then we will use that limit to calculate $\langle T^{\alpha\beta}\rangle$ - only to get zero. This new problem is seen to stem from the fact that the exact solution (3.1) describes a ground state of the branes in a vacuum. However, we will see that on the fluid-gravity limit of the duality also other solutions are allowed. Finally, we will get the $\langle T^{\alpha\beta}\rangle$ up to the zeroth order by choosing a special solution. It will be seen that the more terms one wants to calculate for the stress-energy tensor, the more complicated a metric one needs. However, in order to get the metric one only has to use ordinary general relativity. In the next chapter, this process will be done explicitly in detail up to the first order.

3.3.1 Modifying the duality: the large t'Hooft limit

On the hydrodynamic limit the coupling constant of the field theory has to be very big in order to keep the fluid in local thermodynamic equilibrium. Therefore, we need only the $g_{Eff} = \infty$ -limit of the duality, with g_{Eff} being the effective coupling constant for the field theory. As the field theory is the low-energy limit of the open strings on the branes, the effective coupling constant would seem to be related with the string coupling constant as

$$g_{Eff} \sim g_{YM} \sim g_{os} \sim \sqrt{g_{cs}} \tag{3.27}$$

using (3.3). This would be bad news, though: for the hydrodynamic limit, one would need the strongly coupled string theory.

[8]From now on we use $M, N, K, \ldots$ as five-indices and $\alpha, \beta, \gamma, \ldots$ as four-indices.

The problem can be solved by taking the *t'Hooft limit*. This is a limit of $\mathcal{N} = 4\ U(N)$ SYM with $N \to \infty$. On this limit, the effective coupling constant of the theory is the t'Hooft coupling $\lambda = g_{YM}^2 N$ which is kept fixed when taking $N \to \infty$. This corresponds to the planar limit of the field theory, planar referring to the Feynman diagrams left in the series expansion. [15] Now we have

$$g_{Eff} \sim \lambda = g_{YM}^2 N \sim g_{cs} N, \tag{3.28}$$

so having λ fixed while taking $N \to \infty$ means that on the string theory side one has $g_{cs} \to 0$, thus the t'Hooft limit corresponds to classical string theory.

When at the t'Hooft limit, we can now go further to the hydrodynamic limit by taking $g_{Eff} \sim \lambda \to \infty$. This is the *large t'Hooft limit.* [16] Remembering the equation (3.2), we see that this means, keeping the curvature parameter R fixed, that

$$4\pi g_{cs} N = \left(\frac{R}{l_s}\right)^4 \to \infty \Rightarrow l_s \to 0. \tag{3.29}$$

This means that at the large t'Hooft limit, only classical supergravity is left on the string theory side. This is a major simplification, as the classical supergravity action is well-known. We can summarize the above discussion on the following table:

Full duality	The $\mathcal{N} = 4\ U(N)$ SYM side	The IIB string theory side
t'Hooft limit:$N \to \infty$, $\lambda = g_{YM}^2 N$ fixed	$\mathcal{N} = 4\ U(N)$ SYM with $N \to \infty$, $g_{Eff} \sim \lambda = g_{YM}^2 N$	Classical string theory ($g_{cs} \to 0$)
large t'Hooft limit: $\lambda \to \infty$	Strongly coupled $\mathcal{N} = 4\ U(N)$ SYM with $N \to \infty$	Classical supergravity ($l_s \to 0$)

Table 3.2: A summary of the above discussion. Taking the large t'Hooft limit simplifies the mathematics considerably, as full string theory is truncated to the classical supergravity. A highly non-trivial fluid-gravity -duality is still left, and that is exactly what is needed here.

3.3.2 First attempt to derive the stress-energy -tensor

We can now try what happens if we compute the stress-energy tensor using the hydrodynamic limit of the duality. The full duality version (3.26) for $\langle T^{\alpha\beta}\rangle$ comes into form

$$\begin{aligned}\langle T^{\alpha\beta}\rangle &= -\frac{2}{\sqrt{-g}}\frac{1}{C_{S^5}}\frac{1}{Z[0]}\frac{\delta Z_{SUGRA}\left[\gamma_{MN}(x)|_{\partial AdS}=g_{\alpha\beta}\right]}{\delta g_{\alpha\beta}}\bigg|_{g_{\alpha\beta}=0}\\ &= -\frac{2}{\sqrt{-g}}\frac{1}{C_{S^5}}\frac{1}{e^{-S_{SUGRA}}}\frac{\delta e^{-S_{SUGRA}}}{\delta g_{\alpha\beta}}\\ &= \frac{2}{\sqrt{-g}}\frac{1}{\int R^5 d\Omega_5}\frac{\delta\int\sqrt{\gamma}\mathcal{L}_{SUGRA}(\gamma_{MN})R^5 d^5x d\Omega_5}{\delta g_{\alpha\beta}}\\ &= \frac{2}{\sqrt{-g}}\frac{\delta\int\sqrt{\gamma}\mathcal{L}_{SUGRA}(\gamma_{MN})d^5x}{\delta g_{\alpha\beta}} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}\end{aligned} \tag{3.30}$$

where we need only the metric part of the supergravity action - we can set all the other fields to zero. On the second line was used the fact that we have classical supergravity, so $Z_{SUGRA} = e^{-S_{SUGRA}}$ and on the third the fact that the metric and therefore also the variation of the action with respect to the boundary metric are independent of the spherical coordinates. Thus we are left with Einstein-Hilbert action [5]:

$$S = \frac{1}{16\pi G}\int_{AdS_5}\sqrt{\gamma}(\mathsf{R}-\Lambda)d^5x \tag{3.31}$$

Here R is used for the Ricci scalar in order to distinguish it from the curvature radius R of the metric (3.11). We already know the metric (3.11) is a solution for the Einstein equation (because it was a solution of the full duality), we just have to relate Λ to R: the Einstein equation (for empty space) is [5]

$$R_{MN} - \frac{1}{2}\mathsf{R}g_{MN} \equiv G_{\alpha\beta} = -\frac{1}{2}\Lambda g_{MN} \tag{3.32}$$

and taking trace on both sides we get

$$\begin{aligned}\mathsf{R} - \frac{5}{2}\mathsf{R} &= -\frac{5}{2}\Lambda \qquad\qquad \Leftrightarrow\\ \Lambda &= \frac{3}{5}\mathsf{R}\end{aligned} \tag{3.33}$$

So we have to obtain the Ricci scalar for the metric (3.11). Using [5]

$$\mathsf{R} = g^{MN}\left(\Gamma^K_{MN,K} - \Gamma^K_{MK,N} + \Gamma^K_{MN}\Gamma^L_{LK} - \Gamma^K_{ML}\Gamma^L_{NK}\right) \tag{3.34}$$

one gets after straightforward calculation

$$\mathsf{R} = -\frac{20}{R^2} \Rightarrow \Lambda = -\frac{12}{R^2} \tag{3.35}$$

However, (3.31) is applicable only for spaces without a boundary. As our AdS_5-space has a boundary and we wish to compute the variation of the action with respect to the boundary metric, the boundary term in the variation does not disappear and we have to take it into account. The generalized Einstein-Hilbert action is with Gibbons-Hawking-York boundary term (with K being the trace of the extrinsic curvature tensor)[9] [5]:

$$S = -\frac{1}{16\pi G}\int_{AdS_5}\sqrt{\gamma}(\mathsf{R}-\Lambda)d^5x - \frac{1}{8\pi G}\int_{\partial AdS_5}\sqrt{-g}Kd^4x \tag{3.36}$$

We could now compute the stress-energy tensor using this action, but then it would be expected that the result will be divergent: nothing comparable to the renormalization procedure needed for the field theory has yet been done. On the AdS side this can be seen from the fact that the boundary is at $r = \infty$ in the coordinates (3.8), and thus the boundary metric induced by (3.11) will be divergent. As mentioned already in [9], the divergences are directly related to the UV divergences of the field theory and therefore should be removed. Balasubramanian and Kraus defined an unambiguous process for defining boundary counterterms added to the action (3.36) in [20]. The counterterms are fixed uniquely by the requirement that they cancel the divergences. As long as the counterterms are local boundary terms, i.e. boundary terms depending only on the boundary metric, this has no effect on the bulk equations of motion. Balasubramanian and Kraus have a rather general treatment, but here we need only their result for the counterterm for AdS_5-space. With it, the complete action becomes

$$\begin{aligned} S &= -\frac{1}{16\pi G}\int_{AdS_5}\sqrt{\gamma}(\mathsf{R}-\Lambda)d^5x - \frac{1}{8\pi G}\int_{\partial AdS_5}\sqrt{-g}Kd^4x \\ &\quad -\frac{1}{8\pi G}\int_{\partial AdS_5}\frac{\sqrt{-3\Lambda}}{2}\sqrt{-g}(1+\frac{1}{\Lambda}\tilde{\mathsf{R}})d^4x, \end{aligned} \tag{3.37}$$

where $\tilde{\mathsf{R}}$ is the Ricci scalar for the boundary metric.

Now we can finally compute the stress-energy tensor. We get (the variation of the bulk part vanishes, as we are considering the classical solutions of Einstein equations)

$$\begin{aligned} T^{\alpha\beta} &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}} \\ &= \frac{2}{\sqrt{-g}}\Bigg[-\frac{1}{8\pi G}(\frac{1}{2}\sqrt{-g}g^{\alpha\beta}K - \sqrt{-g}K^{\alpha\beta}) \\ &\quad -\frac{1}{8\pi G}\left(\frac{1}{2}\frac{\sqrt{-3\Lambda}}{2}\sqrt{-g}g^{\alpha\beta} - \frac{1}{2}\sqrt{\frac{-3}{4\Lambda}}\sqrt{-g}g^{\alpha\beta}\tilde{\mathsf{R}} + \sqrt{\frac{-3}{4\Lambda}}\sqrt{-g}\mathsf{R}^{\alpha\beta}\right)\Bigg] \\ &= \frac{1}{8\pi G}\left[K^{\alpha\beta} - Kg^{\alpha\beta} - \frac{\sqrt{-3\Lambda}}{2}g^{\alpha\beta} - \sqrt{\frac{-3}{\Lambda}}\tilde{G}^{\alpha\beta}\right] \end{aligned} \tag{3.38}$$

[9] The extrinsic curvature tensor is defined as [23] $K^{\alpha\beta} = -\nabla^M\hat{n}^N e^\alpha_M e^\beta_N$, with $\hat{n}^\alpha$ being the outward pointing normal vector to the boundary ∂AdS_5 and e^α_M is the component M of the basis vector α of the boundary metric in the AdS_5-coordinates.

where $\tilde{G}^{\alpha\beta}$ is the Einstein tensor for the boundary metric $g^{\alpha\beta}$. Then, for the metric (3.11), we get

$$
\begin{aligned}
g^{\alpha\beta} &= \gamma^{MN} e^{\alpha}_{M} e^{\beta}_{N} = -\frac{R^2}{r^2} dt^2 + \frac{R^2}{r^2} d\vec{x}^2 && (3.39)\\
\tilde{G}^{\alpha\beta} &= 0 \text{ (as all Christoffel symbols vanish for } g^{\alpha\beta}) && (3.40)\\
n^M &= \nabla^M r = \gamma^{ML} \nabla_L r = \gamma^{ML} (\partial_L r) = \gamma^{Mr} = (\frac{r^2}{R^2}, 0, 0, 0, 0) && (3.41)\\
n_M &= (1, 0, 0, 0, 0) && (3.42)\\
\hat{n}^M &= (\frac{r}{R}, 0, 0, 0, 0) && (3.43)\\
K^{\alpha\beta} &= -\nabla^M \hat{n}^N e^{\alpha}_{M} e^{\beta}_{N} = -\gamma^{ML} \nabla_L \hat{n}^N \delta^{\alpha}_{M} \delta^{\beta}_{N} \\
&= -\gamma^{ML} \left(\partial_L \frac{r}{R} \delta^N_r + \Gamma^N_{KL} \frac{r}{R} \delta^K_r \right) \delta^{\alpha}_{M} \delta^{\beta}_{N} = -\gamma^{\alpha L} \left(\frac{1}{R} \delta^r_L \delta^{\beta}_r + \Gamma^{\beta}_{rL} \frac{r}{R} \right) \\
&= -\gamma^{\alpha L} \left(\Gamma^{\beta}_{rL} \frac{r}{R} \right) = -\gamma^{\alpha\beta} \left(\frac{1}{r} \frac{r}{R} \right) = -\frac{1}{R} g^{\alpha\beta} && (3.44)\\
K &= K^{\alpha}_{\alpha} = -\frac{4}{R} && (3.45)\\
\Longrightarrow T^{\alpha\beta} &= \frac{1}{8\pi G} \left[-\frac{1}{R} g^{\alpha\beta} - (-\frac{4}{R} g^{\alpha\beta}) - \frac{3}{R} g^{\alpha\beta} \right] = 0 && (3.46)
\end{aligned}
$$

So the stress-energy tensor completely vanishes. Our starting point for the whole duality and the resulting calculation for the stress-energy tensor was (3.1), which was an exact string theory solution. However, being exact, it describes the ground state of the stack of branes in empty space [16]. On the fluid-gravity limit, the ground state means that the fluid is at rest and at zero temperature - when compared with the equation derived for the stress-energy tensor earlier (2.32), we see that it really should be zero for this kind of fluid.

In order to get something non-zero out, we have to investigate something else than the ground state. This is possible, because we need only the large t'hooft limit for the calculation, and so demanding the solutions to be valid only at this limit is also sufficient. This opens up new possibilities for the metric as described next.

3.3.3 Solutions allowed at the large t'Hooft limit

The criterium for configurations we have at the large t'Hooft limit is that they have to be solutions for just the supergravity equations instead of full string theory. As shown in e.g. [16], a class of solutions for the metric is

$$
ds^2 = \frac{1}{\sqrt{1 + \frac{R^4}{r^4}}} \left(\left(\frac{r_0^4}{r^4} - 1 \right) dt^2 + \sum_{i=1}^{3} dx_i^3 \right) + \sqrt{1 + \frac{R^4}{r^4}} \left(\frac{1}{\left(1 - \frac{r_0^4}{r^4}\right)} dr^2 + r^2 d\Omega_5^2 \right) \quad (3.47)
$$

From the supergravity point of view, this is the metric part of a non-extremal 3-brane solution. Note that when $r_0 = 0$, (3.47) simplifies to (3.1), which is called extremal from the supergravity point of view. [10] At $r = r_0$, there is a horizon with coordinate singularity. From the string theory point of view, these are approximate solutions for the exited states of the branes. [10]

Now we expect the duality to be valid for (3.47) at the fluid-gravity limit. Every step leading to the full duality conjecture (3.24) can be reproduced for the metric (3.47), with mentioning that the resulting duality will be valid only at the large t'Hooft limit. The space is still divided in two parts with $R \gg r$ and $R \ll r$, and for the duality we need only the area $r \ll R$, where the metric (3.47) is

$$ds^2 = -\frac{r^2}{R^2}\left[\left(1 - \frac{r_0^4}{r^4}\right) dt^2 - \sum_{i=1}^{3} dx_i^3\right] + \frac{R^2}{r^2}\frac{1}{\left(1 - \frac{r_0^4}{r^4}\right)}dr^2 + R^2 d\Omega_5^2 \tag{3.48}$$

and the spherical part is again independent of the other. Noting that when $r \to \infty$, the non-spherical part of the metric approaches the AdS_5-metric and that in these coordinates the conformal boundary is also at $r = \infty$, we conclude that the surface is again the boundary at $r \to \infty$ and that it is again identified with the boundary separating the areas $R \gg r$ and $R \ll r$. Before computing the stress-energy tensor for this metric, we will interpret the parameter r_0. In order to do this, we have to examine black holes in AdS_5-space.

3.3.3.1 Black holes in AdS-space

We want to find a black hole -type solution for asymptotically AdS_5-space. We begin with a spherically symmetric and static ansatz, inspired by the Schwarzschild solution [2]

$$ds^2 = \frac{1}{f(r)}dr^2 - f(r)dt^2 + \frac{r^2}{R^2}\sum_{i=1}^{3} dx_i^2 \tag{3.49}$$

demanding that at $r \to \infty$, $f(r)$ approaches $g(r) = \frac{R^2}{r^2}$ so that the metric is asymptotically of the form (3.11). $f(r)$ can be solved from the Einstein equation

$$R_{MN} + \frac{10}{R^2}g_{MN} \equiv G_{MN} = -\frac{1}{2}\Lambda g_{MN} \tag{3.50}$$

with $\Lambda = -\frac{12}{R^2}$ as in (3.35) (because the situation has to be unchanged at the limit $r \to \infty$). We get for the nonzero Christoffel symbols

$$\Gamma^r_{rr} = \frac{1}{2} g^{rM} \left(g_{Mr,r} + g_{Mr,r} - g_{rr,M} \right) = \frac{1}{2} f(r) \left(-\frac{f'(r)}{f^2(r)} \right) = -\frac{f'(r)}{2f(r)} \tag{3.51}$$

$$\Gamma^r_{tt} = \frac{1}{2} g^{rM} \left(g_{Mt,t} + g_{Mt,t} - g_{tt,M} \right) = \frac{1}{2} f(r) f'(r) \tag{3.52}$$

$$\Gamma^r_{xx} = \frac{1}{2} g^{rM} \left(g_{Mx,x} + g_{Mx,x} - g_{xx,M} \right) = -f(r) \frac{r}{R^2} \tag{3.53}$$

$$\Gamma^t_{rt} = \frac{1}{2} g^{tM} \left(g_{Mr,t} + g_{Mt,r} - g_{rt,M} \right) = \frac{1}{2} \frac{f'(r)}{f(r)} \tag{3.54}$$

$$\Gamma^x_{rx} = \frac{1}{2} g^{xM} \left(g_{Mr,x} + g_{Mx,r} - g_{rx,M} \right) = \frac{1}{2} \frac{R^2}{r^2} 2 \frac{r}{R^2} = \frac{1}{r} \tag{3.55}$$

and for the Ricci tensor components

$$R_{rr} = \Gamma^M_{rr,M} - \Gamma^M_{rM,r} + \Gamma^N_{rr} \Gamma^M_{MN} - \Gamma^N_{rM} \Gamma^M_{rN} = -\frac{f''(r)}{2f(r)} - 3 \frac{f'(r)}{2f(r)} \frac{1}{r} \tag{3.56}$$

$$R_{tt} = \Gamma^M_{tt,M} - \Gamma^M_{tM,t} + \Gamma^N_{tt} \Gamma^M_{MN} - \Gamma^N_{tM} \Gamma^M_{tN} = \frac{1}{2} f(r) f''(r) + \frac{3}{2} f(r) f'(r) \frac{1}{r} \tag{3.57}$$

$$R_{xx} = \Gamma^M_{xx,M} - \Gamma^M_{xM,x} + \Gamma^N_{xx} \Gamma^M_{MN} - \Gamma^N_{xM} \Gamma^M_{xN} = -f'(r) \frac{r}{R^2} - 2f(r) \frac{1}{R^2} \tag{3.58}$$

Other components vanish. Thus, we get for the Ricci scalar

$$\mathsf{R} = -f''(r) - 6f'(r) \frac{1}{r} - 6f(r) \frac{1}{r^2} = -\frac{20}{R^2} \tag{3.59}$$

Finally, the Einstein equation gives the following scalar equations:

$$\frac{3}{2} f'(r) \frac{1}{r} + 3f(r) \frac{1}{r^2} = \frac{6}{R^2} \tag{3.60}$$

$$-\frac{3}{2} f'(r) \frac{1}{r} - 3f(r) \frac{1}{r^2} = -\frac{6}{R^2} \tag{3.61}$$

$$\frac{1}{2} f''(r) r^2 + 2f'(r) r + f(r) = \frac{6}{R^2} r^2 \tag{3.62}$$

The first two differential equations are identical and can easily be solved. The solution is

$$f(r) = \frac{r^2}{R^2} - \frac{C}{r^2} \tag{3.63}$$

and one can check that this also satisfies the third differential equation and the equation for the Ricci scalar (3.59).

Now we have the metric for an asymptotically AdS_5 black hole:

$$ds^2 = \frac{1}{\frac{r^2}{R^2} - \frac{C}{r^2}} dr^2 - \left(\frac{r^2}{R^2} - \frac{C}{r^2}\right) dt^2 + \frac{r^2}{R^2} \sum_{i=3}^{5} dx_i^2, \tag{3.64}$$

which is exactly the metric (3.48) with $CR^2 = r_0^4$ and we can interpret this constant using the definition of the Hawking temperature for black hole event horizons: [24]

$$T = \frac{|f'(r_h)|}{4\pi}, \tag{3.65}$$

where the event horizon is at r_h. For our metric (3.64) $r_h = r_0$, and we get

$$T = \frac{2\frac{r_0}{R^2} + 2\frac{C}{r_0^3}}{4\pi} = \frac{r_0}{R^2\pi} \Longleftrightarrow r_0 = TR^2\pi \tag{3.66}$$

We can now write the metric (3.48) using the temperature interpretation (3.66) for r_0. As already mentioned, in order to calculate the stress-energy tensor we need again only the non-spherical part, which becomes

$$ds^2 = -\frac{r^2}{R^2}\left[\left(1 - \frac{(T\pi)^4 R^8}{r^4}\right) dt^2 - \sum_{i=1}^{3} dx_i^3\right] + \frac{R^2}{r^2}\frac{1}{\left(1 - \frac{(T\pi)^4 R^8}{r^4}\right)} dr^2. \tag{3.67}$$

Defining a new scaled radial coordinate $\rho = R^{-2} r$ and $b = \frac{1}{T\pi}$, this simplifies to

$$ds^2 = -R^2\rho^2\left[\left(1 - \frac{1}{(b\rho)^4}\right) dt^2 - \sum_{i=1}^{3} dx_i^3\right] + \frac{R^2}{\rho^2}\frac{1}{\left(1 - \frac{1}{(b\rho)^4}\right)} d\rho^2. \tag{3.68}$$

R is now an overall scale factor and we are free to choose $R = 1$. Thus the solution is parametrized by only one value, b. Next we will compute the stress-energy tensor using this metric.

3.3.4 The stress-energy tensor at zeroth order

We will now repeat the calculation (3.39)-(3.46) using the metric (3.68). Because of the scaling $R = 1$, we get a multiplicative factor of r^6 in the stress-energy tensor (3.38). [20] We get (renaming for simplicity $\rho = r$ and using the already computed Christoffel symbols (3.51)-(3.55))

$$
\begin{aligned}
g^{\alpha\beta} &= \gamma^{MN} e^{\alpha}_{M} e^{\beta}_{N} = -\frac{1}{r^2\left(1-\frac{1}{(br)^4}\right)}dt^2 + \frac{1}{r^2\left(1-\frac{1}{(br)^4}\right)}d\vec{x}^2 && (3.69)\\
\tilde{G}^{\alpha\beta} &= 0 \text{ (as all Christoffel symbols vanish for } g^{\alpha\beta}) && (3.70)\\
n^M &= \nabla^M r = \gamma^{ML}\left(\partial_L r\right) = \gamma^{Mr} = \left(\left(r^2 - \frac{r^2}{(br)^4}\right), 0, 0, 0, 0\right) && (3.71)\\
n_M &= (1, 0, 0, 0, 0) && (3.72)\\
\hat{n}^M &= (r\sqrt{\left(1-\frac{1}{(br)^4}\right)}, 0, 0, 0, 0) && (3.73)\\
K^{\alpha\beta} &= -\nabla^M \hat{n}^N e^{\alpha}_{M} e^{\beta}_{N} = -\gamma^{ML}\nabla_L \hat{n}^N \delta^{\alpha}_{M}\delta^{\beta}_{N}\\
&= -\gamma^{\alpha L}\left(\partial_L r\sqrt{\left(1-\frac{1}{(br)^4}\right)}\delta^N_r + \Gamma^N_{KL} r\sqrt{\left(1-\frac{1}{(br)^4}\right)}\delta^K_r\right)\delta^{\beta}_N && (3.74)\\
&= -\gamma^{\alpha L}\Gamma^{\beta}_{rL} r\sqrt{\left(1-\frac{1}{(br)^4}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow K^{tt} &= \frac{1+\frac{1}{(br)^4}}{r^2\left(1-\frac{1}{(br)^4}\right)^{\frac{3}{2}}} && (3.75)\\
K^{xx} &= -\frac{1}{r^2\sqrt{\left(1-\frac{1}{(br)^4}\right)}} && (3.76)\\
K = K^{\alpha}_{\alpha} &= \frac{-4+\frac{2}{(br)^4}}{\sqrt{\left(1-\frac{1}{(br)^4}\right)}} && (3.77)\\
\Longrightarrow T^{tt} &= \lim_{r\to\infty}\frac{r^6}{8\pi G}\left[K^{tt} - Kg^{tt} - \frac{\sqrt{-3\Lambda}}{2}g^{tt} - \sqrt{\frac{-3}{\Lambda}}\tilde{G}^{tt}\right] && (3.78)\\
&= \lim_{r\to\infty}\left(\frac{1}{8\pi G}\frac{3}{2}\frac{1}{b^4} + \mathcal{O}\left(\frac{1}{r}\right)\right) = \frac{\pi^3}{16G}3T^4 && (3.79)\\
T^{xx} &= \lim_{r\to\infty}\left(\frac{1}{8\pi G}\frac{1}{2}\frac{1}{b^4} + \mathcal{O}\left(\frac{1}{r}\right)\right) = \frac{\pi^3}{16G}T^4 && (3.80)
\end{aligned}
$$

This can be written as

$$T^{\alpha\beta} = \frac{\pi^3}{16G}T^4\left(4u^{\alpha}u^{\beta} + \eta^{\alpha\beta}\right) \tag{3.81}$$

with $\eta^{\alpha\beta} = \lim_{r\to\infty} r^2 g^{\alpha\beta}$ is the inverse Minkowski metric of the CFT and $u^{\alpha} = (1, 0, 0, 0)$. This is exactly the stress-energy tensor for conformal fluids in zeroth order (2.32), with $C = \frac{\pi^3}{16G}$ and fluid being at rest.

Next we want to get the fluid moving, in order to do this we will introduce the boosted black hole.

3.3.4.1 Boosted black holes

Starting from the metric (3.68), we can change coordinates to (ρ, v, x_1, x_2, x_3) with

$$v = t + y, \qquad dy = \frac{d\rho}{\rho^2 \left(1 - \frac{1}{(b\rho)^4}\right)} \tag{3.82}$$

for which the metric can be written as

$$ds^2 = 2dvd\rho - \rho^2 \left(1 - \frac{1}{(b\rho)^4}\right) dv^2 + \rho^2 \sum_{i=1}^{3} dx_i^3. \tag{3.83}$$

In addition, for simplicity we rename $r = \rho$:

$$ds^2 = 2dvdr - r^2 \left(1 - \frac{1}{(br)^4}\right) dv^2 + r^2 \sum_{i=1}^{3} dx_i^3. \tag{3.84}$$

This can be covariantized [25] by setting $x^\alpha = (v, x_i)$ and using $u^\alpha = (1, 0, 0, 0)$, which both are vectors of four-dimensional Minkowski space (with metric $\eta_{\alpha\beta}$): this gives

$$v = -u_\alpha x^\alpha \quad \Rightarrow \quad dv^2 = u_\alpha u_\beta dx^\alpha dx^\beta \tag{3.85}$$

$$x^i = \left(u^i u_\alpha + \eta^i_\alpha\right) x^\alpha \quad \Rightarrow \quad dx_i dx^i = P_{\alpha\beta} dx^\alpha dx^\beta \tag{3.86}$$

with $P_{\alpha\beta} = u_\alpha u_\beta + \eta_{\alpha\beta}$ as in (2.7). Now the metric will be

$$ds^2 = -2u_\alpha dx^\alpha dr - r^2 \left(1 - \frac{1}{(br)^4}\right) u_\alpha u_\beta dx^\alpha dx^\beta + r^2 P_{\alpha\beta} dx^\alpha dx^\beta \tag{3.87}$$

With the metric in a manifestly covariant form, we can now boost the black hole along the translationally invariant spatial directions x_i by setting u^α to be any constant vector, normalized to $u_\alpha u^\alpha = -1$. As the boundary is a Minkowski space, it is invariant under Lorentz symmetry and so the boosted metric will still be a solution for the Einstein equations, [3] and the stress-energy tensor will still be (3.81), as expected.

We have now derived the expected stress-energy tensor for conformal fluids, in zeroth order, using the AdS/CFT duality. In the next chapter, we will introduce a process, by which the stress-energy tensor can be computed up to arbitrary order using only the same basic general relativity machinery introduced above. Then the stress-energy tensor will be calculated explicitly in the first order.

Chapter 4

Computing the CFT stress-energy tensor

In the previous chapter, we derived the stress-energy tensor for conformal fluids up to zeroth order using the fluid-gravity duality. In this chapter, we describe a process, by which one can find an explicit expression for the stress-energy tensor for conformal fluids up to arbitrary order. Also, the calculation will be done for the first-order terms. This chapter closely follows the original paper of Bhattacharyya, Hubney, Minvalla and Rangamani [26], in which the procedure and results were presented for the first time. Here we try to focus on the core subject and make the notation and calculations more explicit. Some steps are however too complicated and tedious in order to make sense to be written out on the paper, in these cases Maple was used to compute the results. In the original paper, Mathematica was used [27].

4.1 Description of the procedure

4.1.1 General idea

In the previous chapter, we had the metric (3.87) which was a solution of the Einstein equations and thus one was allowed to use the duality and extract the stress-energy tensor (3.81) from it. In order to get more terms in the derivative expansion of the stress-energy tensor, we should have non-constant functions $b(x_\mu), u^\mu(x_\mu)$. Therefore, we promote the constants b, u^μ in the metric to slowly-varying functions of the boundary coordinates. As we want to create a derivative expansion, we will introduce a coefficient ϵ for the boundary coordinates, for bookkeeping purposes (in the end it can be set to 1), so that every derivative brings one ϵ in front of the derived term.

Now we have a metric

$$
\begin{aligned}
ds^2 &= -2u_\alpha\left(x^\mu\right)dx^\alpha dr - r^2\left(1-\frac{1}{(br)^4}\right)u_\alpha\left(x^\mu\right)u_\beta\left(x^\mu\right)dx^\alpha dx^\beta \\
&\quad + r^2 P_{\alpha\beta}\left(x^\mu\right)dx^\alpha dx^\beta.
\end{aligned}
\tag{4.1}
$$

In general, this is not a solution for the Einstein equation (3.32). In order to see what happens when this metric is plugged into the Einstein equation, let us mark the metric (4.1) by g^0_{MN}. This can be Taylor-expanded, for simplicity around the point $x^\mu = 0$ as

$$g^0_{MN} \equiv g^{0,0}_{MN} + \epsilon x^\mu g^{0,1}_{MN,\mu} + \epsilon^2 x^\mu x^\nu g^{0,1}_{MN,\mu,\nu} + \mathcal{O}\left(\epsilon^3\right). \tag{4.2}$$

In the Einstein equation, the Ricci tensor consists of terms with either second derivatives of the metric (type $\Gamma^\alpha_{\beta\gamma,\delta}$) or terms with two times first derivative (type $\Gamma^\alpha_{\beta\gamma}\Gamma^\gamma_{\alpha\beta}$). Reorganizing terms, we can write the left-hand side of the Einstein equation for the metric (4.1) as

$$\begin{aligned} R_{MN}\left[g^0\right] + 4g^0_{MN} &= \\ R_{MN}\left[g^{0,0}\right] + \mathcal{O}(\epsilon) + 4g^{0,0}_{MN} + 4\epsilon x^\mu g^{0,1}_{MN,\mu} + \mathcal{O}\left(\epsilon^2\right) &= \mathcal{O}(\epsilon), \end{aligned} \tag{4.3}$$

as the metric $g^{0,0}_{MN}$ was a solution. Thus, it seems reasonable to guess that the metric g^0_{MN} could be corrected by some other metric ϵg^1_{MN} so that the Einstein equation would be valid up to order ϵ. This process could then be continued as long as one wants: at every step the next order correction to the metric is defined so that it satisfies the Einstein equation up to that order, and the result is the correct metric $g_{MN} = g^0_{MN} + \epsilon g^1_{MN} + \epsilon^2 g^2_{MN} + \ldots$.

We will now take a closer look at what happens when one tries to define a term $\epsilon^n g^n_{MN}$. At this point, one already has the terms $\epsilon^k g^k_{MN}$, $k < n$. The left-hand side of the Einstein equation can be written as (Taylor-expanding all terms of the metric as $g^n_{MN} \equiv g^{n,0}_{MN} + \epsilon x^\mu g^{n,1}_{MN,\mu} + \epsilon^2 x^\mu x^\nu g^{n,2}_{MN,\mu,\nu} + \cdots$)

$$\begin{aligned} &R_{MN}\left[g^0 + \ldots + \epsilon^n g^n\right] + 4\left(g^0_{\alpha MN} + \ldots + \epsilon^n g^n_{MN}\right) = \\ &\quad R_{MN}\left[g^0 + \ldots + \epsilon^{n-1} g^{n-1}\right] + \mathcal{F}_{\alpha MN}\left[\epsilon^n g^n\right] \\ &\quad + 4\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\left(\epsilon^{i+j}\underbrace{x^\mu \cdots x^\nu}_{j \text{ times}} g^{i,j}_{MN,\underbrace{\mu,\cdots,\nu}_{j\text{ times}}}\right) \\ &+ 4\epsilon^n \sum_{i=0}^{n-1}\left(\underbrace{x^\mu \cdots x^\nu}_{n-i \text{ times}} g^{i,n-i}_{MN,\underbrace{\mu,\cdots,\nu}_{n-i\text{ times}}}\right) + 4\epsilon^n g^{n,0}_{MN} + \mathcal{O}\left(\epsilon^{n+1}\right) \end{aligned} \tag{4.4}$$

where

$$\mathcal{F}_{MN} \equiv R_{MN}\left[g^0 + \ldots + \epsilon^n g^n\right] - R_{MN}\left[g^0 + \ldots + \epsilon^{n-1} g^{n-1}\right]. \tag{4.5}$$

In general, it depends on g^k_{MN} and can include any first or second order derivatives. Then using the fact that $g_{MN} = g^0_{MN} + \epsilon g^1_{MN} + \ldots + \epsilon^k g^k_{MN}$ already satisfies the Einstein equation up to order e^k, we can write

$$\begin{aligned} R_{\alpha MN}\left[g^0 + \ldots + \epsilon^{n-1} g^{n-1}\right] &= \\ -4\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\left(\epsilon^{i+j}\underbrace{x^\mu \cdots x^\nu}_{j \text{ times}} g^{i,j}_{MN,\underbrace{\mu,\cdots,\nu}_{j\text{ times}}}\right) + \epsilon^n \mathcal{R}_{MN} + \mathcal{O}\left(\epsilon^{n+1}\right) \end{aligned} \tag{4.6}$$

and replacing this into (4.4) we get

$$R_{MN}\left[g^0+\ldots+\epsilon^n g^n\right]+4\left(g^0_{MN}+\ldots+\epsilon^n g^n_{MN}\right)=$$
$$\mathcal{F}_{MN}\left[\epsilon^n g^n\right]+\epsilon^n\mathcal{R}_{MN}+4\epsilon^n\sum_{i=0}^{n-1}\left(\underbrace{x^\mu\cdots x^\nu}_{n-i\text{ times}}g^{i,n-i}_{MN,\underbrace{\mu,\cdots,\nu}_{n-i\text{ times}}}\right)+4\epsilon^n g^{n,0}_{MN}+\mathcal{O}\left(\epsilon^{n+1}\right) \quad (4.7)$$

This must be of $\mathcal{O}\left(\epsilon^{n+1}\right)$in order to g^n_{MN} be the right correction.

All terms except $\mathcal{F}_{MN}\left[\epsilon^n g^n\right]$and $4\epsilon^n g^{n,0}_{MN}$ depend only on the lower-order corrections of the metric and are thus already known. In general, the Taylor-expansion and the $\mathcal{R}_{MN}$-term coming from the Ricci tensor can both be calculated by computer. These can be called source terms and renamed

$$\mathcal{S}^n_{MN}\equiv-\mathcal{R}_{MN}-4\sum_{i=0}^{n-1}\left(\underbrace{x^\mu\cdots x^\nu}_{n-i\text{ times}}g^{i,n-i}_{MN,\underbrace{\mu,\cdots,\nu}_{n-i\text{ times}}}\right). \quad (4.8)$$

As $\mathcal{F}_{MN}\left[\epsilon^n g^n\right]$ depends on $\epsilon^n g^n_{MN}$ and we want to define g^n_{MN} from the condition that the equation (4.7) is to be zero in order ϵ^n, we are only interested in the first term of the expansion $\mathcal{F}_{MN}\left[\epsilon^n g^n\right]=\epsilon^n\mathcal{F}_0+\epsilon^{n+1}\mathcal{F}_1+\ldots$. This first term can consist of only r-derivatives, as all the others would bring more ϵ-coefficients in the front. Also, it can depend only on the metric $g^{0,0}_{MN}$, as otherwise there would be again too many powers of ϵ. Thus, defining

$$\mathcal{F}_{MN}\left[\epsilon^n g^n\right]+4\epsilon^n g^{n,0}_{MN}\equiv\mathcal{H}_{\alpha MN}\left[\epsilon^n g^n\right]+\mathcal{O}\left(\epsilon^{n+1}\right), \quad (4.9)$$

we can write the equation, from which g^n_{MN} can be solved, as

$$\mathcal{H}_{MN}\left[\epsilon^n g^n\right]=\epsilon^n\mathcal{S}^n_{MN}+\mathcal{O}\left(\epsilon^{n+1}\right). \quad (4.10)$$

Here $\mathcal{H}$ is the same operator, a second-order differential operator with respect to r, in all orders, as it depends only on the metric $g^{0,0}_{MN}$. We see this in the next section when we solve for $\mathcal{H}$. Instead, as seen from the definition (4.8), $\mathcal{S}^n_{MN}$ depends on n and will in general be more complicated for bigger n. However, this source term is in principle known and can be computed from the lower-order metric.

4.1.2 Solving for $\mathcal{H}$

As the operator $\mathcal{H}$ does not depend on x^μ and is a derivative operator only with respect to r, we can solve the equation point by point: first compute the source term S^n_{MN} for some point x^μ and then solve $g^n_{MN}(x^\mu)$ from the equation (4.10). For any point, we can always choose coordinates to set $u^\mu=(1,0,0,0)$ and $b=1$. With this choice, the metric g^0_{MN} is

$$ds^2=2dvdr-r^2\left(1-\frac{1}{r^4}\right)dv^2+r^2dx_idx^i+\mathcal{O}\left(\epsilon\right). \quad (4.11)$$

Before using this to find $\mathcal{H}$, we will fix the gauge. The situation is similar to the so-called hole problem of general relativity: the metric on the surface closing an empty area of space does not define unambiguously the metric inside it: having one metric, all coordinate diffeomorphisms keeping the boundary unchanged also give solutions [2, 5]. In this five-dimensional case, the Einstein equation seems to give 15 scalar equations that would be enough to solve for the metric, but as in the four-dimensional case [5], it turns out that part of them are only constraints for the metric on the boundary. We get only ten equations to be used for solving the metric, and five gauge degrees of freedom are left. As in [26], we choose

$$g^n_{rr} = 0, \qquad g^n_{r\mu} \propto u_\mu, \qquad \left(g^0\right)^{MN} g^n_{NM} = 0 \quad \forall n. \tag{4.12}$$

This gauge is called "background field" gauge, and using it the curves x^μ =constant will be null geodesics, but r will not be an affine parameter for them [3].

Using the definition for the Ricci tensor, we can compute $\mathcal{H}$. One gets (in order to simplify the notation, here we use $G_{MN} = g^{0,0}_{MN}$, known, and $g_{MN} = g^{n,0}_{MN}$, unknown):

$$\begin{aligned}
& \mathcal{H}_{MN}\left[\epsilon^n g^n\right] \\
= \;& \epsilon^n \tfrac{1}{2} G^{RK}\left[g_{KN,M,R} - g_{MN,K,R} - g_{KR,M,N} + g_{RM,K,N}\right] \\
& +\epsilon^n \tfrac{1}{2} g^{RK}\left[G_{KN,M,R} - G_{MN,K,R} - G_{KR,M,N} + G_{RM,K,N}\right] \\
& +\epsilon^n \tfrac{1}{4} G^{RK}\left[G_{KR,L} + G_{KL,R} - G_{RL,K}\right] G^{LS}\left[g_{SN,M} + g_{SM,N} - g_{NM,S}\right] \\
& -\epsilon^n \tfrac{1}{4} G^{RK}\left[G_{KN,L} + G_{KL,N} - G_{NL,K}\right] G^{LS}\left[g_{SR,M} + g_{SM,R} - g_{RM,S}\right] \\
& +\epsilon^n \tfrac{1}{4} G^{RK}\left[g_{KR,L} + g_{KL,R} - g_{RL,K}\right] G^{LS}\left[G_{SN,M} + G_{SM,N} - G_{NM,S}\right] \\
& -\epsilon^n \tfrac{1}{4} G^{RK}\left[g_{KN,L} + g_{KL,N} - g_{NL,K}\right] G^{LS}\left[G_{SR,M} + G_{SM,R} - G_{RM,S}\right] \\
& +\epsilon^n \tfrac{1}{4} g^{RK}\left[G_{KR,L} + G_{KL,R} - G_{RL,K}\right] G^{LS}\left[G_{SN,M} + G_{SM,N} - G_{NM,S}\right] \\
& -\epsilon^n \tfrac{1}{4} g^{RK}\left[G_{KN,L} + G_{KL,N} - G_{NL,K}\right] G^{LS}\left[G_{SR,M} + G_{SM,R} - G_{RM,S}\right] \\
& +\epsilon^n \tfrac{1}{4} G^{RK}\left[G_{KR,L} + G_{KL,R} - G_{RL,K}\right] g^{LS}\left[G_{SN,M} + G_{SM,N} - G_{NM,S}\right] \\
& -\epsilon^n \tfrac{1}{4} G^{RK}\left[G_{KN,L} + G_{KL,N} - G_{NL,K}\right] g^{LS}\left[G_{SR,M} + G_{SM,R} - G_{RM,S}\right] \\
& +4\epsilon^n g^{n,0}_{MN} + \mathcal{O}\left(\epsilon^{n+1}\right)
\end{aligned} \tag{4.13}$$

This looks quite complicated, but fortunately can be simplified drastically using the known metric G_{MN}, the gauge condition and the fact that for the terms of order ϵ^n only $r-$derivatives of g_{MN} are left. The inverse metric G^{MN} needed is

$$G^{\alpha\beta} = \begin{pmatrix} r^2 - \frac{1}{r^2} & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix} \tag{4.14}$$

Using this, the gauge conditions can be expressed as

$$g_{rr} = 0, \qquad g_{r\mu} \propto u_\mu, \qquad 2g_{rv} + \sum_{i=1}^{3} \frac{1}{r^2} g_{ii} = 0. \tag{4.15}$$

As gauge fixing gives five components of the unknown metric g_{MN} (when others are known), we need only ten of the 15 scalar equations from (4.10). It turns out that fortunately, different components of the unknown metric g_{MN} are not mixed by $\mathcal{H}$, except g_{rv}, g_{vv} and $\sum_{i=1}^{3} g_{ii}$ (this stems from the $SO(3)$ symmetry of $G_{\alpha\beta}$ and was expected) [26]. After a tedious calculation, and plugging in the source terms, we get the following group of ten scalar equations containing the components of the unknown metric:

$$\begin{aligned}
\epsilon^n\left(-\frac{d}{dr}\left(2r^4 g_{vr}\right)+\frac{2}{3}\frac{d}{dr}g_{vr}-\frac{d}{dr}\left(r^2 g_{vv}\right)\right) &= \epsilon^n\left(\left(r^2-\frac{1}{r^2}\right)\mathcal{S}^n_{rr}+\mathcal{S}_{vr}\right) \\
&\quad +\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.16)\\
\epsilon^n\left(-\frac{10}{3}\frac{d}{dr}g_{vr}-\frac{2}{3}r\frac{d^2}{dr^2}g_{vr}\right) &= 0+\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.17)\\
\epsilon^n\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left(r^2 g_{vi}\right)\right) &= \epsilon^n\mathcal{S}^n_{ri}+\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.18)\\
\epsilon^n\frac{d}{dr}\left(r^5\left(1-\frac{1}{r^4}\right)\frac{d}{dr}\left(\frac{2\left(g_{ij}-\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}g_{kk}\right)}{r^2}\right)\right) &= \epsilon^n\left(\mathcal{S}^n_{ij}-\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}\mathcal{S}^n_{kk}\right) \\
&\quad +\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.19)
\end{aligned}$$

From these, we can solve the metric g_{MN}. These are called dynamical equations in [26].

Of course, one must ensure that the other five equations are also satisfied by the solved metric. It turns out that one of them is equivalent to the equation (4.17), and the rest four are independent of g_{MN} - they are the aforementioned equations constraining the metric on the boundary. In [26], these were called constraint equations and they can be written simply as

$$\begin{aligned}
0 &= \epsilon^n\left(\left(r^2-\frac{1}{r^2}\right)\mathcal{S}^n_{rv}+\mathcal{S}_{vv}\right)+\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.20)\\
0 &= \epsilon^n\left(\left(r^2-\frac{1}{r^2}\right)\mathcal{S}^n_{ri}+\mathcal{S}_{vi}\right)+\mathcal{O}\left(\epsilon^{n+1}\right) \qquad (4.21)
\end{aligned}$$

As seen next for the first order calculation, these can be interpreted as stress-energy conservation equations for the metric without the n:th correction term, and they are constraint equations in a sense that they restrict the different fluctuation configurations, for which the metric satisfying the Einstein equation exists. The significance of these various equations is summarized below in the table (4.1)

	The gravity side	The hydrodynamic side
Constraint equations	Ensure that a solution exists for the metric	Correspond to the conservation of the stress-energy tensor without the correction the dynamical equations give
Dynamical equations	Give the metric	Give the stress-energy tensor through the duality machinery

Table 4.1: A summary of the above equations and their roles, both in the gravity and hydrodynamic side.

4.2 Metric for the first order calculation

We will now solve $g^{1,0}_{MN}$. In order to compute the source terms (4.8), we need the linear term of the uncorrected metric: $g^{0,1}_{MN}$. Using still the choice of (4.11) we get

$$\begin{aligned} ds^2 &= 2dvdr - r^2\left(1-\frac{1}{r^4}\right)dv^2 + r^2 dx_i dx^i \\ &\quad -2\epsilon x^\lambda\left(\partial_\lambda u_\mu\right)dx^\mu dr - \frac{2}{r^2}\epsilon x^\lambda\left(\partial_\lambda u_\mu\right)dx^\mu dv - 4\frac{x^\lambda\left(\partial_\lambda b\right)}{r^2}dv^2 + \mathcal{O}\left(\epsilon^2\right). \end{aligned} \tag{4.22}$$

The calculations can be made simpler by defining $u^\mu = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta_i}{\sqrt{1-\beta^2}}\right)$ (as in the constant term $g^{0,0}_{MN}$ we already separated v and x_i). After derivation, we set $\beta = 0$ in accordance with the coordinate choice made. We get

$$\begin{aligned} ds^2 &= 2dvdr - r^2\left(1-\frac{1}{r^4}\right)dv^2 + r^2 dx_i dx^i \\ &\quad -2\epsilon x^\lambda\left(\partial_\lambda \beta_i\right)dx^i dr - \frac{2}{r^2}\epsilon x^\lambda\left(\partial_\lambda \beta_i\right)dx^i dv - 4\epsilon\frac{x^\lambda\left(\partial_\lambda b\right)}{r^2}dv^2 + \mathcal{O}\left(\epsilon^2\right). \end{aligned} \tag{4.23}$$

We are now ready to compute the source terms using Maple. Discarding all terms of order ϵ^2, the constraint equations (4.20) and (4.21) become simply

$$\partial_v b = \frac{\partial_i \beta_i}{3} \tag{4.24}$$

$$\partial_i b = \partial_v \beta_i. \tag{4.25}$$

These are exactly the scalar equations from the conservation of the stress-energy tensor in zeroth order (still using the coordinate choice of (4.11)):

$$\begin{aligned}\nabla_\mu T^{\mu\nu} &= -\frac{4}{b^5}(\partial_\mu b)(4u^\mu u^\nu + \eta^{\mu\nu}) + \frac{4(\partial_\mu u^\mu)u^\nu}{b^4} + \frac{4(\partial_\mu u^\nu)u^\mu}{b^4} \\ &= -4(\partial_\mu b)(4u^\mu u^\nu + \eta^{\mu\nu}) + 4(\partial_i\beta_i)\delta^\nu_v + 4(\partial_v u^\nu).\end{aligned} \tag{4.26}$$

Choosing $\nu = v$ gives equation (4.24) and respectively $\nu = i$ gives equation (4.25).

The dynamical equations give (keeping again only ϵ-terms in the sources and using the simplified notation $g^{1,0}_{MN} = g_{MN}$)

$$-\frac{d}{dr}\left(2r^4 g_{vr}\right) + \frac{2}{3}\frac{d}{dr}g_{vr} - \frac{d}{dr}\left(r^2 g_{vv}\right) = -2r^2\partial_i\beta_i \tag{4.27}$$

$$-5\frac{d}{dr}g_{vr} - r\frac{d^2}{dr^2}g_{vr} = 0 \tag{4.28}$$

$$\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left(r^2 g_{vi}\right)\right) = -\frac{3}{r^2}\partial_v\beta_i \tag{4.29}$$

$$\frac{d}{dr}\left(r^5\left(1-\frac{1}{r^4}\right)\frac{d}{dr}\left(\frac{g_{ij} - \frac{1}{3}\delta_{ij}\sum_{k=1}^3 g_{kk}}{r^2}\right)\right) = -3r^2\sigma_{ij} \tag{4.30}$$

where in the last line we have used the definition (2.12) of σ_{ij} applied to the present coordinate choice: $\sigma_{ij} = \left(\partial_i\beta_j + \partial_j\beta_i - \frac{2}{3}\delta_{ij}\partial_k\beta_k\right)$. Beginning to solve the equations from the second one, we get

$$\begin{aligned}-5\frac{d}{dr}g_{vr} - r\frac{d^2}{dr^2}g_{vr} = 0 &\quad\Leftrightarrow \\ -\frac{5}{r}dr = \frac{1}{\frac{dg_{vr}}{dr}}d\frac{dg_{vr}}{dr} &\quad\Leftrightarrow \\ \frac{dg_{vr}}{dr} = C_1 r^{-5} &\quad\Leftrightarrow \\ g_{vr} = \frac{C_2}{r^4} + C_3 &\quad\Leftrightarrow\end{aligned} \tag{4.31}$$

Substituting this into (4.27) we get

$$\begin{aligned}-\frac{d}{dr}\left(r^2 g_{vv}\right) = -2r^2\partial_i\beta_i + 8r^3 C_3 + \frac{8C_2}{3r^5} &\quad\Leftrightarrow \\ g_{vv} = \frac{2r\partial_i\beta_i}{3} - 2r^2 C_3 + \frac{2C_2}{3r^6} + \frac{C_4}{r^2}&\end{aligned} \tag{4.32}$$

and two unknown components of the metric are solved. Using the gauge condition (4.15) we also get $g_{ir} = 0$ and

$$\sum_{i=1}^3 g_{ii} = -2r^2 g_{rv} = -\frac{2C_2}{r^2} - 2r^2 C_3. \tag{4.33}$$

The third equation (4.29) is solved as

$$\begin{aligned}\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left(r^2 g_{vi}\right)\right) &= -\frac{3}{r^2}\partial_v\beta_i &\Leftrightarrow\\ \frac{d}{dr}\left(r^2 g_{vi}\right) &= 3r^2\partial_v\beta_i + D_{1i}r^3 &\Leftrightarrow\\ g_{vi} &= r\partial_v\beta_i + D_{2i}r^2 + \frac{D_{3i}}{r^2} & (4.34)\end{aligned}$$

and the last equation (4.30) gives the rest of the metric components as

$$\begin{aligned}\frac{d}{dr}\left(r^5\left(1-\frac{1}{r^4}\right)\frac{d}{dr}\left(\frac{g_{ij}-\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}g_{kk}}{r^2}\right)\right) &= -3r^2\sigma_{ij} &\Leftrightarrow\\ \frac{d}{dr}\left(\frac{g_{ij}-\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}g_{kk}}{r^2}\right) &= -\frac{r^2}{r^4-1}\sigma_{ij}+\frac{E_{1ij}}{r^5-r} &\Rightarrow\end{aligned}$$

$$\begin{aligned}g_{ij}-\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}g_{kk} =& -\frac{r^2}{4}\ln\left(\frac{r-1}{r+1}\right)\sigma_{ij}-\frac{r^2}{2}\arctan\left(r\right)\sigma_{ij}\\ &+\frac{r^2E_{1ij}}{4}\ln\left(\frac{r^4-1}{r^4}\right)+r^2E_{2ij} &\Leftrightarrow\\ g_{ij} =& \frac{r^2}{4}\ln\left(\frac{\left(r+1\right)^{\frac{E_{1ij}}{\sigma_{ij}}+1}\left(r^2+1\right)\left(r-1\right)^{\frac{E_{1ij}}{\sigma_{ij}}-1}}{r^{4\frac{E_{1ij}}{\sigma_{ij}}}}\right)\sigma_{ij}\\ &-\frac{r^2}{2}\arctan\left(r\right)\sigma_{ij}+r^2E_{2ij}+\frac{1}{3}\delta_{ij}\left(-\frac{2C_2}{r^2}-2r^2C_3\right) & (4.35)\end{aligned}$$

Now the metric is solved. However, there are still seven integration constants left: $C_2, C_3, C_4, D_{2i}, D_{3i}, E_{1ij}$ and E_{2ij}, but we can define them already before computing the stress-energy tensor. In order to have g_{ij} regular everywhere, we must have $E_{1ij} = \sigma_{ij}$. Comparing to the calculation of zeroth order stress-energy tensor in section (3.3.4) we can see that C_3 in g_{vv} and D_{2i} in g_{vi} are multiplying too rapidly growing functions (r^2), and hence they must be set to zero in order to get a finite stress-energy tensor. Also, through this same comparison we find out that when r grows, the term $\frac{C_2}{r^6}$ goes too rapidly to zero in order to make any finite contribution to the stress-energy tensor, and therefore the constant C_2 is insignificant and can be set to zero for simplicity. Further, comparing terms D_{3i} and C_4 to the uncorrected metric (4.23) one observes that they correspond to infinitesimal velocity and temperature variations and are thus forced to zero by our coordinate choice made in (4.11). In order to define the last constant E_{2ij} we have to investigate the behaviour of the correction for g_{ij} at large r. The leading term is $r^2\left(E_{2ij}-\frac{\pi}{4}\sigma_{ij}\right)$, and as reasoned for the terms C_3 and D_{2i}, this grows too fast. Therefore it must be set $E_{2ij} = \frac{\pi}{4}\sigma_{ij}$.

With the integration constants now fixed, we can write the corrected metric as

$$\begin{aligned} ds^2 &= 2dvdr - r^2\left(1-\frac{1}{r^4}\right)dv^2 + r^2 dx_i dx^i \\ &\quad -2\epsilon x^\lambda\left(\partial_\lambda\beta_i\right)dx^i dr - \frac{2}{r^2}\epsilon x^\lambda\left(\partial_\lambda\beta_i\right)dx^i dv - 4\epsilon\frac{x^\lambda\left(\partial_\lambda b\right)}{r^2}dv^2 \\ &\quad +\epsilon\frac{r^2}{2}\left(\ln\left(\frac{(r+1)^2\left(r^2+1\right)}{r^4}\right) - 2\arctan\left(r\right)+\pi\right)\sigma_{ij}dx^i dx^j \\ &\quad +2\epsilon r\partial_v\beta_i dx^i dv + \epsilon\frac{2r\partial_i\beta_i}{3}dv^2 + \mathcal{O}(\epsilon^2). \end{aligned} \tag{4.36}$$

The second line is the linear term of the Taylor expansion for the uncorrected metric. This is still written in the specific coordinates around a fixed point chosen in (4.11). In order to find the global metric, all we have to do is the same covariantization procedure as in section (3.3.4.1): we again write $x^\alpha = (v, x_i)$ and $u^\alpha = (1,0,0,0)$ and note that in the specific coordinates we had $b = 1$. Then we can write the metric as

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2\left(1-\frac{1}{(br)^4}\right)u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu}dx^\mu dx^\nu \\ &\quad +\epsilon\frac{2r\partial_\lambda u^\lambda}{3}u_\mu u_\nu dx^\mu dx^\nu - \epsilon r u^\lambda\partial_\lambda\left(u_\mu u_\nu\right)dx^\mu dx^\nu \\ &\quad +\varepsilon\frac{r^2}{2}b\left[\ln\left(\frac{(br+1)^2\left((br)^2+1\right)}{(br)^4}\right) - 2\arctan\left(br\right)+\pi\right]\sigma_{\mu\nu}dx^\mu dx^\nu. \end{aligned} \tag{4.37}$$

Here $\sigma_{\mu\nu}$ is again (2.12).

4.3 Calculation up to the first order

We will now calculate the stress-energy tensor for the metric (4.37) using the relation (3.38) given by the duality. This should give the stress-energy tensor up to first order in the gradient expansion, and thus during the calculation all higher order terms are dropped. First we will need the induced metric and its inverse:

$$g_{\alpha\beta} = -r^2\left(1-\frac{1}{(br)^4}\right)u_\alpha u_\beta + r^2 P_{\alpha\beta} + \epsilon\frac{2r\partial_\lambda u^\lambda}{3}u_\alpha u_\beta - \epsilon r u^\lambda\partial_\lambda\left(u_\alpha u_\beta\right) + \epsilon F(r)\sigma_{\alpha\beta} \tag{4.38}$$

$$\begin{aligned} g^{\alpha\beta} &= -\frac{1}{f(r)}u^\alpha u^\beta + \frac{1}{r^2}P^{\alpha\beta} - \epsilon\frac{2r\partial_\delta u^\delta}{3f^2(r)}u^\alpha u^\beta - \epsilon\frac{r}{f^2(r)}u^\lambda u^\delta\partial_\delta\left(u_\lambda\right)u^\alpha u^\beta \\ &\quad +\epsilon\frac{1}{rf(r)}\left(P^{\alpha\lambda}u^\delta\partial_\delta\left(u_\lambda\right)u^\beta + u^\alpha u^\delta\partial_\delta\left(u_\kappa\right)P^{\kappa\beta}\right) - \epsilon\frac{1}{r^4}F(r)\sigma^{\alpha\beta} \end{aligned} \tag{4.39}$$

The Christoffel symbols needed for $K^{\alpha\beta} = -\nabla^M \hat{n}^N e^\alpha_M e^\beta_N = -\gamma^{\alpha L}\partial_L \hat{n}^\beta - \gamma^{\alpha L}\Gamma^\beta_{KL}\hat{n}^K$ are

$$\Gamma^\beta_{rr} = 0 \tag{4.40}$$

$$\Gamma^\beta_{r\gamma} = \epsilon\frac{1}{2}\frac{1}{r^2}P^{\beta\alpha}\left(\partial_\alpha u_\gamma - \partial_\gamma u_\alpha\right) + \frac{1}{r}P^\beta_\gamma + \epsilon\frac{1}{2}\frac{1}{r^2}F'(r)\sigma^\beta_\gamma + \epsilon\frac{F(r)}{r^4}\sigma^\beta_\gamma r \tag{4.41}$$

$$\begin{aligned}\Gamma^\beta_{\alpha\gamma} &= \epsilon\frac{1}{2}u^\beta\left(\partial_\alpha u_\gamma - \partial_\gamma u_\alpha\right) + u^\beta\left(r + \frac{1}{b^4 r^3}\right)u_\alpha u_\gamma - u^\beta r P_{\alpha\gamma} \\ &\quad -\epsilon\frac{1}{2}u^\beta\left(\frac{2\partial_\lambda u^\lambda}{3}u_\alpha u_\gamma - u^\lambda\partial_\lambda\left(u_\alpha u_\lambda\right) + F'(r)\sigma_{\alpha\gamma}\right) \\ &\quad +\epsilon\frac{1}{2}\frac{1}{b^4 r^4}P^{\beta\delta}\left(u_\alpha\partial_\gamma u_\delta + u_\gamma\partial_\alpha u_\delta - u_\alpha\partial_\delta u_\gamma - u_\gamma\partial_\delta u_\alpha\right),\end{aligned} \tag{4.42}$$

and the inverse of the bulk metric, which is needed, too, reads

$$\begin{aligned}\gamma^{MN} &= \begin{pmatrix} r^2 - \frac{1}{b^4 r^2} - \epsilon\frac{2r\partial_\lambda u^\lambda}{3} - \epsilon r u^\kappa u^\lambda \partial_\lambda u_\kappa + \epsilon r u^\lambda \partial_\lambda & u^\mu \\ u^\mu & \frac{1}{r^2}P^{\mu\nu} - \epsilon\frac{F(r)}{r^4}\sigma^{\mu\nu}\end{pmatrix} \\ &\equiv \begin{pmatrix} r^2 - \frac{1}{b^4 r^2} + \epsilon r g_1 & u^\mu \\ u^\mu & \frac{1}{r^2}P^{\mu\nu} - \epsilon\frac{F(r)}{r^4}\sigma^{\mu\nu}\end{pmatrix}.\end{aligned} \tag{4.43}$$

For the external curvature tensor, we also need the unit normal vector, that is

$$\hat{n}^r = r\sqrt{\left(1 - \frac{1}{(br)^4}\right)} + \epsilon\frac{r}{2}\left(r^2 - \frac{r^2}{(br)^4}\right)^{-\frac{1}{2}} g_1 \tag{4.44}$$

$$\hat{n}^\alpha = \left[\left(r^2 - \frac{r^2}{(br)^4}\right)^{-\frac{1}{2}} - \epsilon\frac{r}{2}g_1\left(r^2 - \frac{r^2}{(br)^4}\right)^{-\frac{3}{2}}\right]u^\alpha \tag{4.45}$$

With all necessary objects now written down, we can calculate the stress-energy tensor using the formula (3.38). Calculation is again rather tedious, but finally we get (setting the bookkeeping parameter $\epsilon = 1$ as it is not needed any more)

$$\begin{aligned}T^{\alpha\beta} &= \lim_{r\to\infty}\frac{r^6}{8\pi G}\left[K^{\alpha\beta} - K g^{\alpha\beta} - 3g^{\alpha\beta}\right] \\ &= \frac{\pi^3}{16G}T^4\left(3u^\alpha u^\beta + P^{\alpha\beta}\right) - \frac{\pi^2}{8G}T^3\sigma^{\alpha\beta}.\end{aligned} \tag{4.46}$$

This is exactly of the form (2.34), with $\eta = \frac{\pi^2}{16G}T^3$. The entropy density s can be read from the first term [3]: $s = \frac{\pi^3}{4G}T^3$, and we get $\frac{\eta}{s} = \frac{1}{4\pi}$, which is an old result for the $\mathcal{N} = 4$ SYM-theory [28], now reproduced by this method (for the first time in [26]).

Chapter 5

Summary

In this thesis, we have gone through the process of computing the stress-energy tensor for conformal fluids using the AdS/CFT-duality. In the second chapter, it was shown how hydrodynamics is the effective theory for any many-particle - or quantum field theory, and how it boils down to defining the stress-energy tensor. We used the basic method of adding all terms allowed by symmetries in order to create a gradient expansion. This was a valid approach, as at the hydrodynamic limit, the fluctuation wavelength had to be long, and thus the derivative expansion should converge. This was done first for a general relativistic fluid and then for conformal fluids as a special case. It was seen that the expansion is unambiguous up to some coefficients.

For conformal fluids, the definition of the coefficients in gradient expansion was the core subject of this thesis. The process was based on the AdS-CFT -duality, inspired by string theory. In chapter 3, we listed shortly some string theory results and then motivated the duality by analyzing the conformal border of the AdS-space and showing that the symmetries of the theories are identical. After noting the exact formulation of the duality by Witten [9] and Gubser et al. [34], we began to modify it in order to reach the fluid-gravity -correspondence at the large t'Hooft limit. Using the boosted black hole metric, we got the stress-energy tensor for conformal fluids at zeroth order.

In order to compute the stress-energy tensor more accurately in the gradient expansion, we used the process invented by Rangamani et al. [26]: we upgraded the parameters of the boosted black hole to slowly varying functions of the boundary coordinates. This forced us to add corrections to the metric, as an expansion of the boundary derivatives, and it turned out that we can define these corrections up to any order using only the basic Einstein equation of general relativity. The more corrections added to the metric, the more terms would be in the gradient expansion of the stress-energy tensor. In the fourth chapter, we computed the stress-energy tensor for conformal fluids in the first order of the gradient expansion. The result

$$T^{\alpha\beta} = \frac{\pi^3}{16G} T^4 \left(3u^\alpha u^\beta + P^{\alpha\beta}\right) - \frac{\pi^2}{8G} T^3 \sigma^{\alpha\beta} \tag{5.1}$$

reproduced the famous older result for the entropy/viscosity relation: $\frac{\eta}{s} = \frac{1}{4\pi}$. In [26], the stress-energy tensor was computed further, up to second order in gradient expansion.

One can imagine at least three kinds of lines for further research on the topic. The most mathematical one would focus on examining the conjectured duality and trying to prove it. The most straightforward way would be to apply the immediate results. This is challenging, as the fluid-gravity correspondence as represented here applies only to conformal fluids, which are not very realistic. There are some applications concerning quark-gluon plasma, based on the fact that QCD is approximately conformal at high energies. [31] Some of these results have been collected in the review [30] of Casalderrey-Solana et al.. Also, in condensed matter physics some applications have been made e.g. using the scale invariance at critical points. [33] The third way would be to try to expand the duality for conformal field theories with dimensions other than four, or for creating dual representations for non-conformal fluids. One interesting area here would be to examine a gravity dual to the turbulence of real fluids, which has been considered e.g. by C. Eiling et al. [32].

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