6 Kertausosa

1. a) $75^\circ = 75 \cdot \frac{\pi}{180}(\text{rad}) = \frac{5\pi}{12}$

b) $-485^\circ = -485 \cdot \frac{\pi}{180}(\text{rad}) = -\frac{92\pi}{36} = -2\frac{25}{36}\pi$

c) $2160^\circ = 2160 \cdot \frac{\pi}{180}(\text{rad}) = 12\pi$

Vastaus: a) $\frac{5\pi}{12}$ b) $-2\frac{25}{36}\pi$ c) 12π

2. a) $1{,}234 = 1{,}234 \cdot \frac{180^\circ}{\pi} \approx 70{,}7^\circ$

b) $-0{,}775 = -0{,}775 \cdot \frac{180^\circ}{\pi} \approx -44{,}4^\circ$

c) $21{,}824 = 21{,}824 \cdot \frac{180^\circ}{\pi} \approx 1250{,}4^\circ$

Vastaus: a) $70{,}7^\circ$ b) $-44{,}4^\circ$ c) $1250{,}4^\circ$

3. a) $-1450^\circ = 4\cdot(-360^\circ) - 10^\circ$, joten kylki sijaitsee 4. neljänneksessä

b) $\frac{4{,}103}{\pi} \approx 1{,}306$, joten $4{,}103 \approx 1{,}306\pi$

Koska $\pi < 1{,}306\pi < \frac{3\pi}{2}$, kylki on 3.neljänneksessä-

c) $\frac{25}{3}\pi = 8\pi + \frac{\pi}{3}$, joten kylki on 1. neljänneksessä

Vastaus: a) IV b) III c) I

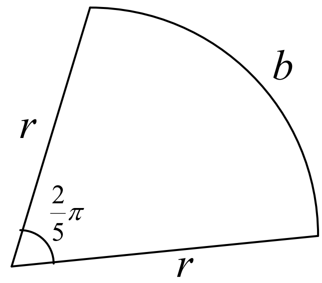
4. a) $-\frac{7}{3}\pi + 2\cdot 2\pi = \frac{5}{3}\pi = 1\frac{2}{3}\pi$

b) $8{,}333 - 2\pi \approx 2{,}050$

c) $2000^\circ - 5\cdot 360^\circ = 200^\circ = 200\cdot\frac{\pi}{180}(rad) = \frac{10}{9}\pi$

Vastaus: a) $1\frac{2}{3}\pi$ b) 2,050 c) $\frac{10}{9}\pi$

5. $b = \frac{2}{5}\pi \cdot r$



Piiri = $2r + b$

$$2r + \frac{2}{5}\pi r = 15$$

$$r(2 + \frac{2}{5}\pi) = 15$$

$$r\left(\frac{10 + 2\pi}{5}\right) = 15 \quad \Big| \cdot \frac{5}{10 + 2\pi}$$

$$r = \frac{75}{10 + 2\pi}$$

$$A = \frac{\frac{2}{5}\pi}{2\pi} \cdot \pi r^2 = \frac{\pi}{5}\left(\frac{75}{10 + 2\pi}\right)^2 = \frac{5625\pi}{5(10 + 2\pi)^2} = \frac{1125\pi}{(10 + 2\pi)^2} \approx 13{,}3$$

Vastaus: $\frac{1125\pi}{(10 + 2\pi)^2} \approx 13{,}3$

6. a) $(\cos 125^\circ, \sin 125^\circ) \approx (-0{,}57;\ 0{,}82)$

b) $\left(\cos(-1{,}656),\ \sin(-1{,}656)\right) \approx (-0{,}09;\ -0{,}996)$

c) $\left(\cos \frac{8}{3}\pi,\ \sin \frac{8}{3}\pi\right) \approx (-0{,}5;\ 0{,}87)$

Vastaus: a) $(-0{,}57;\ 0{,}82)$ b) $(-0{,}09;\ -0{,}996)$ c) $(-0{,}5;\ 0{,}87)$

7. a) $\sin 225^\circ = -\frac{1}{\sqrt{2}}$

b) $\cos(-\tfrac{9}{8}\pi) = \cos\tfrac{9}{8}\pi = -\frac{1}{2}\sqrt{2+\sqrt{2}}$

c) $\sin(-\tfrac{12}{5}\pi) = \sin(-\tfrac{12}{5}\pi + 4\pi) = \sin\tfrac{8}{5}\pi = -\frac{1}{4}\sqrt{10+2\sqrt{5}}$

Vastaus: a) $-\frac{1}{\sqrt{2}}$ b) $-\frac{1}{2}\sqrt{2+\sqrt{2}}$ c) $-\frac{1}{4}\sqrt{10+2\sqrt{5}}$

8.

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$\cos^2\alpha = 1 - \left(\frac{2}{\sqrt{7}}\right)^2$$

$$\cos^2\alpha = 1 - \frac{4}{7}$$

$$\cos^2\alpha = \frac{3}{7}$$

$$\cos\alpha = \pm\frac{\sqrt{3}}{\sqrt{7}}$$

Koska $\frac{7}{2}\pi \le \alpha \le \frac{9}{2}\pi$ on loppukylki 1. tai 4. neljänneksessä ja siten $\cos\alpha > 0$

Siis $\cos\alpha = \frac{\sqrt{3}}{\sqrt{7}}$

Vastaus: $\frac{\sqrt{3}}{\sqrt{7}}$

9. $5\cos^2\alpha + 6\cos\alpha = 8$

Merkitään $\cos\alpha = t$

$5t^2 + 6t - 8 = 0$

$t = \dfrac{-6 \pm \sqrt{6^2 - 4\cdot 5\cdot(-8)}}{2\cdot 5} = \dfrac{-6 \pm 14}{10}$

$t = -2$ tai $t = \dfrac{4}{5}$

Koska $-1 \le \cos\alpha \le 1$, vain $t = \dfrac{4}{5}$ kelpaa eli $\cos\alpha = \dfrac{4}{5}$

$\sin^2\alpha + \cos^2\alpha = 1$

$\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

Kun $-\dfrac{\pi}{2} \le \alpha \le 0$, niin $\sin\alpha < 0$, joten

$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-(\frac{4}{5})^2} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$

Vastaus: $-\dfrac{3}{5}$

10. a) $\tan(\frac{25}{12}\pi) = \tan(2\frac{1}{12}\pi) = \tan(2\pi + \frac{1}{12}\pi) = \tan(\frac{1}{12}\pi) = 2 - \sqrt{3}$

b) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{3}{7} : \left(-\dfrac{2\sqrt{10}}{7}\right) = -\dfrac{3}{7} \cdot \dfrac{7}{2\sqrt{10}} = -\dfrac{3}{2\sqrt{10}}$

Vastaus: a) $2 - \sqrt{3}$ b) $-\dfrac{3}{2\sqrt{10}}$

11. a) $\tan\alpha = 5{,}432 \quad \Big|\tan^{-1}$

$\alpha = 79{,}568...^\circ \approx 79{,}6^\circ$

$\alpha = 79{,}568...^\circ = 79{,}568... \cdot \dfrac{\pi}{180}(rad) = 1{,}3887... \approx 1{,}39$

b) $\tan\alpha = -0{,}436 \quad \Big|\tan^{-1}$

$\alpha_1 = -23{,}557...^\circ$

$\alpha = -23{,}557...^\circ + 180^\circ = 156{,}44...^\circ \approx 156^\circ$

$156{,}44...^\circ = 156{,}44... \cdot \dfrac{\pi}{180}(rad) = 2{,}730... \approx 2{,}73$

Vastaus: a) $79{,}6^\circ \approx 1{,}39(rad)$ b) $156^\circ \approx 2{,}73(rad)$

12. Kun $\frac{3}{2}\pi \le \alpha \le 2\pi$, niin $\sin\alpha < 0$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-(\tfrac{11}{12})^2} = -\sqrt{\tfrac{23}{144}} = -\frac{\sqrt{23}}{12}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = -\frac{\sqrt{23}}{12} : \frac{11}{12} = -\frac{\sqrt{23}}{12}\cdot\frac{12}{11} = -\frac{\sqrt{23}}{12}$$

Vastaus: $-\frac{\sqrt{23}}{12}$

13. $\tan\alpha = 2 \quad \Big| \tan^{-1}$

$\alpha_1 = 63{,}4349...^\circ$

$\alpha = 63{,}4349...^\circ + 180^\circ = 243{,}434...^\circ$

Kehäpiste $(\cos\alpha, \sin\alpha) \approx (-0{,}45; -0{,}89)$

Vastaus: $(-0{,}45; -0{,}89)$

14. Kun $\frac{\pi}{2} \le \alpha \le \pi$, niin $\cos\alpha \le 0$

$$\cos\alpha = -\sqrt{1-\sin^2\alpha}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\sin\alpha}{-\sqrt{1-\sin^2\alpha}} = -\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}$$

15. a) $\sin 7x = \sin(7x + 2\pi) = \sin\left(7(x + \frac{2}{7}\pi)\right)$

Perusjakso on siis $\dfrac{2\pi}{7}$

b) $2\cos\frac{x}{4} = 2\cos(\frac{1}{4}x + 2\pi) = 2\cos\left(\frac{1}{4}(x + 8\pi)\right)$

Perusjakso on siis 8π

c) $\dfrac{\tan 3x}{2} = \dfrac{1}{2}\tan(3x + \pi) = \dfrac{1}{2}\tan\left(3(x + \frac{\pi}{3})\right)$

Perusjakso on siis $\dfrac{\pi}{3}$

Vastaus: a) $\dfrac{2\pi}{7}$ b) 8π c) $\dfrac{\pi}{3}$

16. a) $f(x)=\frac{1}{2}\sin 3x-1$

$$-1\le \sin 3x\le 1 \quad \left|\cdot \tfrac{1}{2}\right.$$

$$-\tfrac{1}{2}\le \tfrac{1}{2}\sin 3x\le \tfrac{1}{2} \quad \left|-1\right.$$

$$-\tfrac{3}{2}\le \tfrac{1}{2}\sin 3x-1\le -\tfrac{1}{2}$$

Arvojoukko on $\left[-\frac{3}{2},-\frac{1}{2}\right]$

b) $g(x)=\dfrac{\cos(-2x)+1}{2}$

$$-1\le \cos(-2x)\le 1 \quad \left|+1\right.$$

$$0\le \cos(-2x)+1\le 2 \quad \left|:2\right.$$

$$0\le \frac{\cos(-2x)+1}{2}\le 1$$

Arvojoukko on $[0,1]$

c) $h(x)=\tan 2x-2$

Arvojoukko sama, kuin tangentin arvojoukko, eli $\mathbf{R}$

Vastaus: a) $\left[-\frac{3}{2},-\frac{1}{2}\right]$ b) $[0,1]$ c) $\mathbf{R}$

17. $f(x)=(\frac{1}{4}\sin 4x+1)(\sin 4x+2)$

Lausekkeet $a(x)=\frac{1}{4}\sin 4x+1$ ja $b(x)=\sin 4x+2$ saavat suurimmat ja pienimmät arvonsa samoissa kohdissa.

$$-1\le \sin 4x\le 1 \quad \Big|\cdot \tfrac{1}{4}$$
$$-\tfrac{1}{4}\le \tfrac{1}{4}\sin 4x\le \tfrac{1}{4} \quad \Big|+1$$
$$\tfrac{3}{4}\le \tfrac{1}{4}\sin 4x+1\le \tfrac{5}{4}$$

$$-1\le \sin 4x\le 1 \quad \Big|+2$$
$$1\le \sin 4x+2\le 3$$

Tulon pienin arvo on $\frac{3}{4}\cdot 1=\frac{3}{4}$

Tulon suurin arvo on $\frac{5}{4}\cdot 3=\frac{15}{4}=3\frac{3}{4}$

Vastaus: Pienin $\frac{3}{4}$ ja suurin $3\frac{3}{4}$

18. Jatkuva ja jaksollinen funktio on esimerkiksi $f(x)=\sin x$
$f(x+2\pi)=f(x)$ kaikilla $x\in \mathbf{R}$

Epäjatkuva ja jaksollinen funktio on esimerkiksi

$$g(x)=\begin{cases}\tan x &, x\ne \frac{\pi}{2}+n\pi\\ 0 &, x=\frac{\pi}{2}+n\pi\end{cases}$$

$g(x+\pi)=g(x)$ kaikilla $x\in \mathbf{R}$

19. a)

$$\sin x = 0{,}33 \quad \Big|\sin^{-1}$$

$$x \approx 19{,}27^\circ + n \cdot 360^\circ \quad \text{tai} \quad x \approx 180^\circ - 19{,}27^\circ + n \cdot 360^\circ$$

$$x \approx 160{,}73^\circ + n \cdot 360^\circ$$

b)

$$2\cos x = -1{,}487 \quad \Big|:2$$

$$\cos x = -0{,}7435 \quad \Big|\cos^{-1}$$

$$x \approx \pm 138{,}03^\circ + n \cdot 360^\circ$$

c)

$$\tan x = 8 \quad \Big|\tan^{-1}$$

$$x \approx 82{,}87^\circ + n \cdot 180^\circ$$

Vastaus: a) $x \approx 19{,}27^\circ + n \cdot 360^\circ$ tai $x \approx 160{,}73^\circ + n \cdot 360^\circ$

b) $x \approx \pm 138{,}03^\circ + n \cdot 360^\circ$

c) $x \approx 82{,}87^\circ + n \cdot 180^\circ$

20. a)

$$4\sin x = \sqrt{6} + \sqrt{2} \quad |: 4$$

$$\sin x = \tfrac{1}{4}(\sqrt{6} + \sqrt{2})$$

$$x = \tfrac{7\pi}{12} + n2\pi \text{ tai } x = \pi - \tfrac{7\pi}{12} + n2\pi$$

$$x = \tfrac{5\pi}{12} + n2\pi$$

b)

$$7\cos x + \sqrt{6} = 3\cos x + \sqrt{2}$$

$$4\cos x = -(\sqrt{6} - \sqrt{2})$$

$$\cos x = -\tfrac{1}{4}(\sqrt{6} - \sqrt{2})$$

$$x = \pm\tfrac{7\pi}{12} + n2\pi$$

c)

$$\sqrt{3}\sin x + \cos x = 0$$

$$\sqrt{3}\sin x = -\cos x \quad |: \sqrt{3}\cos x \neq 0$$

$$\tan x = -\tfrac{1}{\sqrt{3}}$$

$$x = \tfrac{5\pi}{6} + n\pi$$

Vastaus: a) $x = \tfrac{7\pi}{12} + n2\pi$ tai $x = \tfrac{5\pi}{12} + n2\pi$

b) $x = \pm\tfrac{7\pi}{12} + n2\pi$

c) $x = \tfrac{5\pi}{6} + n\pi$

21. a)

$$\sin 4x = \sin(7x+2)$$

$$4x = 7x + 2 + n2\pi \quad \text{tai} \quad 4x = \pi - (7x+2) + n2\pi$$

$$-3x = 2 + n2\pi \quad \text{tai} \quad 11x = \pi - 2 + n2\pi$$

$$x = -\tfrac{2}{3} + n\tfrac{2\pi}{3} \quad \text{tai} \quad x = \tfrac{\pi-2}{11} + n\tfrac{2\pi}{11}$$

b)

$$\cos 2x = \sin(2x+1)$$

$$\cos 2x = \cos(\tfrac{\pi}{2} - (2x+1))$$

$$\cos 2x = \cos(\tfrac{\pi}{2} - 2x - 1)$$

$$2x = \tfrac{\pi}{2} - 2x - 1 + n2\pi \quad \text{tai} \quad 2x = -(\tfrac{\pi}{2} - 2x - 1) + n2\pi$$

$$4x = \tfrac{\pi}{2} - 1 + n2\pi \quad \text{tai} \quad 0 = -\tfrac{\pi}{2} + 1 + n2\pi$$

$$x = \tfrac{\pi}{8} - \tfrac{1}{4} + n\tfrac{\pi}{2} \quad \quad \text{Ei ratkaisua}$$

Vastaus: a) $x = -\tfrac{2}{3} + n\tfrac{2\pi}{3}$ tai $x = \tfrac{\pi-2}{11} + n\tfrac{2\pi}{11}$

b) $x = \tfrac{\pi}{8} - \tfrac{1}{4} + n\tfrac{\pi}{2}$

22. a)

$$3\tan(2x+\tfrac{\pi}{2})=\sqrt{3}$$

$$\tan(2x+\tfrac{\pi}{2})=\frac{\sqrt{3}}{3}$$

$$\tan(2x+\tfrac{\pi}{2})=\frac{1}{\sqrt{3}}$$

$$2x+\tfrac{\pi}{2}=\tfrac{\pi}{6}+n\pi \quad \Big| -\tfrac{\pi}{2}$$

$$2x=-\tfrac{\pi}{3}+n\pi \quad \Big| :2$$

$$x=-\tfrac{\pi}{6}+n\tfrac{\pi}{2}$$

b)

$$2\cos(x^2-\tfrac{\pi}{6})=1 \quad \Big| :2$$

$$\cos(x^2-\tfrac{\pi}{6})=\tfrac{1}{2}$$

$$x^2-\tfrac{\pi}{6}=\tfrac{\pi}{3}+n2\pi \quad \text{tai} \quad x^2-\tfrac{\pi}{6}=-\tfrac{\pi}{3}+n2\pi$$

$$x^2=\underbrace{\tfrac{\pi}{2}+n2\pi}_{\geq 0,\ kun\ n\geq 0} \quad \text{tai} \quad x^2=\underbrace{-\tfrac{\pi}{6}+n2\pi}_{\geq 0,\ kun\ n\geq 1}$$

$$x=\pm\sqrt{\tfrac{\pi}{2}+n2\pi},\ n\in\{0,\ 1,\ 2,\ \ldots\} \quad \text{tai} \quad x=\pm\sqrt{-\tfrac{\pi}{6}+n2\pi}\ ,\ n\in\{1,\ 2,\ 3,\ldots\}$$

Vastaus: a) $x=-\tfrac{\pi}{6}+n\tfrac{\pi}{2}\ ,\ n\in\mathbf{Z}$

b) $x=\pm\sqrt{\tfrac{\pi}{2}+n2\pi},\ n\in\mathbf{N}$ tai $x=\pm\sqrt{-\tfrac{\pi}{6}+n2\pi}\ ,\ n\in\mathbf{Z}_+$

23. $\cos x = -\frac{2}{3}$ ja $0 \le x \le \pi$

Kulma x on siis I tai II neljänneksessä, jolloin $\sin x \ge 0$.

$$\cos 2x = 2\cos^2 x - 1 = 2 \cdot (-\tfrac{2}{3})^2 - 1 = 2 \cdot \tfrac{4}{9} - 1 = \tfrac{8}{9} - 1 = -\tfrac{1}{9}$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin x = \pm\sqrt{1 - \cos^2 x}$$

Koska $\sin x \ge 0$, niin $\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - (-\frac{2}{3})^2} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

$$\sin 2x = 2\sin x \cos x = 2 \cdot \tfrac{\sqrt{5}}{3} \cdot (-\tfrac{2}{3}) = -\tfrac{4\sqrt{5}}{9}$$

Vastaus: $\cos 2x = -\frac{1}{9}$ ja $\sin 2x = -\frac{4\sqrt{5}}{9}$

24.

$$2\sin 2x = \sqrt{10 - 2\sqrt{5}}\cos x$$

$$2 \cdot 2\sin x \cos x - \sqrt{10 - 2\sqrt{5}}\cos x = 0$$

$$\cos x(4\sin x - \sqrt{10 - 2\sqrt{5}}) = 0$$

$\cos x = 0$ tai $4\sin x = \sqrt{10 - 2\sqrt{5}}$

$x = \frac{\pi}{2} + n\pi$ tai $\sin x = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$

$x = \frac{\pi}{5} + n2\pi$ tai $x = \pi - \frac{\pi}{5} + n2\pi$

$x = \frac{4\pi}{5} + n2\pi$

Vastaus: $x = \frac{\pi}{2} + n\pi$ tai $x = \frac{\pi}{5} + n2\pi$ tai $x = \frac{4\pi}{5} + n2\pi$

25.

$$\sin 2x \cos 2x = -\frac{\sqrt{3}}{4} \quad \Big| \cdot 2$$

$$2\sin 2x \cos 2x = -\frac{\sqrt{3}}{2}$$

$$\sin(2 \cdot 2x) = -\frac{\sqrt{3}}{2}$$

$$\sin 4x = -\frac{\sqrt{3}}{2}$$

$$4x = \frac{4\pi}{3} + n2\pi \quad \text{tai} \quad 4x = \pi - \frac{4\pi}{3} + n2\pi$$

$$x = \frac{\pi}{3} + n\frac{\pi}{2} \quad \text{tai} \quad 4x = -\frac{\pi}{3} + n2\pi$$

$$x = -\frac{\pi}{12} + n\frac{\pi}{2}$$

Vastaus: $x = \frac{\pi}{3} + n\frac{\pi}{2}$ tai $x = -\frac{\pi}{12} + n\frac{\pi}{2}$

26.

$$4\sin^2 2x-(4+4\sqrt{3})\sin x\cos x+\sqrt{3}=0$$

$$4\sin^2 2x-(2+2\sqrt{3})\cdot 2\sin x\cos x+\sqrt{3}=0$$

$$4\sin^2 2x-(2+2\sqrt{3})\sin 2x+\sqrt{3}=0$$

Merkitään $\sin 2x=t$

$$4t^2-(2+\sqrt{3})t+\sqrt{3}=0$$

$$t=\frac{2+2\sqrt{3}\pm\sqrt{(2+\sqrt{2})^2-4\cdot 4\cdot\sqrt{3}}}{2\cdot 4}=\frac{2+2\sqrt{3}\pm\sqrt{4+4\sqrt{2}+2-16\sqrt{3}}}{8}$$

$$t=\frac{2+2\sqrt{3}\pm\sqrt{4-8\sqrt{3}+12}}{8}=\frac{2+2\sqrt{3}\pm\sqrt{(2-2\sqrt{3})^2}}{8}$$

$$t=\frac{2+2\sqrt{3}\pm(2-2\sqrt{3})}{8}$$

$$t=\frac{2+2\sqrt{3}+2-2\sqrt{3}}{8}\quad\text{tai}\quad t=\frac{2+2\sqrt{3}-2+2\sqrt{3}}{8}$$

$$t=\tfrac{1}{2}\quad\text{tai}\quad t=\tfrac{\sqrt{3}}{2}$$

$$\sin 2x=\tfrac{1}{2}$$

$$2x=\tfrac{\pi}{6}+n2\pi\quad\text{tai}\quad 2x=\pi-\tfrac{\pi}{6}+n2\pi$$

$$x=\tfrac{\pi}{12}+n\pi\quad\text{tai}\quad x=\tfrac{5\pi}{12}+n\pi$$

$$\sin 2x=\tfrac{\sqrt{3}}{2}$$

$$2x=\tfrac{\pi}{3}+n2\pi\quad\text{tai}\quad 2x=\pi-\tfrac{\pi}{3}+n2\pi$$

$$x=\tfrac{\pi}{6}+n\pi\quad\text{tai}\quad x=\tfrac{\pi}{3}+n\pi$$

Vastaus: $x=\tfrac{\pi}{12}+n\pi$ tai $x=\tfrac{5\pi}{12}+n\pi$ tai $x=\tfrac{\pi}{6}+n\pi$ tai $x=\tfrac{\pi}{3}+n\pi$

27. a) $D(\sin(4x+1)) = \cos(4x+1)\cdot 4 = 4\cos(4x+1)$

b) $D(\sin x^2 + \cos x^2) = \cos x^2 \cdot 2x - \sin x^2 \cdot 2x = 2x\cos x^2 - 2x\sin x^2$

c) $D(\tan 6x) = \dfrac{1}{\cos^2 6x}\cdot 6 = \dfrac{6}{\cos^2 6x}$

Vastaus: a) $4\cos(4x+1)$ b) $2x\cos x^2 - 2x\sin x^2$ c) $\dfrac{6}{\cos^2 6x}$

28. a)

$$\begin{aligned} D(x^2\sin^2 x) &= 2x\cdot\sin^2 x + x^2\cdot 2\sin x\cdot\cos x \\ &= 2x\sin^2 x + 2x^2\sin x\cos x \\ &= 2x\sin x(\sin x + x\cos x) \end{aligned}$$

b)

$$\begin{aligned} D\frac{\cos x}{\sin x} &= \frac{-\sin x\cdot\sin x - \cos x\cdot\cos x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x} \end{aligned}$$

c)

$$D(\cos(\tan x)) = -\sin(\tan x)\cdot\frac{1}{\cos^2 x} = -\frac{\sin(\tan x)}{\cos^2 x}$$

Vastaus: a) $2x\sin x(\sin x + x\cos x)$ b) $-\dfrac{1}{\sin^2 x}$ c) $-\dfrac{\sin(\tan x)}{\cos^2 x}$

29. $f(x) = \sin^2 x - \cos^2 x = -(\cos^2 x - \sin^2 x) = -\cos 2x$

$f'(x) = -(-\sin 2x)\cdot 2 = 2\sin 2x$

Tangentin kulmakerroin:
$k_t = f'(\frac{13\pi}{6}) = 2\sin(2\cdot\frac{13\pi}{6}) = 2\sin\frac{13\pi}{3} = 2\sin(4\frac{1}{3}\pi) = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

Normaalin kulmakerroin:
$k_n = -\frac{1}{k_t} = -\frac{1}{\sqrt{3}}$

Normaalin piste:
$\left(\frac{13\pi}{6}, f(\frac{13\pi}{6})\right) = \left(\frac{13\pi}{6}, -\cos(2\cdot\frac{13\pi}{6})\right)$

$= \left(\frac{13\pi}{6}, -\cos(4\frac{1}{3}\pi)\right)$

$= \left(\frac{13\pi}{6}, -\cos\frac{\pi}{3}\right)$

$= \left(\frac{13\pi}{6}, -\frac{1}{2}\right)$

Normaalin yhtälö:
$y - (-\frac{1}{2}) = -\frac{1}{\sqrt{3}}(x - \frac{13\pi}{6})$

$y + \frac{1}{2} = -\frac{1}{\sqrt{3}}x + \frac{13\pi}{6\sqrt{3}}$

$y = -\frac{1}{\sqrt{3}}x + \frac{13\pi}{6\sqrt{3}} - \frac{1}{2}$

Vastaus: $y = -\frac{1}{\sqrt{3}}x + \frac{13\pi}{6\sqrt{3}} - \frac{1}{2}$

30. $g(x) = \cos^2 x - \sin x + 1$

$g'(x) = 2\cos x \cdot (-\sin x) - \cos x = -\cos x(2\sin x + 1) = 0$

$g'(x) = 0$

$\cos x = 0$ tai $2\sin x + 1 = 0$

$x = \frac{\pi}{2} + n2\pi$ tai $2\sin x = -1$

$\sin x = -\frac{1}{2}$

$x = \frac{7\pi}{6} + n2\pi$ tai $x = \pi - \frac{7\pi}{6} + n2\pi$

$x = -\frac{\pi}{6} + n2\pi$

Vastaus: $x = \frac{\pi}{2} + n2\pi$ tai $x = \frac{7\pi}{6} + n2\pi$ tai $x = -\frac{\pi}{6} + n2\pi$

31. $f(x) = A\sin^2 Bx$

$$f'(x) = 2A\sin(Bx)\cdot D(\sin(Bx))$$
$$= 2A\sin(Bx)\cdot\cos(Bx)\cdot B$$
$$= AB\cdot 2\sin(Bx)\cos(Bx)$$
$$= AB\sin(2Bx)$$

Jotta derivaatan suurin arvo olisi 1, pitää olla $AB = 1$ eli $A = \dfrac{1}{B}$

Koska $f'(\frac{3\pi}{4}) = 0$, niin

$$AB\sin(2B\cdot\tfrac{3\pi}{4}) = 0$$
$$\sin(\tfrac{3B\pi}{2}) = 0$$
$$\tfrac{3B\pi}{2} = n\pi \quad \Big|\cdot \tfrac{2}{3\pi}$$
$$B = \tfrac{2n}{3}\ ,\ n\in \mathbf{Z}$$

Koska $B \in [0,\ 2]$ ja $A = \dfrac{1}{B}$, niin kelvollisia ovat

$B = \frac{2}{3}$	$B = \frac{4}{3}$	$B = 2$
$A = \frac{3}{2}$	$A = \frac{3}{4}$	$A = \frac{1}{2}$

Vastaus: $A = \frac{3}{2}$, $B = \frac{2}{3}$ tai $A = \frac{3}{4}$, $B = \frac{4}{3}$ tai $A = \frac{1}{2}$, $B = 2$

32. a) $f(x) = 2\cos x - 3$

suurin arvo, kun $\cos x = 1$ eli $2 \cdot 1 - 3 = -1$
pienin arvo, kun $\cos x = -1$ eli $2 \cdot (-1) - 3 = -5$

b) $g(x) = \sin x \cos x = \frac{1}{2} \cdot 2\sin x \cos x = \frac{1}{2}\sin 2x$

suurin arvo, kun $\sin 2x = 1$ eli $\frac{1}{2} \cdot 1 = \frac{1}{2}$
pienin arvo, kun $\sin 2x = -1$ eli $\frac{1}{2} \cdot (-1) = -\frac{1}{2}$

Vastaus: a) suurin -1 ja pienin -5 b) suurin $\frac{1}{2}$ ja pienin $-\frac{1}{2}$

33. $h(x) = \sqrt{3}\,x + \sin 2x$ välillä $[0, 2\pi]$

$h'(x) = \sqrt{3} + 2\cos 2x$

$$h'(x) = 0$$

$$2\cos 2x = -\sqrt{3}$$

$$\cos 2x = -\tfrac{\sqrt{3}}{2}$$

$$2x = \pm\tfrac{5\pi}{6} + n2\pi$$

$$x = \pm\tfrac{5\pi}{12} + n\pi$$

Välillä $[0,\ 2\pi]$ ovat $x = \frac{5\pi}{12}$, $x = \frac{7\pi}{12}$, $x = \frac{17\pi}{12}$ ja $x = \frac{19\pi}{12}$

	0 – $\frac{5\pi}{12}$	$\frac{5\pi}{12}$ – $\frac{7\pi}{12}$	$\frac{7\pi}{12}$ – $\frac{17\pi}{12}$	$\frac{17\pi}{12}$ – $\frac{19\pi}{12}$	$\frac{19\pi}{12}$ – 2π
$f'(x)$	+	–	+	–	+
$f(x)$	↗	↘	↗	↘	↗

Testipisteet:

$f'(1) = \ldots > 0$

$f'(1{,}5) = \ldots < 0$

$f'(2) = \ldots > 0$

$f'(4{,}5) = \ldots < 0$

$f'(5) = \ldots > 0$

Minimiarvot:

$f(0) = 0$

$f(\frac{7\pi}{12}) = \frac{7\sqrt{3}\pi}{12} - \frac{1}{2} \approx 2{,}67$

$f(\frac{19\pi}{12}) = \frac{19\sqrt{3}\pi}{12} - \frac{1}{2} \approx 8{,}12$

Maksimiarvot:

$f(\frac{5\pi}{12}) = \frac{5\sqrt{3}\pi}{12} + \frac{1}{2} \approx 2{,}77$

$f(\frac{17\pi}{12}) = \frac{17\sqrt{3}\pi}{12} + \frac{1}{2} \approx 8{,}21$

$f(2\pi) = 2\sqrt{3}\,\pi \approx 10{,}9$

Vastaus: Minimiarvot : 0, $\frac{7\sqrt{3}\pi}{12} - \frac{1}{2} \approx 2{,}67$ ja $\frac{19\sqrt{3}\pi}{12} - \frac{1}{2} \approx 8{,}12$

Maksimiarvot: $\frac{5\sqrt{3}\pi}{12} + \frac{1}{2} \approx 2{,}77$, $\frac{17\sqrt{3}\pi}{12} + \frac{1}{2} \approx 8{,}21$ ja $2\sqrt{3}\,\pi \approx 10{,}9$

34. $f(x) = \cos x - \frac{1}{2}\cos 2x$

$$f(x) = \cos x - \frac{1}{2}\cos 2x = \cos x - \frac{1}{2}(2\cos^2 x - 1) = -\cos^2 x + \cos x + \frac{1}{2}$$

Merkitään $\cos x = t$, jolloin
$g(t) = -t^2 + t + \frac{1}{2}$, missä $t \in [-1, 1]$

Fermat'n lauseen mukaan funktio g saa suurimman ja pienimmän arvonsa joko välin päätepisteissä tai derivaatan nollakohdissa.

$$g'(t) = -2t + 1$$
$$g'(t) = 0$$
$$-2t + 1 = 0$$
$$2t = 1$$
$$t = \tfrac{1}{2}$$

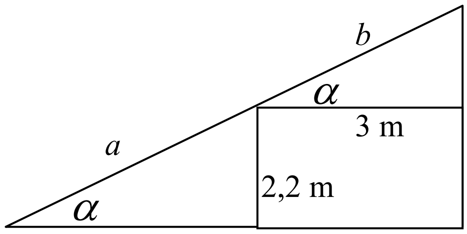
$$g(\tfrac{1}{2}) = -(\tfrac{1}{2})^2 + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{4}$$
$$g(-1) = -(-1)^2 + (-1) + \tfrac{1}{2} = -1\tfrac{1}{2}$$
$$g(1) = -1^2 + 1 + \tfrac{1}{2} = \tfrac{1}{2}$$

Suurin arvo $\frac{3}{4}$, kun
$\cos x = \frac{1}{2}$
$x = \pm\frac{\pi}{3} + n2\pi$

Pienin arvo $-1\frac{1}{2}$, kun
$\cos x = -1$
$x = \pi + n2\pi$

Vastaus: Suurin arvo $\frac{3}{4}$, kun $x = \pm\frac{\pi}{3} + n2\pi$
Pienin arvo $-1\frac{1}{2}$, kun $x = \pi + n2\pi$

35. $\alpha \in]0, \frac{\pi}{2}[$, Tikkaiden pituus $l = a + b$



$$\sin\alpha = \frac{2{,}2}{a} \qquad \cos\alpha = \frac{3}{b}$$

$$a = \frac{2{,}2}{\sin\alpha} \qquad b = \frac{3}{\cos\alpha}$$

Tikkaiden pituus on

$$l(\alpha) = \frac{2{,}2}{\sin\alpha} + \frac{3}{\cos\alpha}$$

$$l'(\alpha) = \frac{0 - 2{,}2\cos\alpha}{\sin^2\alpha} + \frac{0 - 3(-\sin\alpha)}{\cos^2\alpha} = \frac{3\sin\alpha}{\cos^2\alpha} - \frac{2{,}2\cos\alpha}{\sin^2\alpha}$$

Derivaatan nollakohdat:

$$l'(\alpha) = 0$$

$$\frac{3\sin\alpha}{\cos^2\alpha} = \frac{2{,}2\cos\alpha}{\sin^2\alpha} \quad \Big|\cdot \cos^2\alpha \sin^2\alpha$$

$$3\sin^3\alpha = 2{,}2\cos^3\alpha \quad \Big| : 3\cos^3\alpha$$

$$\frac{\sin^3\alpha}{\cos^3\alpha} = \frac{2{,}2}{3}$$

$$\tan^3\alpha = \frac{2{,}2}{3}$$

$$\tan\alpha = \sqrt[3]{\tfrac{2{,}2}{3}} \quad \Big| \tan^{-1}$$

$$\alpha = 0{,}7337\ldots(\text{rad})$$

Testipisteet:

$l'(0{,}1) \approx -219 < 0$

$l'(1) \approx 6{,}97 > 0$

	0 0,73...	$\frac{\pi}{2}$
$l'(x)$	–	+
$l(x)$	↘	↗

Pienin arvo, kun

$\alpha = 0{,}7337\ldots\text{rad} \approx 42{,}04^\circ$

Tikkaiden pituus $l(0{,}7337\ldots) = 7{,}3247 \approx 7{,}32$ m

Vastaus: 7,32 m

36. a) $a_n = \dfrac{5n+5}{n^2+1}$

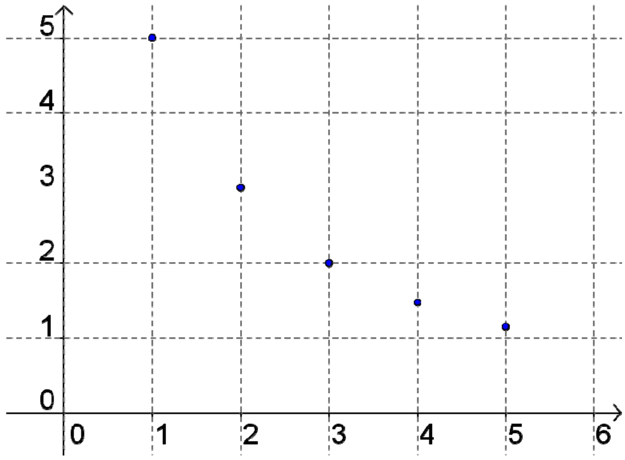
$a_1 = \dfrac{5\cdot 1+5}{1^2+1} = \dfrac{10}{2} = 5$

$a_2 = \dfrac{5\cdot 2+5}{2^2+1} = \dfrac{15}{5} = 3$

$a_3 = \dfrac{5\cdot 3+5}{3^2+1} = \dfrac{20}{10} = 2$

$a_4 = \dfrac{5\cdot 4+5}{4^2+1} = \dfrac{25}{17} = 1\dfrac{8}{17}$

$a_5 = \dfrac{5\cdot 5+5}{5^2+1} = \dfrac{30}{26} = 1\dfrac{2}{13}$



b) $b_n = n\sin\dfrac{n\pi}{4}$

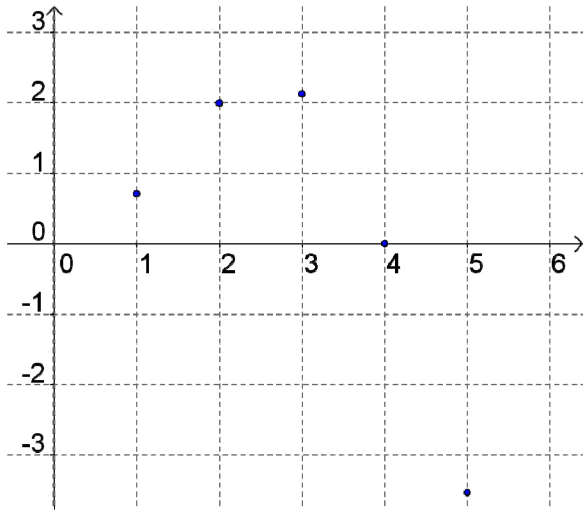
$a_1 = 1\cdot\sin\dfrac{1\cdot\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$a_2 = 2\cdot\sin\dfrac{2\cdot\pi}{4} = 2\sin\dfrac{\pi}{2} = 2$

$a_3 = 3\cdot\sin\dfrac{3\cdot\pi}{4} = 3\sin\dfrac{3\pi}{4} = \dfrac{3}{\sqrt{2}}$

$a_4 = 4\cdot\sin\dfrac{4\cdot\pi}{4} = 4\sin\pi = 0$

$a_5 = 5\cdot\sin\dfrac{5\cdot\pi}{4} = 5\sin\dfrac{5\pi}{4} = -\dfrac{5}{\sqrt{2}}$



37. a) $a_n = 2n^2 - 3n - 15$

$a_n = 12$

$2n^2 - 3n - 15 = 12$

$2n^2 - 3n - 27 = 0$

$$n = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-27)}}{2 \cdot 2} = \frac{3 \pm 15}{4}$$

$n = -3$ tai $n = 4\frac{1}{2}$

Ei ole positiivista kokonaislukuratkaisua eli 12 ei ole jonon jäsen.

b) $b_n = \dfrac{30n + 2}{4n^2 - 2}$

$b_n = 12$

$$\frac{30n + 2}{4n^2 - 2} = 12 \quad \Big| \cdot (4n^2 - 2)$$

$30n + 2 = 48n^2 - 24$

$48n^2 - 30n - 26 = 0$

$$n = \frac{-(-30) \pm \sqrt{(-30)^2 - 4 \cdot 48 \cdot (-26)}}{2 \cdot 48} = \frac{30 \pm \sqrt{5897}}{96} \notin \mathbf{Z}$$

Ei kokonaislukuratkaisua eli 12 ei ole jonon jäsen.

Vastaus: a) Ei ole b) Ei ole

38. a) 0, 3, 8, 15, 24, 35, … Esimerkiksi $a_n = n^2 - 1$

b) 0, −1, 0, 1, 0, −1, 0, 1, ... Esimerkiksi $b_n = \cos(\frac{\pi}{2} + n \cdot \frac{\pi}{2})$

tai $b_n = -\sin(n\frac{\pi}{2})$

39. a) $a_n = \dfrac{n^3}{2n^2+2}$

Tutkitaan funktion $f(x) = \dfrac{x^3}{2x^2+2}$, $x \geq 1$, $x \in \mathbf{R}$ kulkua.

$$f'(x) = \frac{3x^2 \cdot (2x^2+2) - x^3 \cdot 4x}{(2x^2+2)^2} = \frac{6x^4+6x^2-4x^4}{(2x^2+2)^2} = \frac{2x^4+6x^2}{(2x^2+2)^2}$$

$f'(x) > 0$ kaikilla $x \geq 1$.
f on siis aidosti kasvava, joten myös a_n on aidosti kasvava.

b) $b_n = \dfrac{n^2-8n+16}{2n^2}$ Tukitaan peräkkäisten jäsenten erotusta

$b_{n+1} - b_n$

$$= {}^{n^2)}\frac{(n+1)^2 - 8(n+1) + 16}{2(n+1)^2} - {}^{(n+1)^2)}\frac{n^2-8n+16}{2n^2}$$

$$= \frac{n^2\left(n^2+2n+1-8n-8+16\right) - (n^2+2n+1)(n^2-8n+16)}{2n^2(n+1)^2}$$

$$= \frac{n^2\left(n^2-6n+9\right) - \left(n^4-8n^3+16n^2+2n^3-16n^2+32n+n^2-8n+16\right)}{2n^2(n+1)^2}$$

$$= \frac{n^4-6n^3+9n^2-(n^4-6n^3+n^2+24n+16)}{2n^2(n+1)^2}$$

$$= \frac{8n^2-24n-16}{2n^2(n+1)^2} = \frac{8(n^2-3n-2)}{\underbrace{2n^2(n+1)^2}_{>0,\ \text{kun}\ n\geq 1}}$$

Kun $n = 1$, niin $n^2 - 3n - 2 = 1^2 - 3 \cdot 1 - 2 = -4 < 0$
Kun $n = 4$, niin $n^2 - 3n - 2 = 4^2 - 3 \cdot 4 - 2 = 2 > 0$
Jono ei siis ole kasvava eikä vähenevä

Vastaus: a) aidosti kasvava b) ei kumpaakaan

40. $c_n = \dfrac{n^2 - 12n + 34}{2n^2 + 2}$

Väite: $c_n < \frac{1}{2}$ eli $c_n - \frac{1}{2} < 0$ kaikilla $n > 10$

Todistus:

$$c_n - \frac{1}{2} = \frac{n^2 - 12n + 34}{2n^2 + 2} - {}^{n^2+1)}\frac{1}{2} = \frac{n^2 - 12n + 34 - n^2 - 1}{2n^2 + 2} = \frac{-12n + 33}{\underbrace{2n^2 + 2}_{>0 \text{ kaikilla } n}}$$

Kun $n > 10$, niin $c_n - \frac{1}{2} < -12 \cdot 10 + 33 = -87 < 0$

Siis $c_n - \frac{1}{2} < 0$ kaikilla $n > 10$ □

Tutkitaan funktion $f(x) = \dfrac{x^2 - 12x + 34}{2x^2 + 2}$, $x \geq 1$, $x \in \mathbf{R}$ kulkua.

$$f'(x) = \frac{(2x - 12) \cdot (2x^2 + 2) - (x^2 - 12x + 34) \cdot 4x}{(2x^2 + 2)^2}$$

$$= \frac{4x^3 + 4x - 24x^2 - 24 - 4x^3 + 48x^2 - 136x}{(2x^2 + 2)^2} = \frac{24x^2 - 132x - 24}{(2x^2 + 2)^2}$$

$f'(x) = 0$

$24x^2 - 132x - 24 = 0$

$$x = \frac{-(-132) \pm \sqrt{(-132)^2 - 4 \cdot 24 \cdot (-24)}}{2 \cdot 24} = \frac{132 \pm 12\sqrt{137}}{48} = \frac{11}{4} \pm \frac{1}{4}\sqrt{137}$$

Vain $x = \frac{11}{4} + \frac{1}{4}\sqrt{137} \approx 5{,}7$ kelpaa ($x > 1$)

	1	$\frac{11}{4} + \frac{1}{4}\sqrt{137} \approx 5{,}7$
$f'(x)$	–	+
$f(x)$	↘	↗

Testipisteet:
$f'(2) \approx -1{,}9 < 0$

$f'(6) \approx 0{,}009 > 0$

Kulkukaavion mukaan pienin jäsen on joko c_5 tai c_6.

$$c_5 = \frac{5^2 - 12 \cdot 5 + 34}{2 \cdot 5^2 + 2} = -\frac{1}{52}$$

$$c_6 = \frac{6^2 - 12 \cdot 6 + 34}{2 \cdot 6^2 + 2} = -\frac{1}{37}$$

Eli pienin jäsen on $-\frac{1}{37}$

Kulkukaavion, ja alussa todistetun mukaan, suurin jäsen on c_1, jos $c_1 > \frac{1}{2}$

$$c_1 = \frac{1^2 - 12 \cdot 1 + 34}{2 \cdot 1^2 + 2} = \frac{23}{4} = 5\tfrac{3}{4} > \tfrac{1}{2}$$

Siis suurin jäsen on $c_1 = 5\frac{3}{4}$

Vastaus: Suurin jäsen $5\frac{3}{4}$ ja pienin $-\frac{1}{37}$

41. (a_n) aritmeettinen jono $a_n = a_1 + (n-1) \cdot d$

a) $a_3 = 5$ ja $a_{10} = 40$

$$- \begin{cases} a_1 + 2 \cdot d = 5 \\ a_1 + 9 \cdot d = 40 \end{cases}$$

$$-7d = -35$$

$$d = 5$$

$$a_1 = 5 - 2d = 5 - 2 \cdot 5 = -5$$

$$a_{20} = -5 + 19 \cdot 5 = 90$$

b) $a_6 = 10$ ja $a_{10} = 6$

$$
-\begin{cases} a_1 + 5 \cdot d = 10 \\ a_1 + 9 \cdot d = 6 \end{cases}
$$

$$-4d = 4$$

$$d = -1$$

$a_1 = 10 - 5d = 10 - 5 \cdot (-1) = 15$

$a_{20} = 15 + 19 \cdot (-1) = -4$

Vastaus: a) $d = 5$ ja $a_{20} = 90$ b) $d = -1$ ja $a_{20} = -4$

42. a) $a_n = (n+1)(n-1) - (n-1)^2 + 3 = n^2 - 1 - n^2 + 2n - 1 + 3 = 2n + 1$

$a_{n+1} - a_n = 2(n+1) + 1 - (2n+1) = 2n + 2 + 1 - 2n - 1 = 2 = \text{vakio}$

Jono on aritmeettinen

b) $b_n = n(2-n) = 2n - n^2$

$$
\begin{aligned}
b_{n+1} - b_n &= 2(n+1) - (n+1)^2 - (2n - n^2) \\
&= 2n + 2 - n^2 - 2n - 1 - 2n + n^2 \\
&= -2n + 1
\end{aligned}
$$

Ei ole vakio

Jono ei ole aritmeettinen

Vastaus: a) kyllä b) ei

43. (b_n) aritmeettinen jono, $b_n = b_1 + (n-1)d$

$b_2 = 200$ ja $b_{11} = 192$

$$
- \begin{cases} b_1 + d = 200 \\ b_1 + 10d = 192 \end{cases}
$$

$$-9d = 8$$

$$d = -\tfrac{8}{9}$$

$b_1 = 200 - d = 200 - (-\tfrac{8}{9}) = 200\tfrac{8}{9}$

$b_n = 200\tfrac{8}{9} + (n-1)(-\tfrac{8}{9}) = 201\tfrac{7}{9} - \tfrac{8}{9}n$

$$b_n \geq 50$$

$$201\tfrac{7}{9} - \tfrac{8}{9}n \geq 50$$

$$\tfrac{8}{9}n \leq 151\tfrac{7}{9} \quad \Big|: \tfrac{8}{9}$$

$$n \leq 170\tfrac{3}{4}$$

Siis 170 jäsenistä ovat vähintään 50

Vastaus: 170 kpl

44. $(b_n) = (2^{a_n})$ on aritmeettinen, $a_1 = 4$ ja $a_2 = 6$

$b_1 = 2^{a_1} = 2^4 = 16$

$b_2 = 2^{a_2} = 2^6 = 64$

Koska (b_n) on aritmeettinen, niin

$$b_2 = b_1 + d$$

$$16 + d = 64$$

$$d = 48$$

$$b_n = b_1 + (n-1)d = 16 + (n-1)\cdot 48 = 48n - 32$$

$$2^{a_n} = 48n - 32$$

$$\lg 2^{a_n} = \lg(48n - 32)$$

$$a_n \lg 2 = \lg(48n - 32)$$

$$a_n = \frac{\lg(48n - 32)}{\lg 2}$$

Vastaus: $a_n = \dfrac{\lg(48n - 32)}{\lg 2}$

45. Jono 3, $2x$, x^2, ... on aritmeettinen. Peräkkäisten jäsenten erotus on vakio, joten

$$x^2 - 2x = 2x - 3$$

$$x^2 - 4x + 3 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{4 \pm 2}{2}$$

$x = 1$ tai $x = 3$

Jos $x = 1$, niin jono on 3, 2, 1, … ($d = -1$), joten
$a_5 = 3 + 4 \cdot (-1) = -1$

Jos $x = 3$, niin jono on 3, 6, 9, …($d = 3$), joten
$a_5 = 3 + 4 \cdot 3 = 15$

Vastaus: $a_5 = -1$ tai $a_5 = 15$

46. a) $1, -1, 1, -1, 1, ...$ voi olla geometrinen, sillä

$$\frac{-1}{1} = \frac{1}{-1} = \frac{-1}{1} = \frac{1}{-1} = -1$$

b) 2, 4, 6, 8, 10, … ei voi olla geometrinen, sillä

$$\frac{4}{2} = 2 \text{ ja } \frac{6}{4} = \frac{3}{2}$$

c) 4 375, 6 125, 8 575, 12 005, 16 810, … ei voi olla geometrinen

$$\frac{6\,125}{4\,375} = \frac{7}{5} \text{ ja } \frac{16\,810}{12\,005} = \frac{3\,362}{4\,401} \neq \frac{7}{5}$$

d) $b_n = \dfrac{4^{2n}}{8^{2n-1}}$, joten $b_{n+1} = \dfrac{4^{2(n+1)}}{8^{2(n+1)-1}} = \dfrac{4^{2n+2}}{8^{2n+1}}$

$$\frac{b_{n+1}}{b_n} = \frac{4^{2n+2}}{8^{2n+1}} : \frac{4^{2n}}{8^{2n-1}} = \frac{4^{2n+2}}{8^{2n+1}} \cdot \frac{8^{2n-1}}{4^{2n}} = 4^{2n+2-2n} \cdot 8^{2n-1-(2n+1)}$$

$$= 4^2 \cdot 8^{-2}$$

$$= \tfrac{16}{64}$$

$$= \tfrac{1}{4} = \text{vakio}$$

Jono on geometrinen

Vastaus: a) voi olla b) ei c) ei d) on

47. Jono on geometrinen

a) $a_4 = 6$ ja $a_{10} = 20$

$$a_4 \cdot q^6 = a_{10}$$

$$q^6 = \frac{a_{10}}{a_4} = \frac{20}{6} = \frac{10}{3}$$

$$q = \pm\sqrt[6]{\tfrac{10}{3}}$$

$$a_1 \cdot q^3 = 6$$

$$a_1 = \frac{6}{(\pm\sqrt[6]{\frac{10}{3}})^3} = \pm\frac{6}{\left((\frac{10}{3})^{\frac{1}{6}}\right)^3} = \pm\frac{6}{(\frac{10}{3})^{\frac{1}{2}}}$$

$$a_1 = \pm\tfrac{6\sqrt{3}}{\sqrt{10}}$$

$\sqrt{10})$

$$a_{31} = \tfrac{6\sqrt{3}}{\sqrt{10}} \cdot (\sqrt[6]{\tfrac{10}{3}})^{30} = \tfrac{6\sqrt{3}}{\sqrt{10}} \cdot \left((\tfrac{10}{3})^{\frac{1}{6}}\right)^{30} = \tfrac{6\sqrt{3}}{\sqrt{10}} \cdot (\tfrac{10}{3})^5 = \tfrac{200\,000\sqrt{3}}{81\sqrt{10}} = \tfrac{20\,000\sqrt{30}}{81}$$

tai

$$a_{31} = -\tfrac{6\sqrt{3}}{\sqrt{10}} \cdot (-\sqrt[6]{\tfrac{10}{3}})^{30} = -\tfrac{20\,000\sqrt{30}}{81}$$

b) $b_4 = 100$ ja $b_9 = 50$

$$b_4 \cdot q^5 = b_9$$

$$q^5 = \frac{b_9}{b_4} = \frac{50}{100} = \frac{1}{2}$$

$$q = \sqrt[5]{\tfrac{1}{2}} = (\tfrac{1}{2})^{\frac{1}{5}}$$

$$b_1 \cdot q^3 = 100$$

$$b_1 = \frac{100}{\left((\frac{1}{2})^{\frac{1}{5}}\right)^3} = \frac{100}{(\frac{1}{2})^{\frac{3}{5}}}$$

$$b_1 = 100 \cdot 2^{\frac{3}{5}}$$

$$b_{31} = 100 \cdot 2^{\frac{3}{5}} \cdot \left((\tfrac{1}{2})^{\frac{1}{5}}\right)^{30} = 100 \cdot 2^{\frac{3}{5}} \cdot (\tfrac{1}{2})^6 = 100 \cdot \frac{2^{\frac{3}{5}}}{2^6} = \frac{100}{2^{5\frac{2}{5}}} = \frac{100}{32\sqrt[5]{2^2}} = \frac{25}{8\sqrt[5]{4}}$$

Vastaus: a) $a_{31} = \pm\tfrac{20\,000\sqrt{30}}{81}$ b) $b_{31} = \dfrac{25}{8\sqrt[5]{4}}$

48. Jono $4, 1-2x, x^2, ...$ on geometrinen

Peräkkäisten jäsenten suhde on vakio:

$$\frac{1-2x}{4}=\frac{x^2}{1-2x} \qquad x \neq \frac{1}{2}, \text{ sillä muutoin jono ei ole geometrinen}$$

$$(1-2x)^2=4x^2$$

$$1-4x+4x^2=4x^2$$

$$4x=1$$

$$x=\tfrac{1}{4}$$

Jono on siis $4, \frac{1}{2}, \frac{1}{16}, ...$ ja siten $q=\dfrac{\frac{1}{2}}{4}=\frac{1}{8}$

$$\frac{a_{20}}{a_{10}}=\frac{4\cdot(\frac{1}{8})^{19}}{4\cdot(\frac{1}{8})^{9}}=(\tfrac{1}{8})^{10}=0{,}000\ 000\ 000\ 931...\approx 0{,}000\ 000\ 09\ \%$$

Siis a_{20} on $100\ \% - 0{,}000\ 000\ 09\ \% = 99{,}999\ 999\ 91\ \%$ suurempi

Vastaus: a_{20} on $99{,}999\ 999\ 91\ \%$ suurempi

49.

Vuosia päivämäärästä 1.1.2012	Rahaa patjassa (€)
1	$0{,}95 \cdot 10\,000$
2	$0{,}95^2 \cdot 10\,000$
3	$0{,}95^3 \cdot 10\,000$
…	
n	$0{,}95^n \cdot 10\,000$

n. vuoden kuluttua talletuksesta patjassa on rahaa
$a_n = 0{,}95^n \cdot 10\,000$ €

Rahamäärä 2.1.2021 on $a_9 = 0{,}95^9 \cdot 10\,000 \approx 6\,302{,}49$ €

Alittaa 1 000 €, kun

$$a_n < 1\,000$$

$$0{,}95^n \cdot 10\,000 < 1\,000$$

$$0{,}95^n < 0{,}1 \quad |\lg \text{ (aidosti kasvava)}$$

$$\lg 0{,}95^n < -1$$

$$n \lg 0{,}95 < -1 \quad |:\lg 0{,}95 \ (<0)$$

$$n > 44{,}890...$$

Siis 1 000 € alittuu 1.1.(2012 + 45) = 1.1.2057

Vastaus: 6 302,49 €
1.1.2057 tapahtuvan noston jälkeen

50. a)

$$\sum_{k=1}^{6}\frac{2k-1}{2k}=\frac{2\cdot 1-1}{2\cdot 1}+\frac{2\cdot 2-1}{2\cdot 2}+...+\frac{2\cdot 6-1}{2\cdot 6}$$

$$=\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\frac{7}{8}+\frac{9}{10}+\frac{11}{12}$$

$$=4\tfrac{31}{40}$$

b)

$$\sum_{j=3}^{9}\frac{(-1)^j}{j}=\frac{(-1)^3}{3}+\frac{(-1)^4}{4}+...+\frac{(-1)^9}{9}$$

$$=-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{8}-\frac{1}{9}$$

$$=-\tfrac{619}{2520}$$

c) $\sum_{i=2}^{8}(-2)^i(-3)^{i-1}$

$$(-2)^i(-3)^{i-1}=(-1\cdot 2)^i(-1\cdot 3)^{i-1}=(-1)^i(-1)^{i-1}\cdot 2^i\cdot 3^{i-1}$$

$$=(-1)^{2i-1}\cdot 2^i\cdot\frac{3^i}{3}=-1\cdot\tfrac{1}{3}\cdot(2\cdot 3)^i=-\tfrac{1}{3}\cdot 6^i$$

$$\sum_{i=2}^{8}(-2)^i(-3)^{i-1}=-\tfrac{1}{3}\cdot\sum_{i=2}^{8}6^i=-\tfrac{1}{3}\cdot(6^2+6^3+6^4+...+6^8)$$

$$=-\tfrac{1}{3}\cdot 2\ 015\ 532$$

$$=-671\ 844$$

Vastaus: a) $4\tfrac{31}{40}$ b) $-\tfrac{619}{2520}$ c) $-671\ 844$

51. Aritmeettisia summia $S_n = \dfrac{a_1 + a_n}{2} \cdot n$

a) $\underbrace{6 + 13 + 20 + ... + 111}_{\frac{6+111}{7}+1=16 \text{ kpl}} = \dfrac{6+111}{2} \cdot 16 = 936$

b) $0\underbrace{-100 - 91 - 82 - ... + 80}_{\frac{80-(-100)}{9}+1=21 \text{ kpl}} = \dfrac{-100+80}{2} \cdot 21 = -210$

Vastaus: a) 936 b) -210

52. (a_n) aritmeettinen jono, $a_3 = 30$ ja $a_{12} = -120$

$$\sum_{k=10}^{30} a_k = \frac{a_{10} + a_{30}}{2} \cdot 21$$

$$\begin{cases} a_1 + 2d = 30 \\ a_1 + 11d = -120 \end{cases}$$

$-$

$-11d = 150$

$d = -16\tfrac{2}{3}$

$a_1 = 20 - 2d$

$a_1 = 20 - 2 \cdot (-16\tfrac{2}{3})$

$a_1 = 63\tfrac{1}{3}$

$$\sum_{k=10}^{30} a_k = \frac{a_{10} + a_{30}}{2} \cdot 21$$

$$= \frac{(63\tfrac{1}{3} + 9 \cdot (-16\tfrac{2}{3})) + (63\tfrac{1}{3} + 29 \cdot (-16\tfrac{2}{3}))}{2} \cdot 21$$

$$= -5320$$

Vastaus: -5320

53. $a_n = 5n - 10$ aritmeettinen jono

$a_1 = 5 \cdot 1 - 10 = -5$

$$a_1 + a_2 + ... + a_n \leq 1000$$

$$\frac{-5 + 5n - 10}{2} \cdot n \leq 1000 \quad |\cdot 2$$

$$(5n - 15)n \leq 2000$$

$$5n^2 - 15n - 2000 \leq 0 \quad |:5$$

$$n^2 - 3n - 400 \leq 0$$

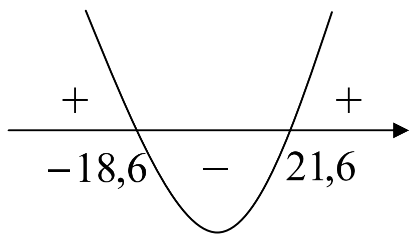
Nollakohdat:

$n^2 - 3n - 400 = 0$

$$n = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-400)}}{2 \cdot 1} = \frac{3 \pm \sqrt{1609}}{2}$$

$n \approx -18{,}6$ tai $n \approx 21{,}6$

Siis $1 \leq n \leq 21$



Suurin n:n arvo on siis 21

Vastaus: 21

54. Rekursiivinen määritelmä $\begin{cases} a_1 = 2 \\ a_{n+1} = a_n - n \quad , n = 1, 2, 3, ... \end{cases}$.

$$a_2 = a_1 - 1 = 2 - 1$$

$$a_3 = a_2 - 2 = 2 - 1 - 2$$

$$a_4 = a_3 - 3 = 2 - 1 - 2 - 3$$

$$a_5 = a_4 - 4 = 2 - 1 - 2 - 3 - 4$$

...

$$a_n = 2 - (1 + 2 + 3 + ... + (n - 1))$$

$$a_{200} = 2 - (\underbrace{1 + 2 + 3 + ... + 199}_{\text{aritmeettinen summa}}) = 2 - \frac{1 + 199}{2} \cdot 199 = -19\ 898$$

Vastaus: $-19\ 898$

55. Aritmeettinen jono (c_n), $\sum_{k=1}^{10} c_k = -10$ ja $\sum_{k=21}^{30} c_k = 30$

$$\begin{cases} \dfrac{c_1 + c_{10}}{2} \cdot 10 = -10 \\ \dfrac{c_{21} + c_{30}}{2} \cdot 10 = 30 \end{cases}$$

$$\begin{cases} 5(c_1 + c_{10}) = -10 & |:5 \\ 5(c_{21} + c_{30}) = 30 & |:5 \end{cases}$$

$$\begin{cases} c_1 + c_1 + 9 \cdot d = -2 \\ c_1 + 20 \cdot d + c_1 + 29 \cdot d = 6 \end{cases}$$

$$- \begin{cases} 2c_1 + 9d = -2 \\ 2c_1 + 49d = 6 \end{cases}$$

$$-40d = -8$$

$$d = \tfrac{1}{5}$$

$$2c_1 = -2 - 9d$$

$$2c_1 = -2 - 9 \cdot \tfrac{1}{5}$$

$$2c_1 = -\tfrac{19}{5} \quad |:2$$

$$c_1 = -\tfrac{19}{10}$$

$$\sum_{k=1}^{50} c_k = \frac{c_1 + c_{50}}{2} \cdot 50 = \frac{-\tfrac{19}{10} + (-\tfrac{19}{10}) + 49 \cdot \tfrac{1}{5}}{2} \cdot 50 = 3 \cdot 50 = 150$$

Vastaus: 150

56. a) Geometrinen summa $3+9+27+...+3\ 486\ 784\ 401$

$$q=\frac{9}{3}=3$$

$$a_n=3\ 486\ 784\ 401$$

$$3\cdot 3^{n-1}=3\ 486\ 784\ 401$$

$$3^n=3\ 486\ 784\ 401$$

$$\lg 3^n=\lg 3\ 486\ 784\ 401$$

$$n\lg 3=\lg 3\ 486\ 784\ 401$$

$$n=\frac{\lg 3\ 486\ 784\ 401}{\lg 3}=20$$

$$S_{20}=\frac{3\cdot(1-3^{20})}{1-3}=5\ 230\ 176\ 600$$

b) Geometrinen summa $0{,}99+0{,}99^2+0{,}99^3+...+0{,}99^{15}$

$a_1=0{,}99$, $q=0{,}99$ ja $n=15$

$$S_{15}=\frac{0{,}99\cdot(1-0{,}99^{15})}{1-0{,}99}=99(1-0{,}99^{15})\approx 13{,}9$$

Vastaus: a) 5 230 176 600 b) $99(1-0{,}99^{15})\approx 13{,}9$

57. a)

$$11^{4k-2} = \tfrac{1}{11^2}\cdot(11^4)^k = \tfrac{1}{121}\cdot 14641^k = \tfrac{14641}{121}\cdot 14641^{k-1} = 121\cdot 14\,641^{k-1}$$

Geometrinen jono ($a_n = a_1\cdot q^{n-1}$)

$$\sum_{k=1}^{20} 11^{4k-2} = \sum_{k=1}^{20} 121\cdot 14641^{k-1} = \frac{121\cdot(1-14641^{20})}{1-14641} \approx 1{,}7\cdot 10^{81}$$

b)

$$\frac{4^{2k}}{8^{2k-1}} = \frac{4^{2k}}{8^{-1}\cdot 8^{2k}} = 8\cdot(\tfrac{4^2}{8^2})^k = 8\cdot(\tfrac{1}{4})^k = 8\cdot\tfrac{1}{4}\cdot(\tfrac{1}{4})^{k-1} = 2\cdot(\tfrac{1}{4})^{k-1}$$

Geometrinen jono
$a_{20} = 2\cdot(\tfrac{1}{4})^{19}$, $q = \tfrac{1}{4}$, $n = 50-20+1 = 31$

$$\sum_{k=20}^{50} \frac{4^{2k}}{8^{2k-1}} = \sum_{k=20}^{50} 2\cdot(\tfrac{1}{4})^{k-1} = \frac{2\cdot(\tfrac{1}{4})^{19}\cdot(1-(\tfrac{1}{4})^{31})}{1-\tfrac{1}{4}} \approx 9{,}7\cdot 10^{-12}$$

Vastaus: a) $1{,}7\cdot 10^{81}$ b) $9{,}7\cdot 10^{-12}$

58. $c_k = 10 \cdot 1{,}0025^{k-1}$ on geometrinen jono

$c_1 = 10$, $q = 1{,}0025$

$$\sum_{k=1}^{n} 10 \cdot 1{,}0025^{k-1} \geq 1000$$

$$\frac{10 \cdot (1 - 1{,}0025^n)}{1 - 1{,}0025} \geq 1000 \quad |:10$$

$$\frac{1 - 1{,}0025^n}{-0{,}0025} \geq 100 \quad | \cdot (-0{,}0025)$$

$$1 - 1{,}0025^n \leq -0{,}25$$

$$1{,}0025^n \geq 1{,}25 \quad |\lg \text{ (aidosti kasvava)}$$

$$\lg 1{,}0025^n \geq \lg 1{,}25$$

$$n \lg 1{,}0025 \geq \lg 1{,}25 \quad |: \lg 1{,}0025 \ (> 0)$$

$$n \geq \frac{\lg 1{,}25}{\lg 1{,}0025}$$

$$n \geq 89{,}36...$$

Pienin kelpaava n on siis 90

Vastaus: $n = 90$

59. $k + \frac{1}{2}k + (\frac{1}{2})^2 k + (\frac{1}{2})^3 k + ... + (\frac{1}{2})^{19} k = 10$

Geometrinen summa, $a_1 = k$, $q = \frac{1}{2}$, $n = 20$

$$S_{20} = 10$$

$$\frac{k(1-(\frac{1}{2})^{20})}{1-\frac{1}{2}} = 10$$

$$2k(1-(\frac{1}{2})^{20}) = 10 \quad |:2$$

$$k(1-(\frac{1}{2})^{20}) = 5$$

$$k = \frac{5}{1-(\frac{1}{2})^{20}}$$

$$k \approx 5{,}000005$$

Vastaus: $k \approx 5{,}000005$

60. Rekursiivinen määritelmä $\begin{cases} a_1 = 2 \\ a_{n+1} = 2a_n - 1 \text{ , } n = 1,2,3,... \end{cases}$

$$a_2 = 2a_1 - 1 = 2 \cdot 2 - 1 = 2^2 - 1$$

$$a_3 = 2a_2 - 1 = 2 \cdot (2^2 - 1) - 1 = 2^3 - 2 - 1$$

$$a_4 = 2a_3 - 1 = 2 \cdot (2^3 - 2 - 1) - 1 = 2^4 - 2^2 - 2 - 1$$

$$a_5 = 2^5 - 2^3 - 2^2 - 2 - 1$$

...

$$a_n = 2^n - 2^{n-2} - 2^{n-3} - ... - 2^2 - 2 - 1 = 2^n - (\underbrace{2^{n-2} + 2^{n-3} + ... + 2^2 + 2 + 1}_{\text{geometrinen summa}})$$

$$a_n = 2^n - \frac{1 \cdot (1 - 2^{n-1})}{1 - 2}$$

$$= 2^n + 1 - 2^{n-1}$$

$$= 2 \cdot 2^{n-1} - 2^{n-1} + 1$$

$$= 2^{n-1} + 1$$

$$a_{50} = 2^{49} + 1 \approx 5{,}6 \cdot 10^{14}$$

Vastaus: $a_{50} = 2^{49} + 1 \approx 5{,}6 \cdot 10^{14}$

TESTI 1

1. a) $D(3\sin 4x) = 3D(\sin 4x) = 3\cos 4x \cdot D4x = 12\cos 4x$

b)

$$\begin{aligned} D(x\tan x) &= Dx\tan x + xD\tan x \\ &= \tan x + x(1+\tan^2 x) \\ &= \tan x + x + x\tan^2 x \end{aligned}$$

c)

$$\begin{aligned} D(\cos^2(-x+1)+x) &= D(\cos^2(-x+1)) + Dx \\ &= 2\cos(-x+1)D\cos(-x+1)+1 \\ &= 2\cos(-x+1)[-\sin(-x+1)D(-x+1)]+1 \\ &= 2\cos(-x+1)\sin(-x+1)+1 \\ &= \sin[2(-x+1)]+1 \\ &= \sin(-2x+2)+1 \end{aligned}$$

2. a)

$$\sin 5\alpha = \sin 45°$$

$5\alpha = 45° + n360° \ |:5$ tai $5\alpha = 180° - 45° + n360° \ |:5$

$\alpha = 9° + n72°$ tai $\alpha = 27° + n72°, \quad n \in Z$

b)

$$\cos x(\cos 3x + 1) = 0$$

$\cos x = 0$ tai $\cos 3x + 1 = 0$

$x = \dfrac{\pi}{2} + n\pi$ $\qquad \cos 3x = -1$

$$3x = \pi + n2\pi \ |:3$$

$$x = \frac{\pi}{3} + n\frac{2\pi}{3}, \quad n \in Z$$

c) $\sin\alpha = \cos\alpha \;\; |: \cos\alpha \neq 0$

$$\frac{\sin\alpha}{\cos\alpha} = 1$$

$$\tan\alpha = 1$$

$$\alpha = \frac{\pi}{4} + n\pi, \;\; n \in Z$$

Tutkitaan erikseen, jos $\cos\alpha = 0$. Tällöin $\alpha = \frac{\pi}{2} + n\pi, n \in Z$.
Sijoittamalla yhtälöön $\cos\alpha = 0$ saadaan

$$\sin\alpha = 0$$

$$\alpha = n\pi, \;\; n \in Z$$

Siis kosinin nollakohdat $\alpha = \frac{\pi}{2} + n\pi, n \in Z$ eivät ole yhtälön ratkaisuja.

Vastaus: a) $\alpha = 9^\circ + n72^\circ$ tai $\alpha = 27^\circ + n72^\circ, \quad n \in Z$

b) $x = \frac{\pi}{2} + n\pi$ tai $x = \frac{\pi}{3} + n\frac{2\pi}{3}, \quad n \in Z$

c) $\alpha = \frac{\pi}{4} + n\pi, \;\; n \in Z$

3. Aritmeettisen jonon yleinen termi on $a_n = a_1 + (n-1)d$

$$\begin{cases} a_5 = a_1 + (5-1)d = 2 \\ a_{23} = a_1 + (23-1)d = -23 \end{cases}$$

$$\begin{cases} a_1 + 4d = 2 \\ a_1 + 22d = -23 \;\; |\cdot(-1) \end{cases}$$

$$\begin{cases} a_1 + 4d = 2 \\ -a_1 - 22d = 23 \end{cases}$$

$$-18d = 25 \quad |:(-18)$$

$$d = -\frac{25}{18}$$

Sijoitetaan saatu arvo jompaan kumpaan yhtälöön.

$$\begin{aligned} a_1 &= -4d + 2 \\ &= -4 \cdot (-\frac{25}{18}) + 2 \\ &= \frac{68}{9} \end{aligned}$$

Yleinen termi

$$\begin{aligned} a_n &= \frac{68}{9} + (n-1)\left(-\frac{25}{18}\right) \\ &= \frac{68}{9} - \frac{25}{18}n + \frac{25}{18} \\ &= -\frac{25}{18}n + \frac{161}{18} \end{aligned}$$

b) Geometrisessä jonossa -8, - 4, - 2 ensimmäinen jäsen $a_1 = -8$ ja $q = \frac{1}{2}$.

Geometrisen jonon yleinen termi $a_n = a_1 q^{n-1} = -8 \cdot \left(\frac{1}{2}\right)^{n-1}$.

Sadas jäsen on $a_{100} = -8 \cdot \left(\frac{1}{2}\right)^{100-1} = -8 \cdot \left(\frac{1}{2}\right)^{99} = -\frac{2^3}{2^{99}} = -\frac{1}{2^{96}} = -1{,}26... \cdot 10^{-29}$.

Vastaus: a) $a_n = -\frac{25}{18}n + \frac{161}{18}$ b) $a_{100} = -8 \cdot \left(\frac{1}{2}\right)^{99} = -1{,}26... \cdot 10^{-29}$

4. $f(x) = 2\sin x + \cos 2x$

Funktio on jaksollinen, jatkuva funktio välillä $[0, 2\pi]$, joten voidaan käyttää Fermat'n lausetta.
Derivoidaan *f*.

$$\begin{aligned} f'(x) &= 2\cos x - 2\sin 2x \\ &= 2\cos x - 2 \cdot 2\sin x \cos x \\ &= 2\cos x(1 - 2\sin x)) \end{aligned}$$

Derivaatan nollakohdat

$$2\cos x(1 - 2\sin x) = 0)$$

$$\begin{aligned} &2\cos x = 0 && \text{tai} && 1 - 2\sin x = 0 \\ &\cos x = 0 && \text{tai} && 2\sin x = 1 \ |:2 \\ &x = \frac{\pi}{2} + n\pi && \text{tai} && \sin x = \frac{1}{2} \\ & && && x = \frac{\pi}{6} + n2\pi,\ n \in Z \ \text{ tai} \\ & && && x = \frac{5}{6}\pi + n2\pi,\ n \in Z \end{aligned}$$

Derivaatan nollakohdista välille $]0, 2\pi[$ kuuluvat $x = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, $x = \frac{5}{6}\pi$ ja $x = \frac{3}{2}\pi$.

Tutkitaan funktion arvot derivaatan nollakohdissa ja suljetun välin päätepisteissä (Fermat).

$$f(0) = 2\sin 0 + \cos 2 \cdot 0 = 1$$

$$f(\frac{\pi}{6}) = 2\sin\frac{\pi}{6} + \cos\left(2 \cdot \frac{\pi}{6}\right) = \frac{3}{2}$$

$$f(\frac{\pi}{2}) = 2\sin\frac{\pi}{2} + \cos\left(2 \cdot \frac{\pi}{2}\right) = 1$$

$$f(\frac{5}{6}\pi) = 2\sin\frac{5}{6}\pi + \cos\left(2 \cdot \frac{5}{6}\pi\right) = \frac{3}{2}$$

$$f(\frac{3}{2}\pi) = 2\sin\frac{3}{2}\pi + \cos\left(2 \cdot \frac{3}{2}\pi\right) = -3$$

$f(2\pi) = 2\sin 2\pi + \cos(2 \cdot 2\pi) = 1$

Funktion *f* suurin arvo on siis $\frac{3}{2}$ ja pienin arvo -3.
Näin ollen jatkuvan funktion arvojoukko on $\left[-3, \frac{3}{2}\right]$.

$f(-x) = 2\sin(-x) + \cos(-2x) = -2\sin x + \cos 2x$

Koska $f(x) \neq f(-x)$, niin funktion ei ole parillinen.
Koska $-f(x) \neq f(-x)$, niin funktio ei ole pariton.
Funktio ei siis ole parillinen eikä pariton.

Vastaus: pienin arvo -3, suurin arvo $\frac{3}{2}$, arvojoukko $\left[-3, \frac{3}{2}\right]$, ei parillinen eikä pariton

5. Tutkitaan puuston määrää vuoden 1988 alusta alkaen vuosittain.

Aika	Puuston määrä (m^3)
1988 alku	100
1989 alku (+1)	$1{,}17 \cdot 100 - 16$
1990 alku (+2)	$1{,}17(1{,}17 \cdot 100 - 16) - 16 = 1{,}17^2 \cdot 100 - 1{,}17 \cdot 16 - 16$
1991 alku (+3)	$1{,}17 \cdot (1{,}17^2 \cdot 100 - 1{,}17 \cdot 16 - 16) - 16 = 1{,}17^3 \cdot 100 - 1{,}17^2 \cdot 16 - 1{,}17 \cdot 16 - 16$
+ *n* vuotta 1988 alusta alkaen	$1{,}17^n \cdot 100 - 1{,}17^{n-1} \cdot 16 - \ldots - 1{,}17 \cdot 16 - 16$

n vuoden jälkeen 1988 alusta alkaen puuta on yli 120 m^3. Saadaan siis epäyhtälö

$$1{,}17^n \cdot 100 - 1{,}17^{n-1} \cdot 16 - \ldots - 1{,}17 \cdot 16 - 16 > 120$$

$$1{,}17^n \cdot 100 - \left(1{,}17^{n-1} \cdot 16 + \ldots + 1{,}17 \cdot 16 + 16\right) > 120$$

$a_1 = 16, q = 1{,}17,$ summassa termejä n kpl

$$1{,}17^n \cdot 100 - \frac{16\left(1 - 1{,}17^n\right)}{\underbrace{1 - 1{,}17}_{-0{,}17}} > 120 \quad \Big| \cdot 0{,}17$$

$$1{,}17^n \cdot 17 + 16\left(1 - 1{,}17^n\right) > 20{,}4$$

$$1{,}17^n \cdot 17 - 1{,}17^n \cdot 16 + 16 > 20{,}4$$

$$1{,}17^n > 4{,}4 \quad \Big| \ln \quad \text{aidosti kasvava}$$

$$n \ln 1{,}17 > \ln 4{,}4 \quad \Big| : \ln 1{,}17$$

$$n > \frac{\ln 4{,}4}{\ln 1{,}17} = 9{,}44\ldots$$

Koska $n \in Z_+$, niin kun $n \geq 10$, puuta on yli 120 m^3. Tämä saavutetaan vuoden 1998 alussa (1988 + 10).

Vastaus: vuoden 1998 alussa

6. Tutkitaan tuottoa, pääomaa ja jaettuja apurajoja (€) vuosittain alkaen 1993.

Aika	**Tuotto (milj.)**	**Pääoma (milj.)**	**Apurahat (milj.)**
1993 alku		10	
1994 alku (+1)	$0{,}10 \cdot 10$	$1{,}05 \cdot 10$	$0{,}05 \cdot 10$
1995 alku (+2)	$0{,}10 \cdot 1{,}05 \cdot 10$	$1{,}05^2 \cdot 10$	$0{,}05 \cdot 1{,}05 \cdot 10$
1996 alku (+3)	$0{,}10 \cdot 1{,}05^2 \cdot 10$	$1{,}05^3 \cdot 10$	$0{,}05 \cdot 1{,}05^2 \cdot 10$
n vuotta eteenpäin 1993 alusta (+n)	$0{,}10 \cdot 1{,}05^{n-1} \cdot 10$	$1{,}05^n \cdot 10$	$0{,}05 \cdot 1{,}05^{n-1} \cdot 10$

Kun pääoma on kaksinkertaistunut, se on 20 milj. eli

$$1{,}05^n \cdot 10 = 20$$
$$1{,}05^n = 2$$
$$n \ln 1{,}05 = \ln 2 \,|: \ln 1{,}05$$
$$n = \frac{\ln 2}{\ln 1{,}05} = 14{,}206... \approx 15$$

Pääoma on kaksinkertaistunut vuoden 2007 lopussa (2008 alussa).

Tähän mennessä (n =15) on jaettu apurahoja (milj.) yhteensä
$0{,}05 \cdot 10 + 0{,}05 \cdot 1{,}05 \cdot 10 + \ldots + 0{,}05 \cdot 1{,}05^{14} \cdot 10$.

Vuosittain jaetut apurahat muodostavat geometrisen summan, jossa
$a_1 = 0{,}05 \cdot 10$ ja $q = 1{,}05$.

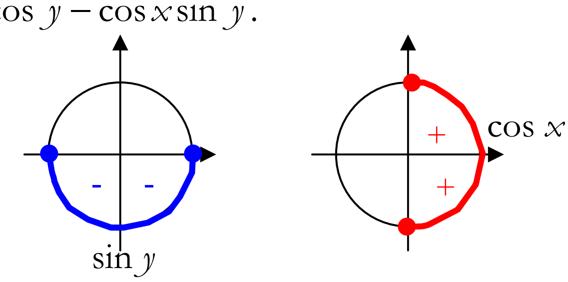
$$S = 0{,}05 \cdot 10 \cdot \frac{1 - 1{,}05^{15}}{1 - 1{,}05} = 10789281{,}79 \approx 10{,}8 \text{ (milj.)}$$

Vastaus: 2007 lopussa, 10,8 milj.

7. Taulukkokirjan mukaan $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

Tiedetään, että $\sin x = \frac{1}{4}$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

ja $\cos y = -\frac{1}{3}$, $\pi \le y \le 2\pi$.



Sijoitetaan lähtötiedot muunnoskaavaan

$$\sin x \cos y - \cos x \sin y = \frac{1}{4} \cdot \left(-\frac{1}{3}\right) - \cos x \cdot \sin y$$

Lähtötietojen avulla pitää määrittää siis arvot $\cos y$ ja $\cos x$.
Yksikköympyrän avulla voidaan päätellä, että $\sin y < 0$ ja $\cos x > 0$.

Lasketaan ensin arvo lausekkeelle sin y.
Peruskaavan avulla saadaan

$$\sin^2 y + \cos^2 y = 1$$
$$\sin^2 y = 1 - \cos^2 y$$
$$\sin y = \pm\sqrt{1 - \left(-\frac{1}{3}\right)^2}$$
$$\sin y = -\sqrt{\frac{8}{9}}$$
$$\sin y = -\frac{\sqrt{8}}{3}$$

sin y <0, kun $\pi \le y \le 2\pi$

Vastaavasti saadaan arvo lausekkeelle cos x.

$$\sin^2 x + \cos^2 x = 1$$
$$\cos^2 x = 1 - \sin^2 x$$
$$\cos x = \pm\sqrt{1 - \left(\frac{1}{4}\right)^2}$$
$$\cos x = \sqrt{\frac{15}{16}}$$
$$\cos x = \frac{\sqrt{15}}{4}$$

cos x >0, kun $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Sijoitetaan saadut arvot lausekkeeseen

$$\sin x \cos y - \cos x \sin y = \frac{1}{4}\cdot\left(-\frac{1}{3}\right) - \frac{\sqrt{15}}{4}\cdot\left(-\frac{\sqrt{8}}{3}\right)$$
$$= -\frac{1}{12} + \frac{\sqrt{120}}{12}$$
$$= -\frac{1}{12} + \frac{2\sqrt{30}}{12} = -\frac{1}{12} + \frac{\sqrt{30}}{6} \approx 0{,}83$$

Vastaus: $-\frac{1}{12} + \frac{\sqrt{30}}{6} \approx 0{,}83$

TESTI 2

1. a) $x \in \left[\frac{\pi}{2}, \pi\right]$, joten $\cos x < 0$

$$\sin^2 x + \cos^2 x = 1$$

$$\left(\frac{1}{3}\right)^2 + \cos^2 x = 1 \quad \left| -\left(\frac{1}{3}\right)^2\right.$$

$$\cos^2 x = \frac{8}{9} \quad |\sqrt{\ }$$

$$|\cos x| = \sqrt{\frac{8}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

Koska $\cos x < 0$, niin $\cos x = -\frac{2\sqrt{2}}{3}$

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{3} : \left(-\frac{2\sqrt{2}}{3}\right) = -\frac{1}{3} \cdot \frac{3}{2\sqrt{2}} = -\frac{1}{2\sqrt{2}}$$

b)

$$\frac{\tan x \cdot \sin y}{\cos 2z} = \frac{\tan\left(\frac{\pi}{3}\right) \cdot \sin\left(-\frac{\pi}{4}\right)}{\cos\left(2 \cdot \left(-\frac{5\pi}{6}\right)\right)} \qquad \sin(-x) = -\sin x$$

$$= \frac{\tan\left(\frac{\pi}{3}\right) \cdot \left(-\sin\left(\frac{\pi}{4}\right)\right)}{\cos\left(-\frac{5\pi}{3}\right)} \qquad \cos(-x) = \cos x$$

$$= \frac{\tan\left(\frac{\pi}{3}\right) \cdot \left(-\sin\left(\frac{\pi}{4}\right)\right)}{\cos\left(\frac{5\pi}{3}\right)} = -\frac{\sqrt{3} \cdot \frac{1}{\sqrt{2}}}{\frac{1}{2}} \overset{\sqrt{2})}{=} -\frac{\sqrt{3}}{\sqrt{2}} \cdot 2 = -\sqrt{6}$$

Vastaus: a) $\cos x = -\frac{2\sqrt{2}}{3}$ ja $\tan x = -\frac{1}{2\sqrt{2}}$ b) $-\sqrt{6}$

2. a) Jono $2,\ \frac{3}{2},\ \ldots$ on aritmeettinen, joten $d = a_2 - a_1 = \frac{3}{2} - 2 = -\frac{1}{2}$

$$a_n = a_1 + (n-1)d$$

$$a_{20} = 2 + 19 \cdot \left(-\frac{1}{2}\right) = -\frac{15}{2}$$

Aritmeettinen summa: $\displaystyle\sum_{n=1}^{20} a_n = \frac{a_1 + a_{20}}{2} \cdot 20 = \frac{2 + \left(-\frac{15}{2}\right)}{2} \cdot 20 = -55$

b) Jono $2,\ \frac{3}{2},\ \ldots$ on geometrinen, joten $q = \dfrac{a_2}{a_1} = \dfrac{\frac{3}{2}}{2} = \dfrac{3}{4}$

$$\sum_{n=1}^{20} a_n = \frac{a_1(1-q^{20})}{1-q} = \frac{2\left(1 - \left(\frac{3}{4}\right)^{20}\right)}{1 - \frac{3}{4}} = 7{,}9746\ldots \approx 7{,}97$$

Vastaus: a) -55 b) $7{,}97$

3. a)

$$\tan x = \sqrt{3}$$

$$x = \frac{\pi}{3} + n\pi,\ n \in \mathbf{Z}$$

b)

$$\sin x = \sin(x - \frac{\pi}{3})$$

$$x = x - \frac{\pi}{3} + n2\pi \quad \text{tai} \quad x = \pi - (x - \frac{\pi}{3}) + n2\pi$$

$$0 = -\frac{\pi}{3} + n2\pi \qquad \text{tai} \quad x = \pi - x + \frac{\pi}{3} + n2\pi$$

Epätosi

$$2x = \frac{4\pi}{3} + n2\pi \quad |:2$$

$$x = \frac{2\pi}{3} + n\pi,\ n \in \mathbf{Z}$$

c)

$$\sin x + \sqrt{3}\cos x = 0 \quad |:\cos x \neq 0$$

$$\frac{\sin x}{\cos x} + \sqrt{3} = 0$$

$$\tan x = -\sqrt{3}$$

$$x = \frac{2\pi}{3} + n\pi,\ n \in \mathbf{Z}$$

Jos $\cos x = 0$, niin

$$\sin x + \sqrt{3} \cdot 0 = 0$$

$$\sin x = 0$$

Epätosi, koska sini ja kosini eivät ole samanaikaisesti nollia.

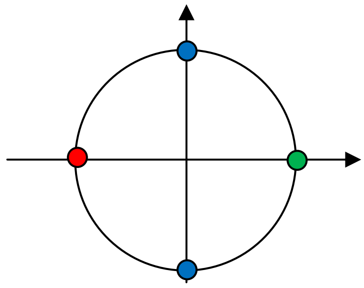
Vastaus: a) $x = \frac{\pi}{3} + n\pi,\ n \in \mathbf{Z}$

b) $x = \frac{2\pi}{3} + n\pi,\ n \in \mathbf{Z}$

c) $x = \frac{2\pi}{3} + n\pi,\ n \in \mathbf{Z}$

4. a)

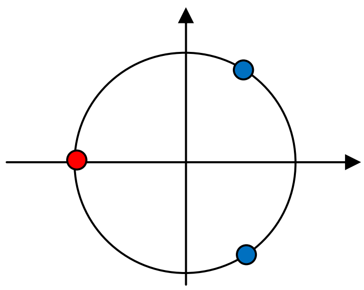
$$\cos^4 x + \sin^2 x - 1 = 0$$
$$\cos^4 x + 1 - \cos^2 x - 1 = 0$$
$$\cos^4 x - \cos^2 = 0$$
$$\cos^2 x(\cos^2 x - 1) = 0$$
$$\cos^2 x = 0 \quad \text{tai} \quad \cos^2 x - 1 = 0$$
$$\cos x = 0 \quad \text{tai} \quad \cos^2 x = 1 \quad \Big| \sqrt{\ }$$
$$x = \frac{\pi}{2} + n\pi \quad \text{tai} \quad \cos x = -1 \quad \text{tai} \quad \cos x = 1$$
$$x = \pi + n2\pi \quad \text{tai} \quad x = n2\pi$$



Yhdistetään vastaukset: $x = n\dfrac{\pi}{2}$, $n \in \mathbf{Z}$

b)

$$\cos x = 2\sin^2 x - 1$$
$$\cos x = 2(1 - \cos^2 x) - 1$$
$$\cos x = 2 - 2\cos^2 x - 1$$
$$2\cos^2 x + \cos x - 1 = 0 \qquad \Big| \cos x = t$$
$$2t^2 + t - 1 = 0$$
$$t = \frac{-1 \pm \sqrt{1^2 - 4\cdot 2\cdot(-1)}}{2\cdot 2} = \frac{-1 \pm 3}{4}$$
$$t = -1 \quad \text{tai} \quad t = \frac{1}{2}$$
$$\cos x = -1 \quad \text{tai} \quad \cos x = \frac{1}{2}$$
$$x = \pi + n2\pi \quad \text{tai} \quad x = \pm\frac{\pi}{3} + n2\pi$$



Yhdistetään vastaukset: $x = \dfrac{\pi}{3} + n\dfrac{2\pi}{3}$

Vastaus: a) $x = n\dfrac{\pi}{2}$, $n \in \mathbf{Z}$ b) $x = \dfrac{\pi}{3} + n\dfrac{2\pi}{3}$, $n \in \mathbf{Z}$

5. $f(x) = \sin 2x + \cos^2 x$

$$f'(x) = \cos 2x \cdot D(2x) + 2\cos x \cdot D(\cos x)$$
$$= 2\cos 2x + 2\cos x(-\sin x)$$
$$= 2\cos 2x - 2\cos x \sin x \qquad \sin 2x = 2\sin x \cos x$$
$$= 2\cos 2x - \sin 2x$$

a)

$$f'\left(\frac{\pi}{3}\right) = 2\cos\left(2 \cdot \frac{\pi}{3}\right) - \sin\left(2 \cdot \frac{\pi}{3}\right)$$
$$= 2\cos\frac{2\pi}{3} - \sin\frac{2\pi}{3}$$
$$= 2 \cdot \left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2} = \frac{-2-\sqrt{3}}{2}$$

b)

$$f'(x) = 0$$
$$2\cos 2x - \sin 2x = 0 \quad |: \cos 2x \neq 0$$
$$2 - \frac{\sin 2x}{\cos 2x} = 0$$
$$\tan 2x = 2$$
$$2x = 1{,}107... + n\pi \quad |: 2$$
$$x = 0{,}5535... + n\frac{\pi}{2} \approx 0{,}55 + n\frac{\pi}{2}$$

Vastaus: a) $f'\left(\frac{\pi}{3}\right) = \frac{-2-\sqrt{3}}{2}$ b) $x = 0{,}55 + n\frac{\pi}{2}$

6. Jonon yleinen jäsen $a_n = n^2 - 113n + 5240$.

a) Luku 2078 on jonon jäsen, jos $a_n = 2078$ jollakin $n \in \mathbf{Z}_+$.

$a_n = 2078$

$n^2 - 113n + 5240 = 2078$

$n^2 - 113n + 3162 = 0$

$$n = \frac{-(-113) \pm \sqrt{(-113)^2 - 4 \cdot 1 \cdot 3162}}{2 \cdot 1} = \frac{113 \pm 11}{2}$$

$n = 51$ tai $n = 62$

Siis 2078 on jonon 51. ja 62. jäsen.

b) Merkitään $f(x) = x^2 - 113x + 5240$ ja etsitään tämän reaaliarvoisen funktion pienin arvo.

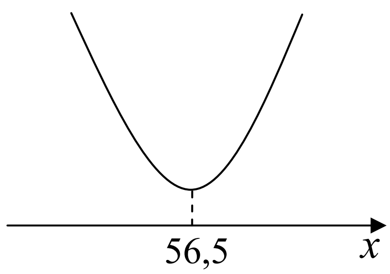
$f'(x) = 2x - 113$

$f'(x) = 0$

$2x - 113 = 0$

$2x = 113$

$x = 56{,}5$



Funktion *f* pienin arvo kohdassa $x = 56{,}5$, joten lukujonon pienin jäsen on a_{56} tai a_{57}

$a_{56} = 56^2 - 113 \cdot 56 + 5240 = 2048$

$a_{56} = 57^2 - 113 \cdot 57 + 5240 = 2048$

Lukujonon pienin jäsen on siis 2048

Vastaus: a) On lukujonon 51. ja 62. jäsen

b) Pienin jäsen 2048

7. a) $f(x) = \cos x - \frac{1}{2}\cos 2x$, kun $x \in [-\pi, \pi]$.

$f'(x) = -\sin x - \frac{1}{2} \cdot (-\sin 2x) \cdot 2 = \sin 2x - \sin x$

Derivaatan nollakohdat:

$\sin 2x - \sin x = 0$

$\sin 2x = \sin x$

$2x = x + n2\pi$ tai $2x = \pi - x + n2\pi$

$x = n2\pi$ tai $3x = \pi + n2\pi$

$x = \frac{\pi}{3} + n\frac{2\pi}{3}$

Välillä $]-\pi, \pi[$ on $x = \frac{\pi}{3}$

$f(-\pi) = \cos(-\pi) - \frac{1}{2}\cos(2(-\pi)) = \cos\pi - \frac{1}{2}\cos 2\pi = -1 - \frac{1}{2} \cdot 1 = -\frac{3}{2}$

$f(\pi) = \cos\pi - \frac{1}{2}\cos 2\pi = -1 - \frac{1}{2} \cdot 1 = -\frac{3}{2}$

$f(\frac{\pi}{3}) = \cos\frac{\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3} = \frac{1}{2} - \frac{1}{2} \cdot \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Fermat'n lauseen mukaan suurin arvo $\frac{3}{4}$ ja pienin $-\frac{3}{2}$

b) $f(x) = \cos^2 x + \sin x + 2 = 1 - \sin^2 x + \sin x + 2 = -\sin^2 x + \sin x + 3$

Merkitään $\sin x = t$.
Koska $-1 \leq \sin x \leq 1$, f saa samat arvot, kuin funktio
$g(t) = -t^2 + t + 3$, kun $-1 \leq t \leq 1$

$$g'(t) = -2t + 1$$
$$g'(t) = 0$$
$$-2t + 1 = 0$$
$$2t = 1$$
$$t = \frac{1}{2}$$

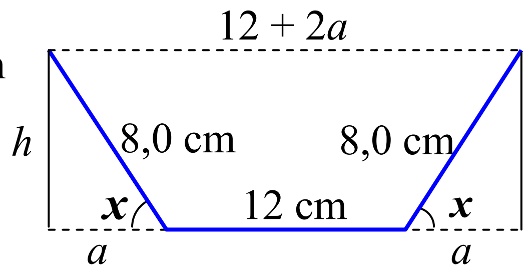
$$g(-1) = -(-1)^2 + (-1) + 3 = 1$$
$$g(1) = -1^2 + 1 + 3 = 3$$
$$f\left(\frac{1}{2}\right) = -\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 3 = 3\frac{1}{4}$$

Fermat'n lauseen mukaan *g*:n (ja siten myös *f*:n) suurin arvo on $3\frac{1}{4}$ ja pienin arvo on 1.

Vastaus: a) Suurin arvo $\frac{3}{4}$ ja pienin $-\frac{3}{2}$

b) Suurin arvo $3\frac{1}{4}$ ja pienin 1

8. Tilavuus on suurin, kun poikkileikkauksen ala on suurimmillaan.



$$\sin x = \frac{h}{8} \quad \text{ja} \quad \cos x = \frac{a}{8}$$

$$h = 8\sin x \qquad a = 8\cos x$$

$$A = \frac{12+12+2a}{2}\cdot h = \frac{24+2a}{2}\cdot h$$

$$A = (12+a)h = 12h + ah$$

$$A(x) = 12\cdot 8\sin x + 8\cos x \cdot 8\sin x$$

$$= 32(3\sin x + 2\sin x\cos x)\ ,\ 0 \le x \le \frac{\pi}{2}$$

$$A'(x) = 32(3(\cos x) + 2(\cos x\cdot\cos x + \sin x(-\sin x)))$$

$$= 32(3\cos x + 2\cos^2 - 2\sin^2 x)$$

$$= 32(3\cos x + 2\cos^2 x - 2(1-\cos^2 x))$$

$$= 32(4\cos^2 x + 3\cos x - 2)$$

Derivaatan nollakohdat:

$$4\cos^2 x + 3\cos x - 2 = 0 \qquad \cos x = t$$

$$4t^2 + 3t - 2 = 0$$

$$t = \frac{-3 \pm \sqrt{3^2 - 4\cdot 4\cdot(-2)}}{2\cdot 4} = \frac{-3\pm\sqrt{41}}{8}$$

$$t = \frac{-3+\sqrt{41}}{8} = 0{,}425\ldots \text{ tai } t = \frac{-3-\sqrt{41}}{8} = -1{,}175\ldots$$

$$\cos x = 0{,}425\ldots \qquad \text{tai} \qquad \cos x = -1{,}175\ldots < -1$$

$$x = \pm 1{,}131\ldots + n2\pi \qquad \text{Ei ratkaisua}$$

Välillä $\left]0, \frac{\pi}{2}\right[$ on $x = 1{,}131\ldots$

$$A(0) = 0$$

$$A\left(\frac{\pi}{2}\right) = 32(3\sin\frac{\pi}{2} + 2\sin\frac{\pi}{2}\cos\frac{\pi}{2}) = 32(3 \cdot 1 + 2 \cdot 0) = 96$$

$$A(1{,}131\ldots) = 111{,}51\ldots$$

Fermat’n lauseen mukaan suurin arvo saadaan, kun

$$x = 1{,}131\ldots\text{rad} = 1{,}131\ldots \cdot \frac{180^\circ}{\pi} \approx 65^\circ$$

Suurin tilavuus:

$$V = A_{\text{pohja}} \cdot 400\text{cm} = 111{,}51\ldots \cdot 400 = 44\ 607{,}93\ldots \approx 45\ 000\,\text{cm}^3 = 45\ l$$

Vastaus: $x = 65^\circ$ ja tilavuus 45 litraa