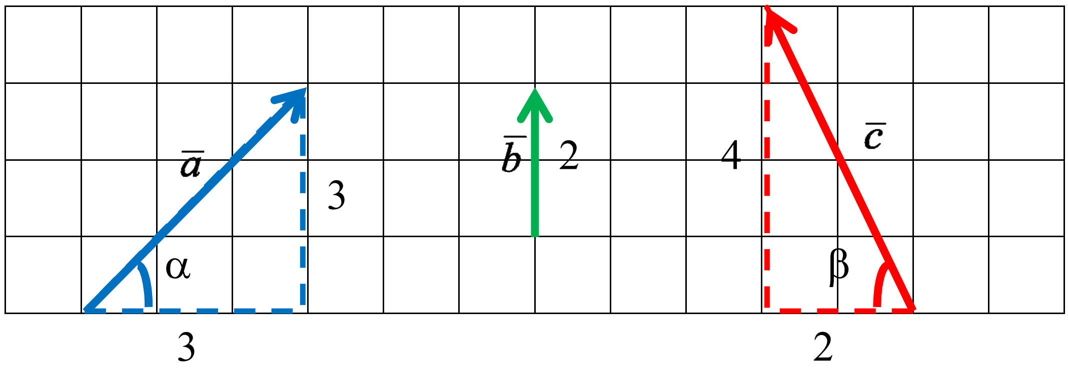
# 4 Kertausosa

**1.**




$$|\bar{a}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

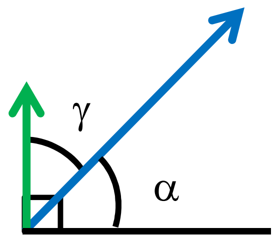
$$|\bar{b}| = 2$$

$$|\bar{c}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$\alpha = 45°$$

$$\tan \beta = \frac{4}{2} = 2$$

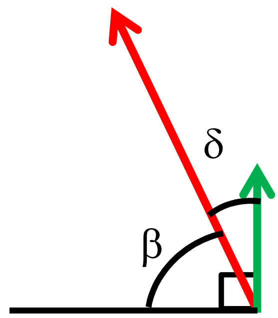
$$\beta = 63{,}434...°$$




Vektorien $\bar{a}$ ja $\bar{b}$ välinen kulma:

$\gamma = 90° - 45°$

$= 45°$

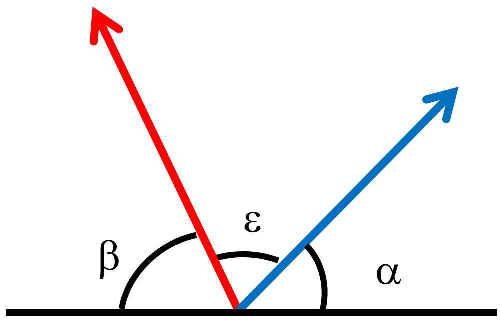



Vektorien $\bar{b}$ ja $\bar{c}$ välinen kulma:

$\delta = 90° - 63{,}434...°$

$= 26{,}56...°$

$\approx 27°$



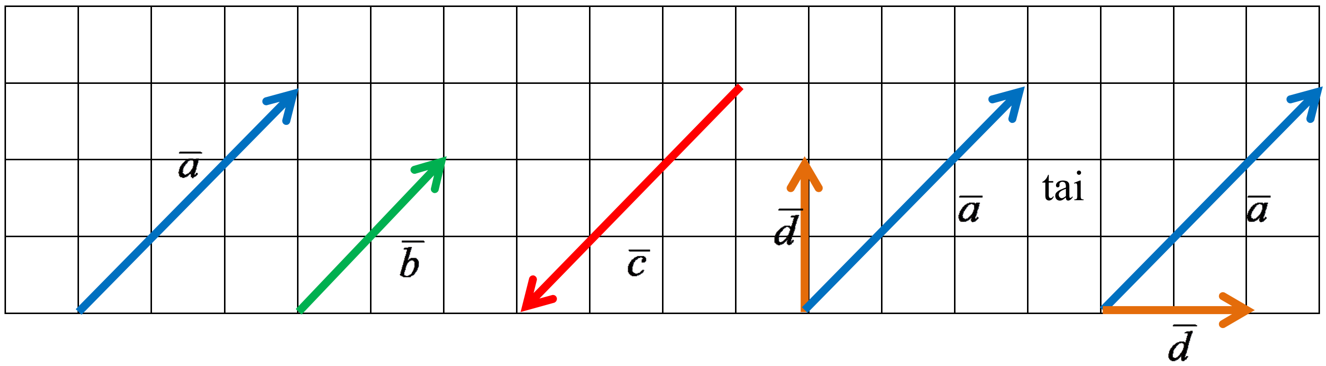

Vektorien $\bar{a}$ ja $\bar{c}$ välinen kulma:

$\varepsilon = 180° - 45° - 63{,}43...°$

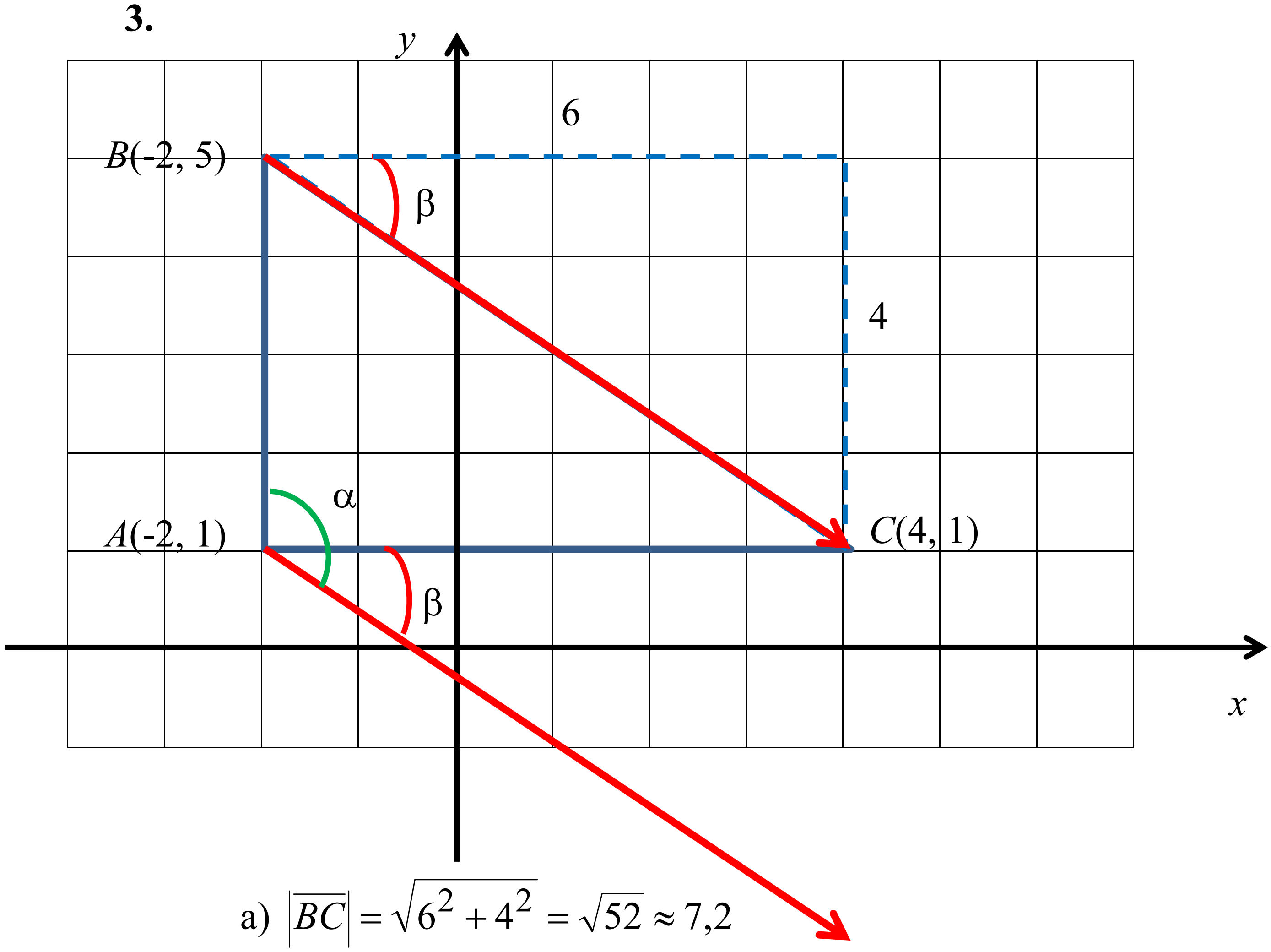
$= 71{,}56...°$

$\approx 72°$

**2.**




**3.**

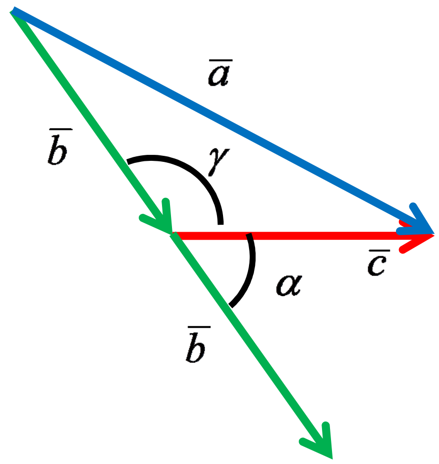



a) $\left|\overline{BC}\right| = \sqrt{6^2 + 4^2} = \sqrt{52} \approx 7{,}2$

b) $\tan \beta = \dfrac{4}{6} \quad \Rightarrow \quad \beta = 33{,}690...°$

$\alpha = \beta + 90° = 33{,}690...° + 90° \approx 124°$

**4.**




$|\overline{a}| = 4{,}8$

$|\overline{b}| = 2{,}9$

$|\overline{c}| = 2{,}4$

Kosinilause:

$$|\overline{a}|^2 = |\overline{b}|^2 + |\overline{c}|^2 - 2|\overline{b}||\overline{c}|\cos\gamma$$

$$2|\overline{b}||\overline{c}|\cos\gamma = |\overline{b}|^2 + |\overline{c}|^2 - |\overline{a}|^2$$

$$\cos\gamma = \frac{|\overline{b}|^2 + |\overline{c}|^2 - |\overline{a}|^2}{2|\overline{b}||\overline{c}|}$$

$$\cos\gamma = \frac{2{,}9^2 + 2{,}4^2 - 4{,}8^2}{2 \cdot 2{,}9 \cdot 2{,}4}$$

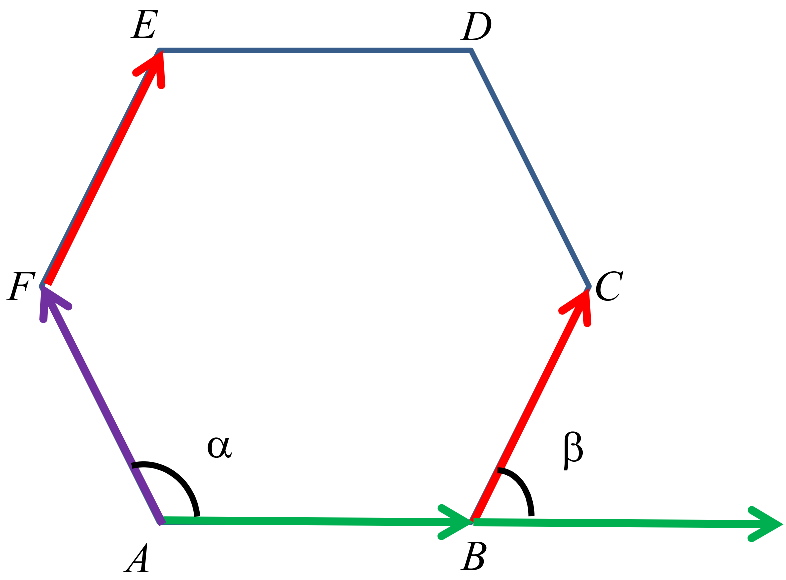
$$\cos\gamma = -0{,}637...$$

$$\gamma = 129{,}58...^\circ$$

Vektorien $\overline{b}$ ja $\overline{c}$ välinen kulma

$$\alpha = 180^\circ - \gamma = 180^\circ - 129{,}58\ldots^\circ = 50{,}41\ldots^\circ \approx 50^\circ$$

**5.**



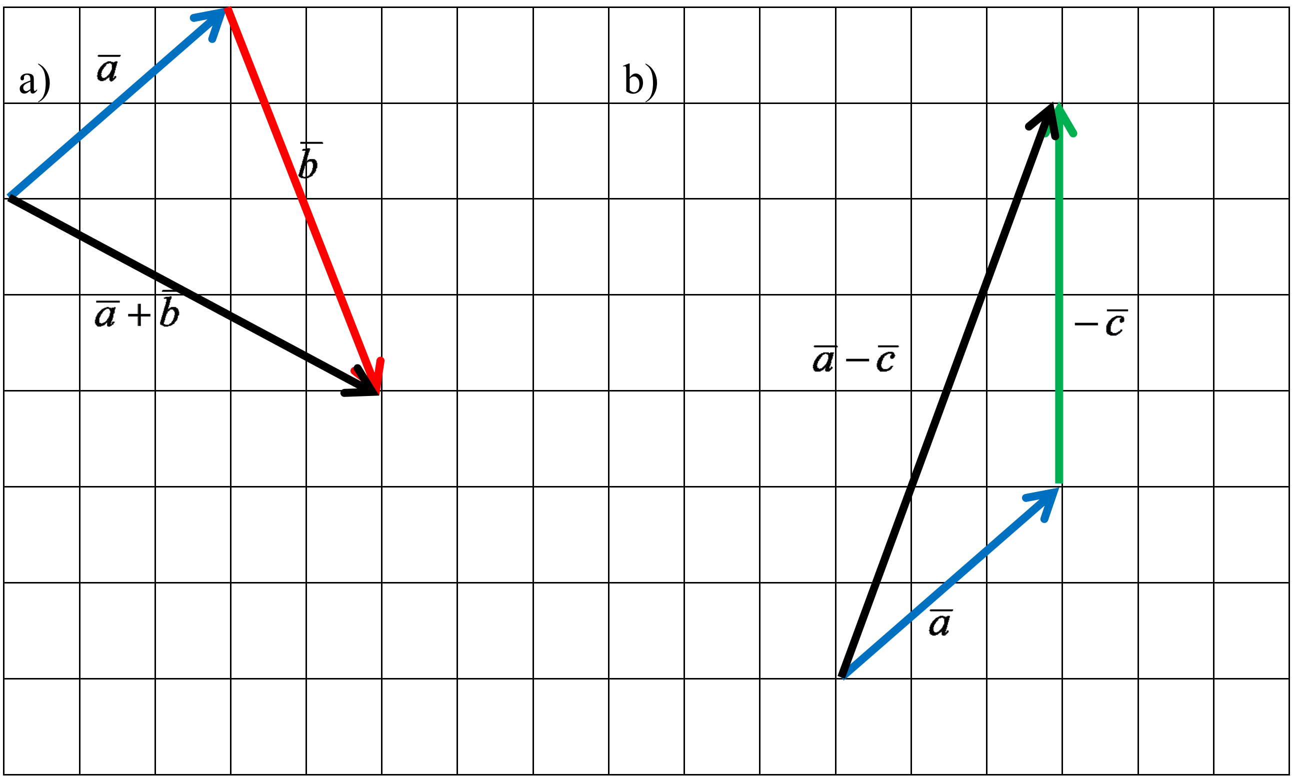

a) $\overline{FE} = \overline{BC}$

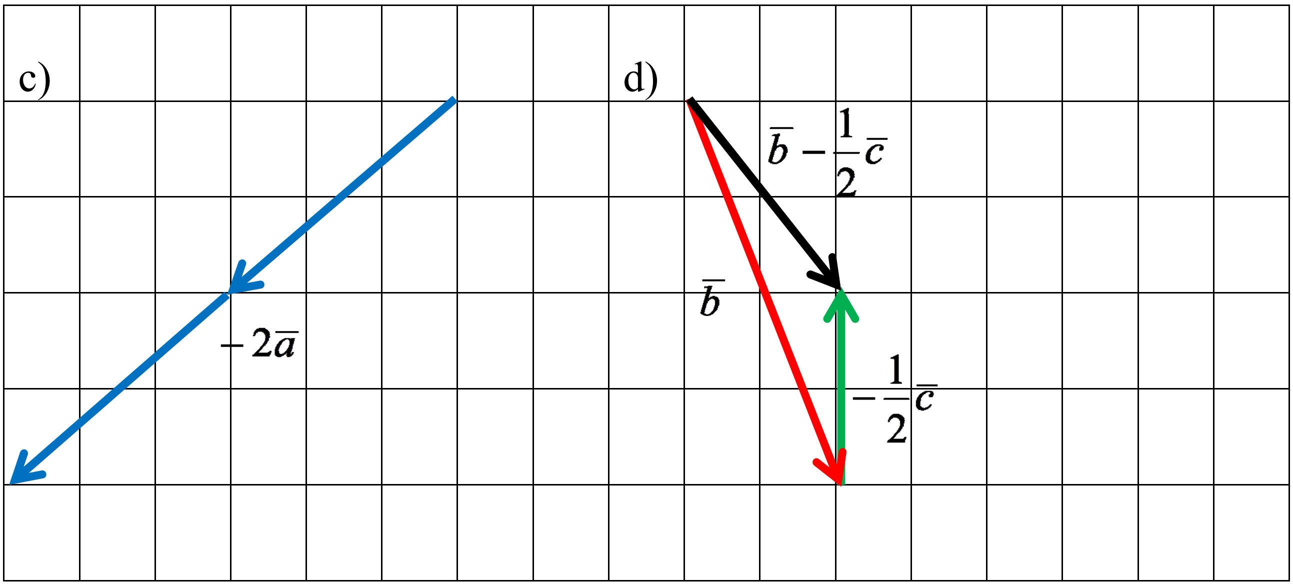
b) Kuusikulmion kulmien summa $(6-2)\cdot 180° = 720°$

Vektorien välinen kulma $\alpha = \dfrac{720°}{6} = 120°$

c) Vektorien välinen kulma $\beta = 180° - 120° = 60°$

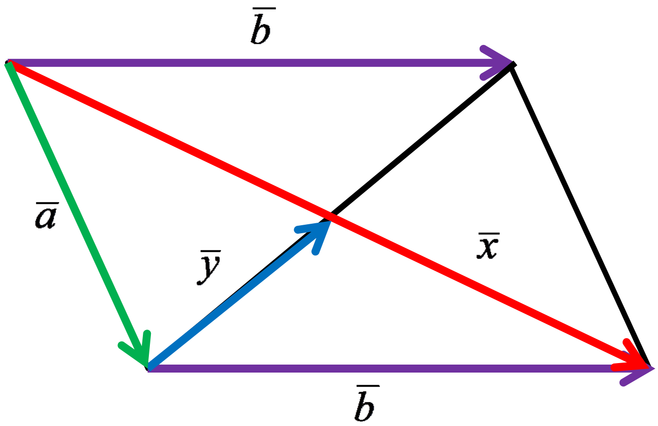
**6.**






**7.** $\overline{x}=\overline{a}+\overline{b}$

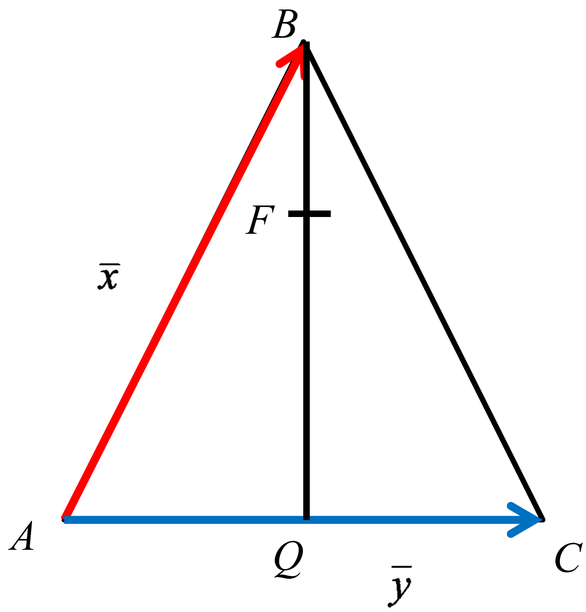
$$\overline{y}=-\overline{a}+\frac{1}{2}\overline{x}$$
$$=-\overline{a}+\frac{1}{2}(\overline{a}+\overline{b})$$
$$=-\frac{1}{2}\overline{a}+\frac{1}{2}\overline{b}$$



**8.** Kolmio on tasakylkinen, joten

$$|\overline{AQ}| = |\overline{QC}| = \frac{1}{2}|AC|$$

$$|\overline{QF}| = \frac{2}{3}|\overline{BQ}|$$



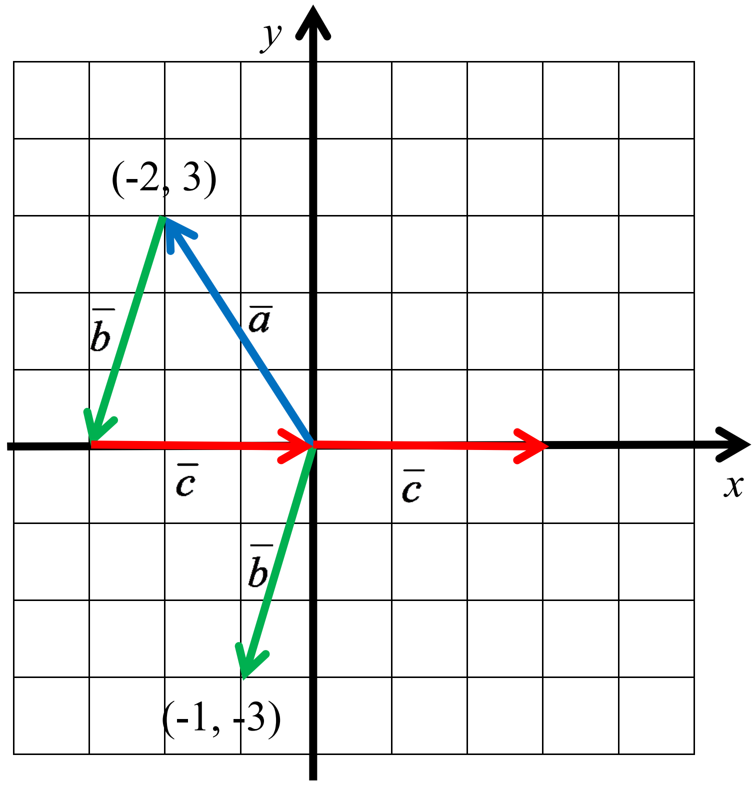

a) $\overline{BQ} = \overline{BA} + \overline{AQ} = -\overline{x} + \frac{1}{2}\overline{y}$

$$\begin{aligned}\overline{FQ} &= \frac{2}{3}\overline{BQ} \\ &= \frac{2}{3}(-\overline{x} + \frac{1}{2}\overline{y}) \\ &= -\frac{2}{3}\overline{x} + \frac{1}{3}\overline{y}\end{aligned}$$

b)

$$\begin{aligned}\overline{CF} &= \overline{CQ} + \overline{QF} \\ &= \overline{CQ} - \overline{FQ} \\ &= -\frac{1}{2}\overline{y} - (-\frac{2}{3}\overline{x} + \frac{1}{3}\overline{y}) \\ &= -\frac{1}{2}\overline{y} + \frac{2}{3}\overline{x} - \frac{1}{3}\overline{y} \\ &= \frac{2}{3}\overline{x} - \frac{5}{6}\overline{y}\end{aligned}$$

**9.** Piirretään vektorit $\overline{a}$ ja $\overline{b}$ peräkkäin ja päätellään kolmas vektori. Siirretään tämän jälkeen päätelty vektori alkamaan origosta.




Kolmannen vektorin loppupiste on (3, 0).

**10.** $\overline{x} = -2\overline{a} + 5\overline{b} \qquad |\overline{x}| = 5$

a) $\overline{x}^0 = \dfrac{1}{|\overline{x}|}\overline{x} = \dfrac{1}{5}(-2\overline{a} + 5\overline{b}) = -\dfrac{2}{5}\overline{a} + \overline{b}$

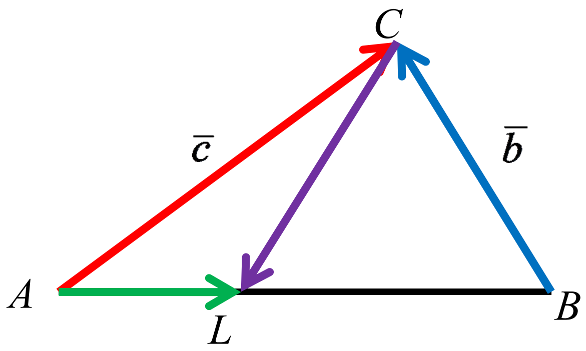
b) $\overline{a} \uparrow\uparrow \overline{x} \qquad |\overline{a}| = 2$

$$\overline{a} = 2\overline{x}^0 = s(-\frac{2}{5}\overline{a} + \overline{b}) = -\frac{4}{5}\overline{a} + 2\overline{b}$$

c) $\overline{b} \uparrow\downarrow \overline{x} \qquad |\overline{b}| = 4$

$$\overline{b} = -4\overline{x}^0 = -4(-\frac{2}{3}\overline{a} + \overline{b}) = \frac{8}{5}\overline{a} - 4\overline{b}$$

**11.** $\overline{AB} = \overline{c} - \overline{b}$




$$\overline{AL} = \frac{3}{8}\overline{AB}$$

$$= \frac{3}{8}(\overline{c} - \overline{b}) = \frac{3}{8}\overline{c} - \frac{3}{8}\overline{b}$$

$$\overline{CL} = -\overline{c} + \overline{AL} = -\overline{c} + \frac{3}{8}\overline{c} - \frac{3}{8}\overline{b} = -\frac{5}{8}\overline{c} - \frac{3}{8}\overline{b}$$

**12.** **Oletus:** $\overline{x} = 5\overline{a} - 9\overline{b}$, $\quad \overline{y} = -7\overline{a} - 3\overline{b}$, $\quad \overline{v} = \frac{1}{3}\overline{a} + 2\overline{b}$

**Väite:** $\overline{x} + \overline{y} \parallel \overline{v}$ eli $\overline{x} + \overline{y} = t\overline{v}$, $\; t \in R$

**Todistus:**

$$\overline{x} + \overline{y} = 5\overline{a} - 9\overline{b} - 7\overline{a} - 3\overline{b} = -2\overline{a} - 12\overline{b}$$

$$\overline{x} + \overline{y} = t\overline{v}$$

$$-2\overline{a} - 12\overline{b} = t(\frac{1}{3}\overline{a} + 2\overline{b})$$

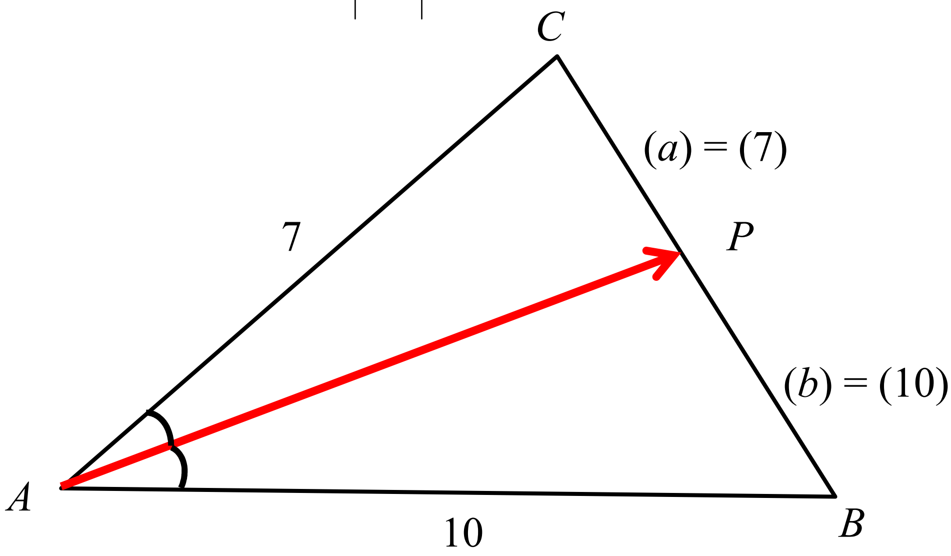
$$-2\overline{a} - 12\overline{b} = \frac{1}{3}t\overline{a} + 2t\overline{b}$$

$$\begin{cases} \frac{1}{3}t = -2 & |\cdot 3 \\ 2t = -12 & |:2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} t = -6 \\ t = -6 \end{cases}$$

$t$:n arvo yksikäsitteinen, joten $\overline{x} + \overline{y} \parallel \overline{v}$ □

**13.** $\overline{AB} = 2\overline{a} + 3\overline{b}$ $\quad |\overline{AB}| = 10$

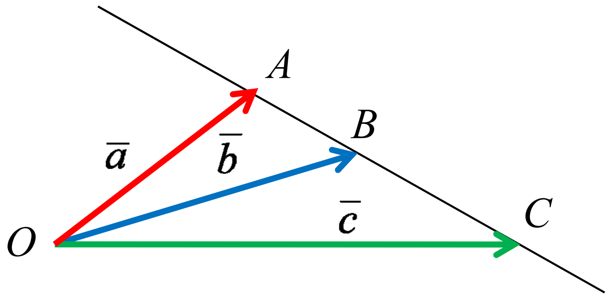
$\overline{AC} = 5\overline{a} - \overline{b}$ $\quad |\overline{AC}| = 7$




$$\frac{a}{b} = \frac{7}{10} \qquad \overline{CP} = \frac{7}{17}\overline{CB}$$

$$\overline{CB} = \overline{CA} + \overline{AB} = -\overline{AC} + \overline{AB} = -5\overline{a} + \overline{b} + 2\overline{a} + 3\overline{b} = -3\overline{a} + 4\overline{b}$$

$$\begin{aligned}\overline{AP} &= \overline{AC} + \frac{7}{17}\overline{CB} \\ &= 5\overline{a} - \overline{b} + \frac{7}{17}(-3\overline{a} + 4\overline{b}) \\ &= 5\overline{a} - \overline{b} - \frac{21}{17}\overline{a} + \frac{28}{17}\overline{b} \\ &= \frac{64}{17}\overline{a} + \frac{11}{17}\overline{b} \\ &= 3\frac{13}{17}\overline{a} + \frac{11}{17}\overline{b}\end{aligned}$$




**14.** $2\bar{a} - 3\bar{b} + \bar{c} = \bar{0}$

$\bar{c} = -2\bar{a} + 3\bar{b}$

$\overline{AB} = -\bar{a} + \bar{b}$

$\overline{AC} = -\bar{a} + \bar{c} = -\bar{a} - 2\bar{a} + 3\bar{b} = -3\bar{a} + \bar{b}$

$3\overline{AB} = 3(-\bar{a} + \bar{b}) = -3\bar{a} + 3\bar{b}$

joten $\overline{AB} \parallel \overline{AC}$

Pisteet *A*, *B* ja *C* ovat samalla suoralla. $\square$

**15.** $\bar{a} - 3\bar{b} = r(2\bar{a} + 3\bar{b}) + s(\bar{a} - 2\bar{b})$

$\bar{a} - 3\bar{b} = 2r\bar{a} + r\bar{b} + s\bar{a} - 2s\bar{b}$

$\bar{a} - 3\bar{b} = (2r + s)\bar{a} + (3r - 2s)\bar{b}$

Komponentteihin jako yksikäsitteinen, joten

$$\begin{cases} 2r + s = 1 & |\cdot 2 \\ 3r - 2s = -3 \end{cases}$$

$$\begin{cases} 4r + 2s = 2 \\ 3r - 2s = -3 \end{cases}$$

$$7r = -1$$

$$r = -\frac{1}{7}$$

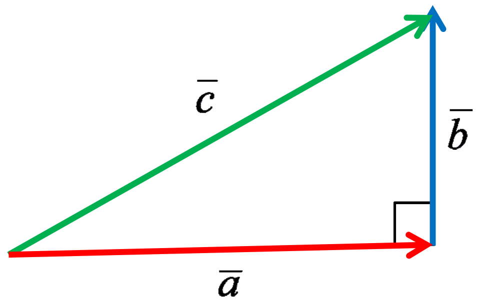
Sijoitetaan ensimmäiseen yhtälöön:

$$2 \cdot \left(-\frac{1}{7}\right) + s = 1 \quad \Leftrightarrow s = 1\frac{2}{7}$$

Vastaus: $\bar{a} - 3\bar{b} = -\frac{1}{7}(2\bar{a} + 3\bar{b}) + 1\frac{2}{7}(\bar{a} - 2\bar{b})$

**16.** $|\bar{c}| = 10\,\text{N}$

$|\bar{b}| = 6\,\text{N}$




$$|\bar{a}|^2 + |\bar{b}|^2 = |\bar{c}|^2$$

$$\begin{aligned} |\bar{a}|^2 &= |\bar{c}|^2 - |\bar{b}|^2 \\ &= 10^2 - 6^2 \\ &= 64 \end{aligned}$$

$$|\bar{a}| = \sqrt{64} = 8$$

Vastaus: 8 N

**17.** a) Vektorit ovat yhdensuuntaisia, jos

$$4\bar{a} - 1\bar{b} = t(2\bar{a} + 5\bar{b}) = 2t\bar{a} + 5t\bar{b}$$

$$\begin{cases} 2t = 4 \\ 5t = -10 \end{cases} \Leftrightarrow \begin{cases} t = 2 \\ t = -2 \neq 2 \end{cases}$$

Vektorit **eivät ole** yhdensuuntaisia.

b) vektorit ovat yhdensuuntaisia, jos
$-4\overline{a}+10\overline{b}=t(2\overline{a}-5\overline{b})$

$$\begin{cases} 2t=-4 \\ -5t=10 \end{cases} \Leftrightarrow \begin{cases} t=-2 \\ t=-2 \end{cases}$$

Vektorit **ovat** yhdensuuntaisia.

**18.** Vektorin $\overline{y}$ suuntainen vektori on muotoa $t\overline{y}$

$t\overline{y}=2\overline{x}-\overline{y}+r(\overline{x}+3\overline{y})$
$t\overline{y}=2\overline{x}-\overline{y}+r\overline{x}+3r\overline{y}$
$t\overline{y}=(2+r)\overline{x}+(-1+3r)\overline{y}$

Komponentteihin jako yksikäsitteinen, joten
$$\begin{cases} 2+r=0 \quad \Leftrightarrow \quad r=-2 \\ -1+3r=t \end{cases}$$

Sijoitetaan *r* = -2 alempaan yhtälöön:
$t=-1+3\cdot(-2)=-7$

Kysytty vektori on $-7\overline{y}$.

**19.** $\overline{RP} = \overline{RB} + \overline{BP}$

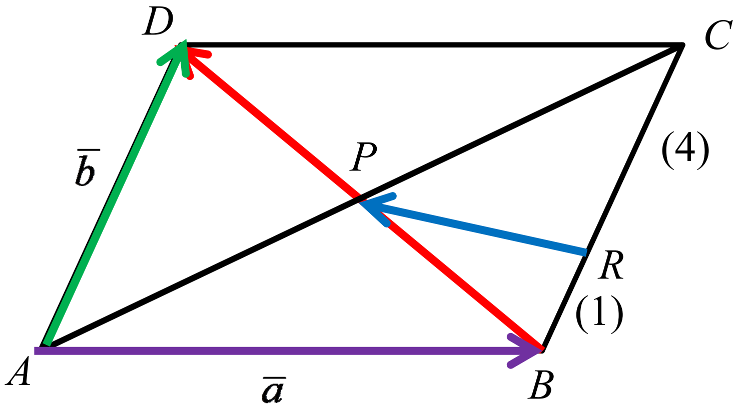
$$= \frac{1}{5}\overline{CB} + \frac{1}{2}\overline{BD}$$

$$= \frac{1}{5}(-\overline{b}) + \frac{1}{2}(\overline{BA} + \overline{AD})$$

$$= -\frac{1}{5}\overline{b} + \frac{1}{2}(-\overline{a} + \overline{b})$$

$$= -\frac{1}{5}\overline{b} - \frac{1}{2}\overline{a} + \frac{1}{2}\overline{b}$$

$$= \frac{3}{10}\overline{b} - \frac{1}{2}\overline{a}$$



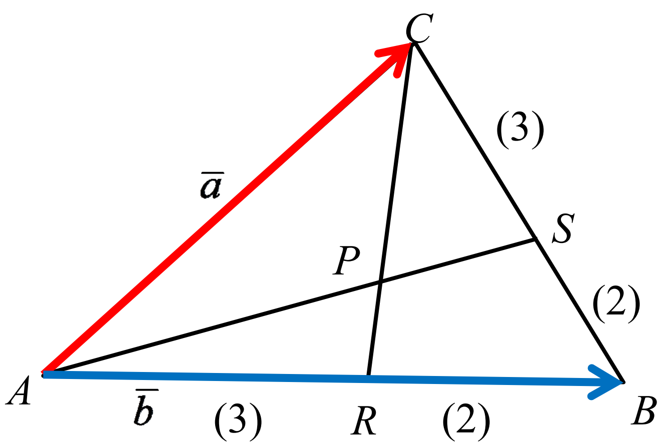

**20.** $\overline{a} = \overline{AC}$

$\overline{b} = \overline{AB}$

$$\overline{BC} = \overline{BA} + \overline{AC}$$

$$= -\overline{AB} + \overline{AC}$$

$$= -\overline{b} + \overline{a}$$




Lausutaan vektori $\overline{AP}$ kahdella eri tavalla:

1)

$$\overline{AP} = t\,\overline{AS} = t(\overline{AB} + \overline{BS}) = t(\overline{b} + \frac{2}{5}\overline{BC})$$

$$= t(\overline{b} + \frac{2}{5}(-\overline{b} + \overline{a}))$$

$$= t(\frac{2}{5}\overline{a} + \frac{3}{5}\overline{b})$$

$$= \frac{2}{5}t\overline{a} + \frac{3}{5}t\overline{b}$$

2)

$$\begin{aligned}\overline{AP} &= \overline{AC} + \overline{CP}\\ &= \overline{a} + s\overline{CR}\\ &= \overline{a} + s(\overline{CA} + \frac{3}{5}\overline{AB})\\ &= \overline{a} + s(-\overline{a} + \frac{3}{5}\overline{b})\\ &= (1-s)\overline{a} + \frac{3}{5}s\overline{b}\end{aligned}$$

Koska kyseessä sama vektori $\overline{AP}$, niin

$$\frac{2}{5}t\overline{a} + \frac{3}{5}t\overline{b} = (1-s)\overline{a} + \frac{3}{5}s\overline{b}$$

$$\begin{cases}\frac{2}{5}t = 1-s\\ \frac{3}{5}t = \frac{3}{5}s \quad \Leftrightarrow \quad t = s\end{cases}$$

Sijoitetaan $t = s$ ylempään yhtälöön:

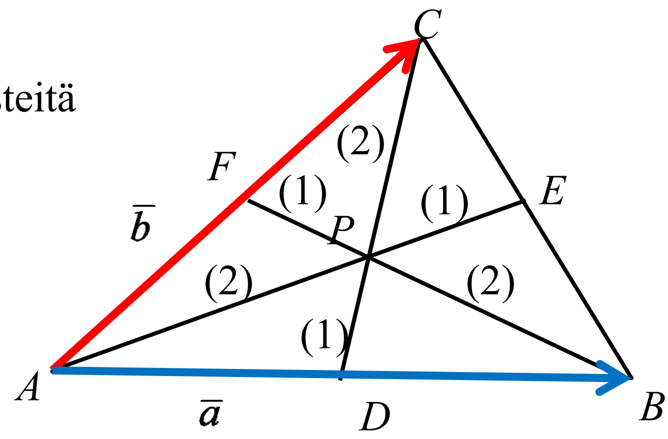
$$\begin{aligned}\frac{2}{5}s &= 1-s\\ \frac{7}{5}s &= 1\\ s &= \frac{5}{7}\end{aligned}$$

Eli $\overline{AP} = \frac{5}{7}\overline{AS}$ ja $\overline{CP} = \frac{5}{7}\overline{CR}$

Piste $P$ jakaa janat $AS$ ja $CR$ suhteessa 5:2.

**21.**

$D$, $E$ ja $F$ ovat sivujen keskipisteitä
$P$ on mediaanien leikkauspiste




$\overline{AB} = \bar{a}$
$\overline{AC} = \bar{b}$

**Väite**: $\overline{PB} + \overline{PA} + \overline{PC} = \bar{0}$

**Todistus:**
Mediaanien leikkauspiste jakaa mediaanit 2:1 kärjestä lukien.

$$\overline{PB} = \frac{2}{3}\overline{FB} = \frac{2}{3}(\overline{FA} + \overline{AB})$$
$$= \frac{2}{3}(-\frac{1}{2}\bar{b} + \bar{a})$$
$$= -\frac{1}{3}\bar{b} + \frac{2}{3}\bar{a}$$

$$\overline{PA} = \frac{2}{3}\overline{EA} = \frac{2}{3}(\frac{1}{2}\overline{CB} + \overline{BA})$$
$$= \frac{2}{3}(\frac{1}{2}(-\bar{b} + \bar{a}) - \bar{a})$$
$$= \frac{2}{3}(-\frac{1}{2}\bar{b} - \frac{1}{2}\bar{a})$$
$$= -\frac{1}{3}\bar{b} - \frac{1}{3}\bar{a}$$

$$\overline{PC} = \frac{2}{3}\overline{DC} = \frac{2}{3}(\overline{DA} + \overline{AC})$$

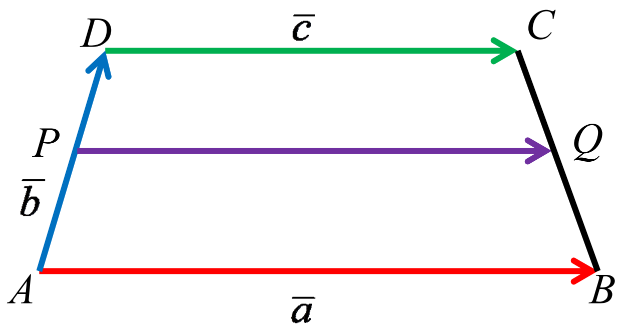
$$= \frac{2}{3}(-\frac{1}{2}\overline{a} + \overline{b})$$

$$= -\frac{1}{3}\overline{a} + \frac{2}{3}\overline{b}$$

$$\overline{PB} + \overline{PA} + \overline{PC} = -\frac{1}{3}\overline{b} + \frac{2}{3}\overline{a} - \frac{1}{3}\overline{b} - \frac{1}{3}\overline{a} - \frac{1}{3}\overline{a} + \frac{2}{3}\overline{b} = \overline{0} \quad \square$$

**22.** $\overline{AB} = \overline{a}$ $\quad \overline{AD} = \overline{b}$ $\quad \overline{DC} = \overline{c}$ $\quad \overline{a} \| \overline{c}$

$\overline{BC} = -\overline{a} + \overline{b} + \overline{c}$




**Väite 1** $\overline{PQ} \parallel \overline{a}$

**Todistus**

$$\overline{PQ} = \overline{PA} + \overline{AB} + \overline{BQ}$$

$$= -\frac{1}{2}\overline{b} + \overline{a} + \frac{1}{2}\overline{BC}$$

$$= -\frac{1}{2}\overline{b} + \overline{a} + \frac{1}{2}(-\overline{a} + \overline{b} + \overline{c})$$

$$= -\frac{1}{2}\overline{b} + \overline{a} - \frac{1}{2}\overline{a} + \frac{1}{2}\overline{b} + \frac{1}{2}c$$

$$= \frac{1}{2}\overline{a} + \frac{1}{2}\overline{c} \qquad | \ \overline{c} \| \overline{a} \Leftrightarrow \overline{c} = t\overline{a}$$

$$= \frac{1}{2}\overline{a} + \frac{1}{2}t\overline{a} = (\frac{1}{2} + \frac{1}{2}t)\overline{a}$$

eli $\overline{PQ} = r\overline{a}$ eli $\overline{PQ} \parallel \overline{a} \quad \square$

**Väite 2** $\left|\overline{PQ}\right| = \frac{1}{2}(|\overline{a}| + |\overline{c}|)$

**Todistus**

$$\overline{PQ} = \frac{1}{2}\overline{a} + \frac{1}{2}\overline{c} = \frac{1}{2}(\overline{a} + \overline{c})$$

Koska $\overline{a} \uparrow\uparrow \overline{c}$, niin $|\overline{a}| + |\overline{c}| = |\overline{a} + \overline{c}|$.

$$\left|\overline{PQ}\right| = \left|\frac{1}{2}(\overline{a} + \overline{c})\right| = \frac{1}{2}|\overline{a} + \overline{c}| = \frac{1}{2}(|\overline{a}| + |\overline{c}|) \square$$

**23.** a) $\overline{OA} = 2\overline{i} - 5\overline{j}$

$\overline{OB} = -\overline{i} + 4\overline{j}$

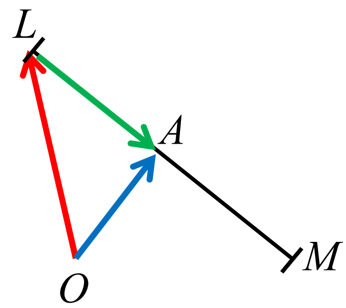
b) $\overline{AB} = (-1-2)\overline{i} + (4-(-5))\overline{j} = -3\overline{i} + 9\overline{j}$

c) $\overline{AB} = \sqrt{(-3)^2 + 9^2} = \sqrt{90} = 3\sqrt{10}$

d) $\overline{AB}^0 = \frac{1}{3\sqrt{10}}(-3\overline{i} + 9\overline{j}) = \frac{-3}{3\sqrt{10}}\overline{i} + \frac{9}{3\sqrt{10}}\overline{j} = -\frac{1}{\sqrt{10}}\overline{i} + \frac{3}{\sqrt{10}}\overline{j}$

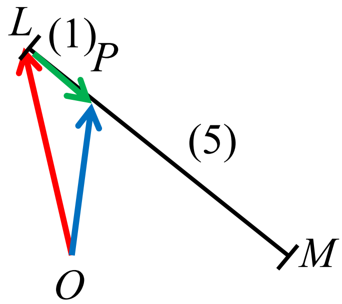
**24.** a) $\overline{LM} = (4-3)\bar{i} + (-6-2)\bar{j} = \bar{i} - 8\bar{j}$

b) Janan keskipisteen $A$ paikkavektori




$$\overline{OA} = \overline{OL} + \overline{LA}$$
$$= \overline{OL} + \frac{1}{2}\overline{LM}$$
$$= 3\bar{i} + 2\bar{j} + \frac{1}{2}(\bar{i} - 8\bar{j})$$
$$= 3\bar{i} + 2\bar{j} + \frac{1}{2}\bar{i} - 4\bar{j}$$
$$= 3\frac{1}{2}\bar{i} - 2\bar{j}$$

c) Muodostetaan pisteen $P$ paikkavektori




$$\overline{OP} = \overline{OL} + \frac{1}{6}\overline{LM}$$
$$= 3\bar{i} + 2\bar{j} + \frac{1}{6}(\bar{i} - 8\bar{j})$$
$$= 3\bar{i} + 2\bar{j} + \frac{1}{6}\bar{i} - \frac{4}{3}\bar{j}$$
$$= 3\frac{1}{6}\bar{i} + \frac{2}{3}\bar{j}$$

Pisteen $P$ koordinaatit $\left(3\frac{1}{6}, \frac{2}{3}\right)$

**25.** Muodostetaan pisteen $A$ paikkavektori.

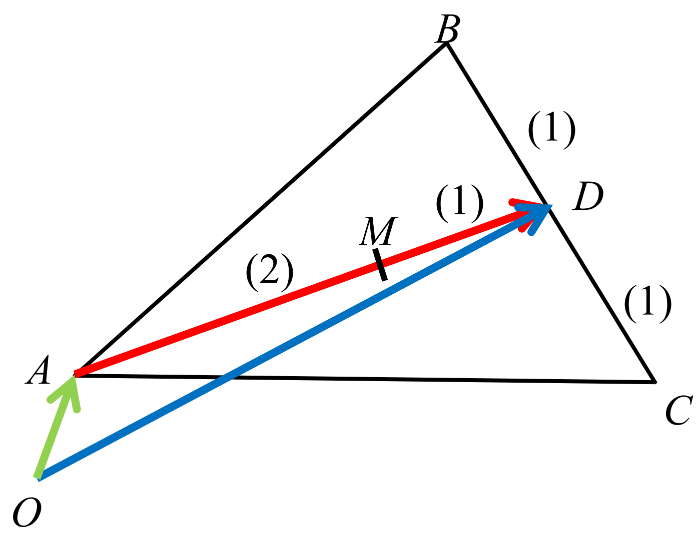
$$\begin{aligned}\overline{OA} &= \overline{OB} + \overline{BA} \\ &= -8\bar{i} + 12\bar{j} + 2\bar{i} + 6\bar{j} \\ &= -6\bar{i} + 18\bar{j}\end{aligned}$$

Paikkavektorin pituus

$$|\overline{OA}| = \sqrt{(-6)^2 + 18^2} = \sqrt{360} = 6\sqrt{10}$$

**26.** $\overline{AD}$ on mediaani, joten

$|\overline{CD}| = |\overline{DB}|$




Mediaanien leikkauspiste $M$ jakaa mediaanin (kärjestä lukien) suhteessa 2:1, joten

$$\overline{AM} = \frac{2}{3}\overline{AD}$$

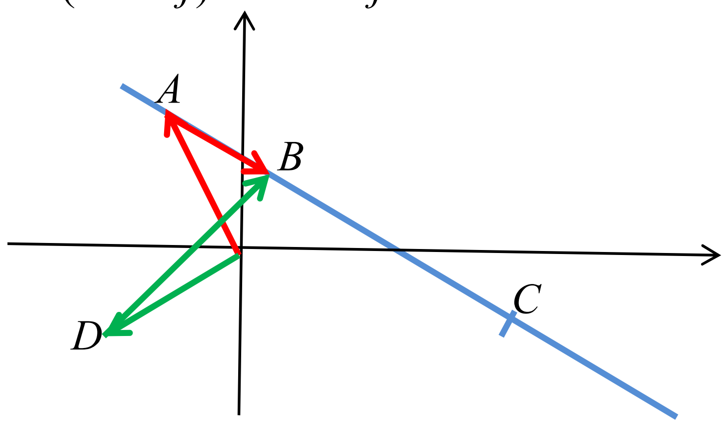
$$\overline{AD} = \frac{3}{2}\overline{AM}$$

$$\begin{aligned}\overline{OD} &= \overline{OA} + \overline{AD} = \overline{OA} + \frac{3}{2}\overline{AM} \\ &= \bar{i} + \bar{j} + \frac{3}{2}\left((4\frac{2}{3} - 1)\bar{i} + (5\frac{2}{3} - 1)\bar{j}\right) \\ &= \bar{i} + \bar{j} + + \frac{11}{2}\bar{i} + 7\bar{j} \\ &= 6\frac{1}{2}\bar{i} + 8\bar{j}\end{aligned}$$

$$\overline{OB} = \overline{OC} + \overline{CB}$$
$$= \overline{OC} + 2\overline{CD}$$
$$= 9\bar{i} + 6\bar{j} + 2\left((6\frac{1}{2} - 9)\bar{i} + (8-6)\bar{j}\right)$$
$$= 4\bar{i} + 10\bar{j}$$

Vastaus: $B = (4,\ 10)$ ja $D = \left(6\frac{1}{2}, 8\right)$

**27.** Vektori $\overline{DB}$ samansuuntainen $3\bar{i} + 4\bar{j}$ vektorin kanssa, joten
$\overline{DB} = t(3\bar{i} + 4\bar{j}) = 3t\bar{i} + 4t\bar{j}$




Muodostetaan pisteen $B$ paikkavektori kahdella tavalla.

**Tapa 1**

$$\overline{OB} = \overline{OA} + \overline{AB} = \overline{OA} + s\overline{AC}$$
$$= -3\bar{i} + 5\bar{j} + s(16\bar{i} - 8\bar{j})$$
$$= (-3 + 16s)\bar{i} + (5 - 8s)\bar{j}$$

**Tapa 2**

$$\overline{OB} = \overline{OD} + \overline{DB} =$$
$$= -5\bar{i} - 5\bar{j} + 3t\bar{i} + 4t\bar{j}$$
$$= (-5 + 3t)\bar{i} + (-5 + 4t)\bar{j}$$

Komponentteihin jako yksikäsitteinen, joten

$$\begin{cases} -3+16s=-5+3t \\ 5-8s-5+4t \end{cases}$$

$$\begin{cases} 16s-3t=-2 \\ -8s-4t=-10 \quad |\cdot 2 \end{cases}$$

$$\begin{cases} 16s-3t=2 \\ -16s-8t=-20 \end{cases}$$

$$-11t=-22 \quad |:(-11)$$

$$t=2$$

Sijoitetaan $t = 2$ pisteen $D$ kautta muodostettuun paikkavektoriin.

$$\begin{aligned} \overline{OB} &= (-5+3\cdot 2)\bar{i}+(-5+4\cdot 2)\bar{j} \\ &= \bar{i}+3\bar{j} \end{aligned}$$

Pisteen $B$ koordinaatit ovat (1, 3).

**28.** a) $\overline{AB}=(-1-0)\bar{i}(3-2)\bar{j}+(1-(-1))\bar{k}=-\bar{i}+\bar{j}+2\bar{k}$

b) $\left|\overline{AB}\right|=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$

c) Olkoon vektorin keskipiste $P$.

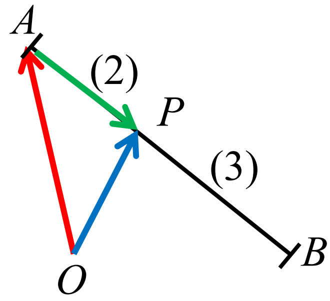
$$\overline{OP} = \overline{OA} + \frac{1}{2}\overline{AB}$$

$$= 2\bar{j} - \bar{k} + \frac{1}{2}(-\bar{i} + \bar{j} + 2\bar{k})$$

$$= 2j - -\bar{k} - \frac{1}{2}\bar{i} + \frac{1}{2}\bar{j} + \bar{k}$$

$$= -\frac{1}{2}\bar{i} + 2\frac{1}{2}\bar{j}$$

Pisteen $P$ koordinaatit ovat $\left(-\frac{1}{2}, 2\frac{1}{2}, 0\right)$.




**29.** Muodostetaan pisteen $P$ paikkavektori.

$$\overline{OP} = \overline{OA} + \overline{AP}$$

$$= \overline{OA} + \frac{2}{5}\overline{AB}$$

$$= 2\bar{i} - 2\bar{j} + \frac{2}{5}\left((-4-2)\bar{i} + (1-(-2))\bar{j} + (-3-0)\bar{k}\right)$$

$$= 2\bar{i} - 2\bar{j} + \frac{2}{5}\left(-6\bar{i} + 3\bar{j} - 3\bar{k}\right)$$

$$= 2\bar{i} - 2\bar{j} - \frac{12}{5}\bar{i} + \frac{6}{5}\bar{j} - \frac{6}{5}\bar{k}$$

$$= -\frac{2}{5}\bar{i} - \frac{4}{5}\bar{j} - \frac{6}{5}\bar{k}$$

Pisteen $P$ koordinaatit ovat $\left(-\frac{2}{5}, -\frac{4}{5}, -\frac{6}{5}\right)$.

**30.** Pisteet $A$, $B$ ja $C$ ovat samalla suoralla, jos

$$\overline{AB} \parallel \overline{AC} \quad \Leftrightarrow \quad \overline{AC} = t\overline{AB}$$

$$\overline{AB} = (1-4)\bar{i} + (-2-0)\bar{j} + (3-2)\bar{k} = -3\bar{i} - 2\bar{j} + \bar{k}$$

$$\overline{AC} = (-11-4)\bar{i}(-10-0)\bar{j} + (7-2)\bar{k} = -15\bar{i} - 10\bar{j} + 5\bar{k}$$

$$5\overline{AB} = 5(-3\bar{i} - 2\bar{j} + \bar{k}) = -15\bar{i} - 10\bar{j} + 5\bar{k} = \overline{AC}$$

Pisteet ovat siis samalla suoralla.

**31.** a) Pallon keskipistemuotoinen yhtälö on:

$$(x-2)^2 + (y-2)^2 + (z-4)^2 = 6^2$$

$$(x-2)^2 + (y-2)^2 + (z-4)^2 = 36$$

b) Piste on pallon pinnalla, jos sen koordinaatit toteuttavat pallon yhtälön.

$$(-2-2)^2 + (4-2)^2 + (0-4)^2 = 36$$

$$16 + 4 + 16 = 36$$

$$36 = 36$$

tosi

Piste on pallon pinnalla.

**32.** Vektori leikkaa $xy$-tason pisteessä $P(x, y, 0)$.
Pisteen $P$ paikkavektori on

$$\overline{OP} = x\bar{i} + y\bar{j}$$

Muodostetaan vektori $\overline{AP}$.

$$\overline{AP} = t\bar{s} = -4t\bar{i} - 5t\bar{j} - t\bar{k}$$

Pisteen $P$ paikkavektori toisella tavalla:

$$\begin{aligned}\overline{OP} &= \overline{OA} + \overline{AP} \\ &= 10\bar{i} + 8\bar{j} + 2\bar{k} - 4t\bar{i} - 5t\bar{j} - t\bar{k} \\ &= (10-4t)\bar{i} + (8-5t)\bar{j} + (2-t)\bar{k}\end{aligned}$$

Komponentteihin jako yksikäsitteinen, joten

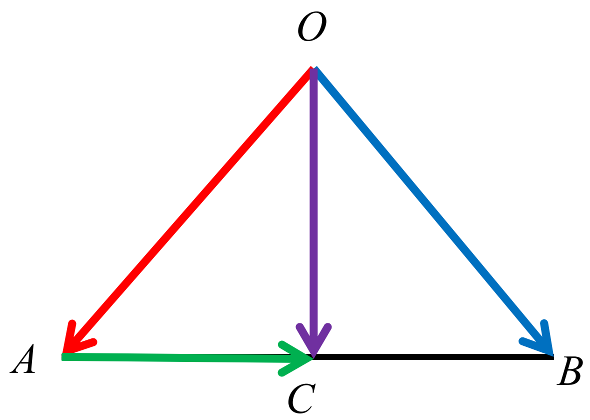
$$\begin{cases} x = 10 - 4t \\ y = 8 - 5t \\ 0 = 2 - t \quad \Rightarrow \quad t = 2 \end{cases}$$

Sijoitetaan $t = 2$ kahteen ylempään yhtälöön.

$$\begin{cases} x = 10 - 4 \cdot 2 = 2 \\ y = 8 - 5 \cdot 2 = -2 \end{cases}$$

Pisteen $P$ koordinaatit ovat (2, -2, 0).

**33.** $O(0, 0, 0)$
$A(1, -5, 2)$
$B(5, 2, -1)$




$$\overline{AB} = 4\bar{i} + 7\bar{j} - 3\bar{k}$$

$$\overline{OA} = \bar{i} - 5\bar{j} + 2\bar{k}$$

$$\overline{OB} = 5\bar{i} + 2\bar{j} - \bar{k}$$

$$\left|\overline{OA}\right| = \left|\overline{OB}\right| = \sqrt{1^2 + (-5)^2 + 2^2} = \sqrt{30}$$

$$\overline{AC} = \frac{1}{2}\overline{AB} = \frac{1}{2}(4\bar{i} - 7\bar{j} - 3\bar{k}) = 2\bar{i} + \frac{7}{2}\bar{j} - \frac{3}{2}\bar{k}$$

$$\overline{OC} = \overline{OA} + \overline{AC} = \bar{i} - 5\bar{j} + 2\bar{k} + 2\bar{i} + \frac{7}{2}\bar{j} - \frac{3}{2}\bar{k} = 3\bar{i} - \frac{3}{2}\bar{j} + \frac{1}{2}\bar{k}$$

$$\left|\overline{OC}\right| = \sqrt{3^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{46}{4}} = \frac{\sqrt{46}}{2}$$

a) Kolmion ala

$$A = \frac{1}{2}\left|\overline{AB}\right| \cdot \left|\overline{OC}\right| = \frac{1}{2}\sqrt{74}\,\frac{\sqrt{46}}{2} = \frac{1}{4}\sqrt{3404} = \frac{\sqrt{851}}{2} \approx 14{,}6$$

b) Merkitään mediaanien leikkauspistettä kirjaimella $P$.

$$\overline{OP} = \frac{2}{3}\overline{OC} = \frac{2}{3}\left(3\bar{i} - \frac{3}{2}\bar{j} + \frac{1}{2}\bar{k}\right) = 2\bar{i} - \bar{j} + \frac{1}{2}\bar{k}$$

Pisteen $P$ koordinaatit ovat $(2, -1, \frac{1}{2})$

**34.** a) $\overline{a} \cdot \overline{b} = -8 \cdot (-1) + 6 \cdot (-2) = -4$

b) $\overline{a} \cdot \overline{b} = 3 \cdot (-1) - 1 \cdot 4 + 2 \cdot (-1) = -9$

**35.** $\left(\overline{a} - 4\overline{b}\right) \cdot \left(-3\overline{a} + 5\overline{b}\right)$

$= \overline{a} \cdot (-3\overline{a}) + \overline{a} \cdot 5\overline{b} - \overline{4} \cdot (3\overline{a}) - 4\overline{b} \cdot 5\overline{b}$

$= -3 \cdot \overline{a}^2 + 5\overline{a} \cdot \overline{b} + 12\overline{b} \cdot \overline{a} - 20\overline{b}^2$

$= -3\left|\overline{a}\right|^2 + 17\overline{a} \cdot \overline{b} - 20\left|\overline{b}\right|^2$

$= -3 \cdot 3^2 + 17 \cdot (-2) - 20 \cdot 5^2$

$= -561$

**36.** $\left|\overline{a}\right| = \left|\overline{b}\right|$

$\left(\overline{a} - \overline{b}\right)\left(\overline{a} + \overline{b}\right)$

$= \overline{a}^2 - \overline{b}^2$

$= \left|\overline{a}\right|^2 - \left|\overline{b}\right|^2$

$= \left|\overline{a}\right|^2 - \left|\overline{a}\right|^2$

$= 0$

**37.**

$$\begin{aligned}
&(3\bar{x}+4\bar{y})^2 \\
&=(3\bar{x})^2+2\cdot 3\bar{x}\cdot 4\bar{y}+(4\bar{y})^2 \\
&=9\bar{x}^2+24\bar{x}\cdot\bar{y}+16\bar{y}^2 \\
&=9|\bar{x}|^2+24\bar{x}\cdot\bar{y}+16|\bar{y}|^2 \\
&=9\cdot 2^2+24\cdot(-5)+16\cdot 4^2 \\
&=172
\end{aligned}$$

$$|3\bar{x}+4\bar{y}|=\sqrt{172}=2\sqrt{43}$$

**38.**

$$\begin{aligned}
(-8\bar{i}+x\bar{j})\cdot(12\bar{i}+(x-2)\bar{j}) &= -72 \\
-8\cdot 12+x(x-2) &= -72 \\
-96+x^2-2x &= -72 \\
x^2-2x-24 &= 0
\end{aligned}$$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 1\cdot(-24)}}{2\cdot 1}$$

$$x=\frac{2\pm 10}{2}$$

$$x=6 \quad \vee \quad x=-4$$

**39.** a) $\cos\alpha = \dfrac{\overline{a}\cdot\overline{b}}{|\overline{a}||\overline{b}|} = \dfrac{12}{7{,}5\cdot 5{,}2} = 0{,}307...$ $\qquad \alpha = 72{,}07...^\circ \approx 72^\circ$

b) $|\overline{a}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$|\overline{b}| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

$\overline{a}\cdot\overline{b} = (-1)\cdot(-2) + 2\cdot(-4) = -6$

$\cos\alpha = \dfrac{-6}{\sqrt{5}\cdot 2\sqrt{5}} = -\dfrac{3}{5}$

$\alpha = 126{,}86...^\circ \approx 127^\circ$

**40.** $\overline{u} = 3\overline{i} + 8\overline{j}$ $\qquad \overline{v} = 4\overline{i} + a\overline{j}$

$$\begin{aligned} \overline{u}\cdot\overline{v} &= 0 \\ 3\cdot 4 + 8a &= 0 \\ 12 + 8a &= 0 \\ 8a &= -12 \\ a &= -\frac{12}{8} = -\frac{3}{2} \end{aligned}$$

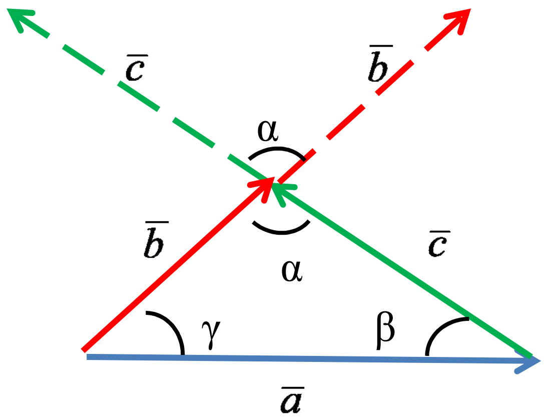
**41.** $\bar{u} = \bar{x} + \bar{y} = (t+5)\bar{i} - 17\bar{j}$

$\bar{v} = \bar{x} - \bar{y} = (t-5)\bar{i} + 2\bar{j}$

$$\bar{u} \cdot \bar{v} = 0$$
$$(t+5)(t-5) - 17 \cdot 2 = 0$$
$$t^2 - 25 - 34 = 0$$
$$t^2 = 59$$
$$t = \pm\sqrt{59}$$

**42.** $\bar{a} = -7\bar{i} - 6\bar{j}$ $\qquad$ $\bar{b} = 2\bar{i} - 5\bar{j}$

$$\begin{aligned}\bar{c} &= -\bar{a} + \bar{b} \\ &= 7\bar{i} + 6\bar{j} + 2\bar{i} - 5\bar{j} \\ &= 9\bar{i} + \bar{j}\end{aligned}$$




$$\cos\gamma = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} = \frac{-7 \cdot 2 - 6 \cdot (-5)}{\sqrt{(-7)^2 + (-6)^2}\sqrt{2^2 + (-5)^2}} = \frac{16}{\sqrt{85}\sqrt{29}}$$
$$= 0{,}322\ldots$$

$$\gamma = 71{,}200\ldots° \approx 71{,}2°$$

$$\cos\alpha = \frac{\bar{b} \cdot \bar{c}}{|\bar{b}||\bar{c}|} = \frac{2 \cdot 9 - 5 \cdot 1}{\sqrt{29}\sqrt{9^2 + 1^2}} = \frac{13}{\sqrt{29}\sqrt{82}}$$
$$= 0{,}266\ldots$$

$$\alpha = 74{,}538\ldots° \approx 74{,}5°$$

$$\beta \approx 180° - 71{,}2° - 74{,}5° = 34{,}3°$$

**43.** $|\overline{a}_1| = |\overline{a}_2| = a$ ja $\overline{a}_1 \cdot \overline{a}_2 = 0$

$\overline{r} = \frac{1}{2}\overline{a}_1 - (1-k)\overline{a}_2$ ja $|\overline{r}| = ka$

$$\begin{aligned}
|\overline{r}|^2 &= \overline{r} \cdot \overline{r} \\
&= \left(\frac{1}{2}\overline{a}_1 - (1-k)\overline{a}_2\right) \cdot \left(\frac{1}{2}\overline{a}_1 - (1-k)\overline{a}_2\right) \\
&= \frac{1}{4}|\overline{a}_1|^2 - \frac{1}{2}(1-k)\underbrace{(\overline{a}_1 \cdot \overline{a}_2)}_{=0} - \frac{1}{2}(1-k)\underbrace{(\overline{a}_1 \cdot \overline{a}_2)}_{=0} + (1-k)^2|\overline{a}_2|^2 \\
&= \frac{1}{4}|\overline{a}_1|^2 + (1-k)^2|\overline{a}_2|^2 \\
&= \frac{1}{4}a^2 + (1-k)^2 a^2 \\
&= a^2\left(\frac{1}{4} + (1-k)^2\right)
\end{aligned}$$

Toisaalta

$|\overline{r}|^2 = (ka)^2 = k^2a^2$

Saadaan yhtälö

$$a^2\left(\frac{1}{4} + (1-k)^2\right) = k^2a^2 \quad |: a^2 \neq 0$$

$$\frac{1}{4} + 1 - 2k + k^2 = k^2$$

$$2k = \frac{5}{4} \quad |: 2$$

$$k = \frac{5}{8}$$

**44.** $A = (2, 1, 4)$ $B = (-1, 3, 1)$

$\bar{s} = \overline{AB} = -3\bar{i} + 2\bar{j} - 3\bar{k}$

$$\begin{aligned}\overline{OP} &= \overline{OA} + t\bar{s} \\ &= 2\bar{i} + \bar{j} + 4\bar{k} + t(-3\bar{i} + 2\bar{j} - 3\bar{k}) \\ &= (2-3t)\bar{i} + (1-2t)\bar{j} + (4-3t)\bar{k}\end{aligned}$$

**45.** a) Suoran suuntavektori esim. $\bar{s} = 5\bar{i} + 3\bar{j} - 4\bar{k}$

Suoran piste saadaan, kun annetaan parametrille $t$ jokin arvo.

Esim. kun $t = 1$

$$\begin{cases} x = 2 + 5 = 7 \\ y = 3 \\ z = 1 - 4 = -3 \end{cases}$$

Piste (7, 3, -3) on suoralla.

b) Suoran suuntavektori esim. $\bar{s} = -\bar{i} + \bar{j} + 2\bar{k}$

Suoran piste saadaan, kun annetaan parametrille $t$ jokin arvo.

Esim. kun $t = -1$

$$\begin{cases} x = -5 + 1 = -4 \\ y = -1 - 1 = -2 \\ z = 10 - 2 = 8 \end{cases}$$

Piste (-4, -2, 8) on suoralla.

**46.** Suoran suuntavektori saadaan kulmakertoimen avulla.

Suoran kulmakerroin $k = -6 = \frac{-6}{1}$.

Suoran suuntavektori $\bar{s} = \bar{i} - 6\bar{j}$.

Valitaan suoralta jokin piste.
Suora leikkaa $y$-akselin pisteessä $A(0, 5)$.

Suoran vektoriyhtälö on

$$\begin{aligned}\overline{OP} &= \overline{OA} + t\bar{s}\\ &= 5\bar{j} + t(\bar{i} - 6\bar{j})\\ &= t\bar{i} + (5 - 6t)\bar{j}\end{aligned}$$

Parametriesitys

$$\begin{cases} x = t \\ y = 5 - 6t \end{cases}$$

**47.** $xz$-tason leikkauspisteessä $y = 0$.

$$\begin{aligned} 2-8t &= 0 \\ -8t &= -2 \\ t &= \frac{1}{4} \end{aligned}$$

Sijoitetaan saatu $t$:n arvo koordinaattien lausekkeisiin.

$$\begin{cases} x = -12 + 3 \cdot \frac{1}{4} = -9 \\ y = 0 \\ z = 3 - 4 \cdot \frac{1}{4} = 2 \end{cases}$$

Suora leikkaa $xz$-tason pisteessä (-9, 0, 2).

**48.** Suoran suuntavektori

$\bar{s} = \overline{AB} = -2\bar{i} - 2\bar{j} + 4\bar{k}$

Valitaan suoran tunnetuksi pisteeksi $A(4, 0, 1)$

Suoran parametriesitys:

$$\begin{cases} x = 4 - 2t \\ y = 0 - 2t \\ z = 1 + 4t \end{cases} \Leftrightarrow \begin{cases} x = 4 - 2t \\ y = -2t \\ z = 1 + 4t \end{cases}$$

Sijoitetaan pisteen (6, 2, _3) koordinaatit parametriesitykseen.

$$\begin{cases} 6 = 4 - 2t \\ 2 = -2t \quad \Leftrightarrow t = -1 \\ -3 = 1 + 4t \end{cases}$$

Sijoitetaan $t$ = -1 kahteen muuhun yhtälöön.

$6 = 4 - 2 \cdot (-1)$ $\quad\quad$ $-3 = 1 + 4 \cdot (-1)$

$6 = 6$ $\quad\quad$ $-3 = -3$

tosi $\quad\quad$ tosi

Piste (6, 2, -3) on suoralla.

**49.** a) Kirjoitetaan suorien yhtälöt ensin ratkaistussa muodossa.

$$2x - 5y + 1 = 0 \qquad x + 2y + 2 = 0$$

$$5y = 2x + 1 \qquad 2y = -x - 2$$

$$y = \frac{2}{5}x + \frac{1}{5} \qquad y = -\frac{1}{2}x - 1$$

Kulmakertoimien avulla saadaan suorien suuntavektorit.

$$\bar{s}_1 = 5\bar{i} + 2\bar{j} \qquad \bar{s}_2 = 2\bar{i} - \bar{j}$$

$$|\bar{s}_1| = \sqrt{5^2 + 2^2} = \sqrt{29} \qquad |\bar{s}_2| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

$$\bar{s}_1 \cdot \bar{s}_2 = 5 \cdot 2 + 2 \cdot (-1) = 8$$

$$\cos\alpha = \frac{8}{\sqrt{29}\sqrt{5}} = 0{,}664...$$

$$\alpha = 48{,}366...° \approx 48°$$

b) Suorien suuntavektorit:

$$\bar{s}_1 = -3\bar{i} + \bar{j} + \bar{k} \qquad \bar{s}_2 = -4\bar{i} + \bar{j} + 5\bar{k}$$

$$|\bar{s}_1| = \sqrt{(-3)^2 + 1^2 + 1^2} = \sqrt{11} \qquad |\bar{s}_2| = \sqrt{(-4)^2 + 1^2 + 5^2} = \sqrt{42}$$

$$\bar{s}_1 \cdot \bar{s}_2 = -3 \cdot (-4) + 1 \cdot 1 + 1 \cdot 5 = 18$$

$$\cos\alpha = \frac{18}{\sqrt{11}\sqrt{42}} = 0{,}837...$$

$$\alpha = 33{,}129...° \approx 33°$$

**50.** Leikkauspisteessä koordinaatit samat:

$$\begin{cases} 3 + 3t = 6 - s \\ -4 + 3t = -2 - 4s \\ \frac{13}{9} - t = 1 + 2s \end{cases} \Leftrightarrow \begin{cases} 3t + s = 3 \\ 3t + 4s = 2 \quad |\cdot(-1) \\ -t - 2s = -\frac{4}{9} \end{cases}$$

$$\begin{cases} 3t + s = 3 \\ -3t - 4s = -2 \end{cases}$$

$$-3s = 1$$

$$s = -\frac{1}{3}$$

$$3t + \left(-\frac{1}{3}\right) = 3$$

$$3t = \frac{10}{3}$$

$$t = \frac{10}{9}$$

Sijoitetaan saadut arvot kolmanteen yhtälöön:

$$-\frac{10}{9}-2\cdot\left(-\frac{1}{3}\right)=-\frac{4}{9}$$

$$-\frac{10}{9}+\frac{2}{3}=-\frac{4}{9}$$

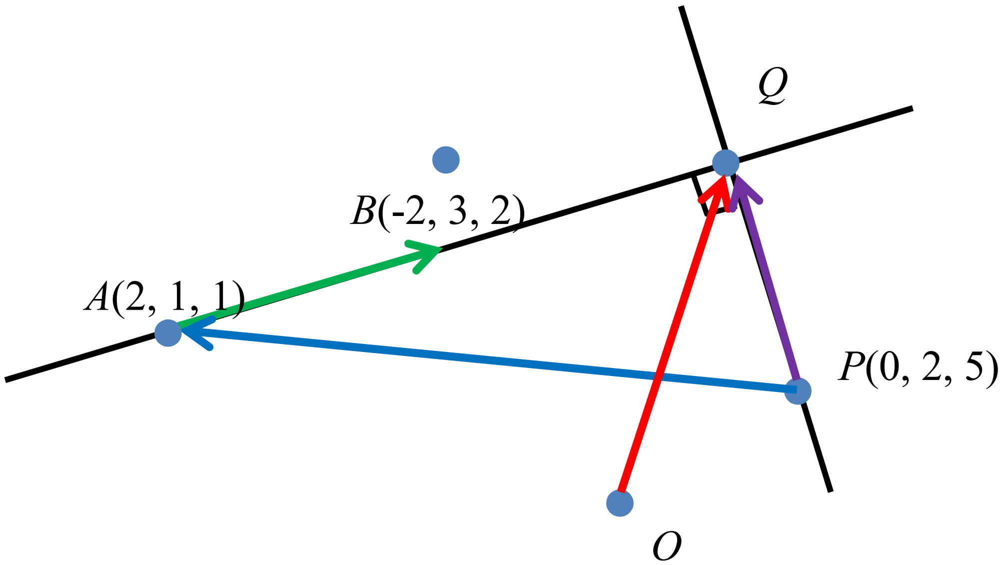
$$-\frac{4}{9}=-\frac{4}{9}$$

Eli leikkauspisteessä $s=-\frac{1}{3}$, $t=\frac{10}{9}$.

Leikkauspisteen koordinaatit:

$$\begin{cases} x=6-s=6-\left(-\frac{1}{3}\right)=6\frac{1}{3} \\ y=-2-4s=-2-4\cdot\left(-\frac{1}{3}\right)=-\frac{2}{3} \\ z=1+2s=1+2\cdot\left(-\frac{1}{3}\right)=\frac{1}{3} \end{cases}$$

**51.**




$$\overline{PA} = 2\bar{i} - \bar{j} - 4\bar{k}$$

$$\bar{s} = \overline{AB} = -4\bar{i} + 2\bar{j} + \bar{k}$$

$$\begin{aligned}\overline{PQ} &= \overline{PA} + ts \\ &= 2\bar{i} - \bar{j} + 4\bar{k} + t(-4\bar{i} + 2\bar{j} + \bar{k}) \\ &= (2-4t)\bar{i} + (-1+2t)\bar{j} + (-4+t)\bar{k}\end{aligned}$$

$$\begin{aligned}\overline{PQ} \cdot \bar{s} &= 0 \\ -4(2-4t) + 2(-1+2t) + 1(-4+t) &= 0 \\ -8 + 16t - 2 + 4t - 4 + t &= 0 \\ 21t &= 14 \\ t &= \frac{14}{21} = \frac{2}{3}\end{aligned}$$

$$\begin{aligned}\overline{PQ} &= \left(2 - 4 \cdot \frac{2}{3}\right)\bar{i} + \left(-1 + 2 \cdot \frac{2}{3}\right)\bar{j} + \left(-4 + \frac{2}{3}\right)\bar{k} \\ &= -\frac{2}{3}\bar{i} + \frac{1}{3}\bar{j} - \frac{10}{3}\bar{k}\end{aligned}$$

Valitaan suoran *n* suuntavektoriksi $3 \cdot \overline{PQ} = -2\overline{i} + \overline{j} - 10\overline{k}$.
Piste (0, 2, 5) on suoralla *n*.

Suoran parametriesitys:

$$\begin{cases} x = -2t \\ y = 2 + t \\ z = 5 - 10t \end{cases}$$

Leikkauspisteen paikkavektori

$$\begin{aligned} \overline{OQ} &= \overline{OP} + \overline{PQ} \\ &= 2\overline{j} + 5\overline{k} - \frac{2}{3}\overline{i} + \frac{1}{3}\overline{j} - \frac{10}{3}\overline{k} \\ &= -\frac{2}{3}\overline{i} + 2\frac{1}{3}\overline{j} + 1\frac{2}{3}\overline{k} \end{aligned}$$

Leikkauspisteen koordinaatit ovat $\left(-\frac{2}{3}, 2\frac{1}{3}, 1\frac{2}{3}\right)$.

**52.** Muodostetaan suorien parametriesitykset:

Pekka

$$\begin{cases} x = 10 - 10t \\ y = 5 + t \\ z = 2 - t \end{cases}$$

Heikki

$$\begin{aligned} x &= 8 - 4u \\ y &= 12 - 3u \\ z &= 3 - u \end{aligned}$$

Leikkauspisteessä:

$$\begin{cases} 10-10t=8-4u \\ 5+t=12-3u \\ 2-t=3-u \end{cases} \Leftrightarrow \begin{cases} -10t+4u=-2 \\ t+3u=7 \\ -t+u=1 \end{cases}$$

Muodostetaan kahdesta alimmasta yhtälöstä yhtälöpari ja ratkaistaan se.

$$\begin{cases} t+3u=7 \\ -t+u=1 \end{cases}$$

$$4u=8$$

$$u=2$$

$$t=7-3u=7-3\cdot 2=1$$

Tutkitaan toteuttavatko saadut arvot ensimmäisen yhtälön.

$$-10t+4u=-2$$

$$-10\cdot 1+4\cdot 2=-2$$

$$-2=-2$$

tosi

Sijoitetaan $t = 1$ Pekan yhtälöön:

$$\begin{cases} x=10-10\cdot 1=0 \\ y=5+1=6 \\ z=2-1=1 \end{cases}$$

Saalis on pisteessä (0, 6, 1).

**53.** $A(2, 0, 1)$, $B(-2, 2, 0)$, $C(1, 3, 1)$

$$\overline{AB} = -4\bar{i} + 2\bar{j} - \bar{k}$$

$$\overline{AC} = -\bar{i} + 3\bar{j}$$

Tason vektoriyhtälö:

$$\begin{aligned}
&\overline{OA} + s\overline{AB} + r\overline{AC} \\
&= 2\bar{i} + \bar{k} + s(-4\bar{i} + 2\bar{j} - \bar{k}) + r(-\bar{i} + 3\bar{j}) \\
&= (2 - 4s - r)\bar{i} + (2s + 3r)\bar{j} + (1 - s)\bar{k}
\end{aligned}$$

Tason parametriesitys:

$$\begin{cases} x = 2 - 4s - r \\ y = 2s + 3r \\ z = 1 - s \end{cases}$$

**54.** $x = 2 - r \quad \Leftrightarrow \quad r = 2 - x$

Sijoitetaan tämä toiseen yhtälöön ja ratkaistaan *s*.

$$y = 1 - 2s + 2 - x$$
$$2s = 3 - x - y$$
$$s = \frac{3}{2} - \frac{1}{2}x - \frac{1}{2}y$$

Sijoitetaan $r = 2 - x$ ja $s = \frac{3}{2} - \frac{1}{2}x - \frac{1}{2}y$ kolmanteen yhtälöön.

$$z = 3\left(\frac{3}{2} - \frac{1}{2}x - \frac{1}{2}y\right) - 2(2 - x)$$
$$z = \frac{9}{2} - \frac{3}{2}x - \frac{3}{2}y - 4 + 2x \qquad |\cdot 2$$
$$2z = 9 - 3x - 3y - 8 + 2x$$
$$x + 3y + 2z - 1 = 0$$

**55.** Tason $3x + 3y - z - 4 = 0$ eräs normaalivektori on
$\overline{n} = 3\overline{i} + 3\overline{j} - \overline{k}$ .

Tämä on myös yhdensuuntaisen tason normaalivektori.

Taso kulkee pisteen $A(-1, 0, 2)$ kautta. Jos piste $B(2, -2, 5)$ on tasossa, niin
$\overline{AB} \cdot \overline{n} = 0$

$\overline{AB} = 3\overline{i} - 2\overline{j} + 3\overline{k}$

$\overline{AB} \cdot \overline{n} = 3 \cdot 3 - 2 \cdot 3 + 3 \cdot (-1) = 0$

Piste (2, -2, 5) on siis tasossa.

**56.** $A(1, 1, 3)$
Suoralla piste $B(-1, -4, 1)$.

Tason suuntavektori $\overline{s}_1 = \overline{AB} = -2\overline{i} - 5\overline{j} - 2\overline{k}$ .

Valitaan suoran suuntavektori tason toiseksi suuntavektoriksi
$\overline{s}_2 = -\overline{i} + 3\overline{j} - 6\overline{k}$

Tason vektoriyhtälö

$$
\begin{aligned}
\overline{OP} &= \overline{OA} + t\overline{s}_1 + r\overline{s}_2 \\
&= \overline{i} + \overline{j} + 3\overline{k} + t(-2\overline{i} - 5\overline{j} - 2\overline{k}) + r(-\overline{i} + 3\overline{j} - 6\overline{k}) \\
&= (1 - 2t - r)\overline{i} + (1 - 5t + 3r)\overline{j} + (3 - 2t - 6r)\overline{k}
\end{aligned}
$$

Parametriesitys:

$$\begin{cases} x = 1-2t-r \quad \Leftrightarrow \quad r = 1-2t-x \\ y = 1-5t+3r \\ z = 3-2t-6r \end{cases}$$

Sijoitetaan $r$ = 1- 2$t$ – $x$ kahteen muuhun yhtälöön:

$$\begin{cases} y = 1-5t+3(1-2t-x) \\ z = 3-2t-6(1-2t-x) \end{cases}$$

$$\begin{cases} y = 1-5t+3-6t-3x \\ z = 3-2t-6+12t+6x \end{cases}$$

$$\begin{cases} y = 4-11t-3x \quad |\cdot 10 \\ z = -3+10t+6x \quad |\cdot 11 \end{cases}$$

$$\begin{cases} 10y = 40-110t-30x \\ 11z = -33+110t+66x \end{cases}$$

$$10y+11z = 7+36x$$

$$-36x+10y+11z-7=0$$

**57.** Sijoitetaan suoran parametriesityksestä koordinaattien lausekkeet tason yhtälöön.

$$2\cdot(-t)-(-2+t)-2(2+2t)+1=0$$

$$-2t+2-t-4-4t+1=0$$

$$-7t=1$$

$$t=-\frac{1}{7}$$

Sijoitetaan saatu $t$:n arvo suoran parametriesitykseen.

$$\begin{cases} x = -t = \frac{1}{7} \\ y = -2 + t = -2 + \left(-\frac{1}{7}\right) = -2\frac{1}{7} \\ z = 2 + 2t = 2 + 2 \cdot \left(-\frac{1}{7}\right) = 1\frac{5}{7} \end{cases}$$

**58.** Suoran parametriesitys:

$$\begin{cases} x = 2 + 2t \\ y = 2 - t \\ z = 2 + 3t \end{cases}$$

Leikkauspisteessä

$$2 + 2t - (2 - t) + 2 + 3t - 1 = 0$$
$$2 + 2t - 2 + t + 2 + 3t - 1 = 0$$
$$6t = -1$$
$$t = -\frac{1}{6}$$

Sijoitetaan saatu $t$:n arvo suoran parametriesitykseen:

$$\begin{cases} x = 2 + 2t = 2 + 2 \cdot \left(-\frac{1}{6}\right) = 1\frac{2}{3} \\ y = 2 - t = 2 - \left(-\frac{1}{6}\right) = 2\frac{1}{6} \\ z = 2 + 3t = 2 + 3 \cdot \left(-\frac{1}{6}\right) = 1\frac{1}{2} \end{cases}$$

**59.** Suoran suuntavektori

$$\bar{s}_1 = (0-3)\bar{i} + (2-2)\bar{j} + (-2-(-1))\bar{k}$$
$$= -3\bar{i} - \bar{k}$$

$$|\bar{s}_1| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$$

Pisteiden projektiopisteet *xy*-tasossa: (3, 2) ja (0, 2)
Projektiopisteiden kautta kulkeva suuntavektori:

$$\bar{s}_2 = (0-3)\bar{i} + (2-2)\bar{j} = -3\bar{i}$$

$$|\bar{s}_2| = \sqrt{(-3)^2} = 3$$

$$\bar{s}_1 \cdot \bar{s}_2 = -3 \cdot (-3) = 9$$

Lasketaan vektorien välinen kulma.

$$\cos\alpha = \frac{\bar{s}_1 \cdot \bar{s}_2}{|\bar{s}_1||\bar{s}_2|} = \frac{9}{\sqrt{10} \cdot 3} = 0{,}948...$$

$$\alpha = 18{,}43...°$$
$$\approx 18{,}4°$$

**60.** Suoran parametriesitys:

$$\begin{cases} x = 3 + t \\ y = -2 + t \\ z = 2 + 5t \end{cases}$$

Suoran suuntavektori $\bar{s} = \bar{i} + \bar{j} + 5\bar{k}$ .

Suoran ja tason välinen kulma saadaan suuntavektorin ja normaalivektorin välisestä kulmasta.

$$|\bar{s}| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}$$

$$|\bar{n}| = \sqrt{(-4)^2 + 2^2 + 1^2} = \sqrt{21}$$

$$\bar{s} \cdot \bar{n} = 1 \cdot (-4) + 1 \cdot 2 + 5 \cdot 1 = 3$$

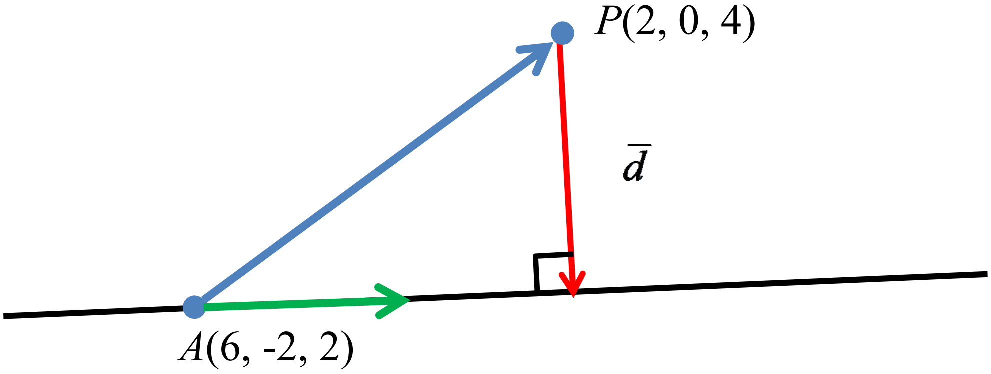
$$\cos\alpha = \frac{\bar{s} \cdot \bar{n}}{|\bar{s}||\bar{n}|} = \frac{3}{\sqrt{27}\sqrt{21}} = 0{,}125...$$

$$\alpha = 82{,}76...° \approx 83°$$

Koska kulma on terävä, suoran ja tason välinen kulma on

$$90° - 83° = 7°$$

**61.**



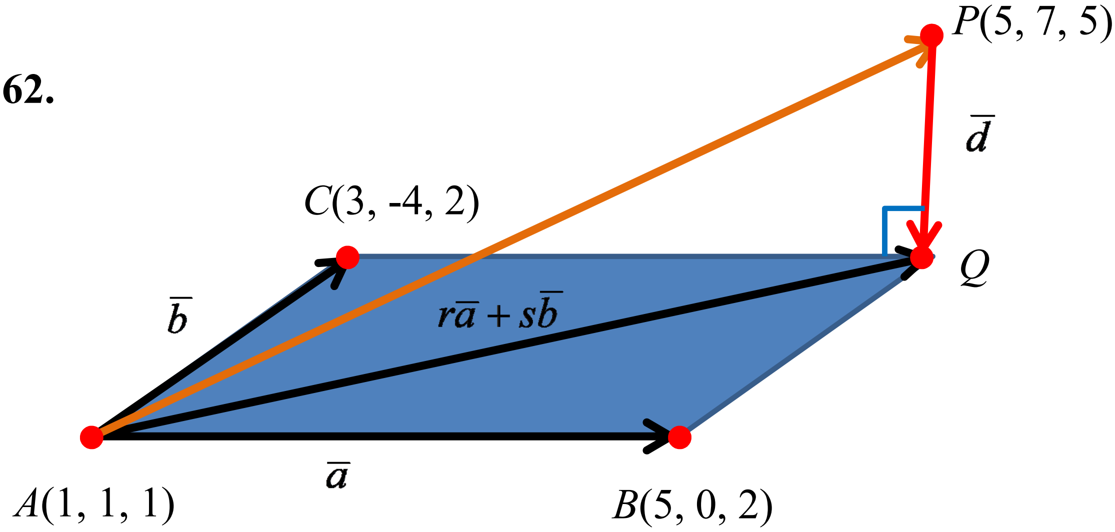

$$\begin{aligned}\bar{d} &= \overline{PA} + t\bar{s} \\ &= 4\bar{i} - 2\bar{j} - 2\bar{k} + t(\bar{i} + 2\bar{j} - \bar{k}) \\ &= (4+t)\bar{i} + (-2+2t)\bar{j} + (-2-t)\bar{k}\end{aligned}$$

$$\begin{aligned}\bar{d} \cdot \bar{s} &= 0 \\ (4+t)\cdot 1 + (-2+2t)\cdot 2 + (-2-t)\cdot(-1) &= 0 \\ 4 + t - 4 + 4t + 2 + t &= 0 \\ 6t &= -2 \\ t &= -\frac{1}{3}\end{aligned}$$

$$\begin{aligned}\bar{d} &= (4 - \frac{1}{3})\bar{i} + (-2 + 2\cdot\left(-\frac{1}{3}\right))\bar{j} + (2 - \left(-\frac{1}{3}\right))\bar{k} \\ &= \frac{11}{3}\bar{i} - \frac{8}{3}\bar{j} - \frac{5}{3}\bar{k}\end{aligned}$$

$$|\bar{d}| = \sqrt{\left(\frac{11}{3}\right)^2 + \left(-\frac{8}{3}\right)^2 + \left(-\frac{5}{3}\right)^2} = \sqrt{\frac{210}{9}} = \frac{\sqrt{210}}{3}$$

**62.**




$$\overline{PA} = -4\bar{i} - 6\bar{j} - 4\bar{k}$$
$$\bar{a} = \overline{AB} = 4\bar{i} - \bar{j} + \bar{k}$$
$$\bar{b} = \overline{AC} = 2\bar{i} - 5\bar{j} + \bar{k}$$

$$\begin{aligned}\bar{d} &= \overline{PA} + \overline{AQ} \\ &= \overline{PA} + r\bar{a} + s\bar{b} \\ &= -4\bar{i} - 6\bar{j} - 4\bar{k} + r(4\bar{i} - \bar{j} + \bar{k}) + s(2\bar{i} - 5\bar{j} + \bar{k}) \\ &= (-4 + 4r + 2s)\bar{i} + (-6 - r - 5s)\bar{j} + (-4 + r + s)\bar{k}\end{aligned}$$

$$\begin{cases}\bar{d} \perp \bar{a} \\ \bar{d} \perp \bar{b}\end{cases} \Leftrightarrow \begin{cases}\bar{d} \cdot \bar{a} = 0 \\ \bar{d} \cdot \bar{b} = 0\end{cases}$$

$$\begin{cases}4 \cdot (-4 + 4r + 2s) - 1 \cdot (-6 - r - 5s) + 1 \cdot (-4r + s) = 0 \\ 2 \cdot (-4 + 4r + 2s) - 5(-6 - r - 5s) + 1 \cdot (-4 + r + s) = 0\end{cases}$$

$$\begin{cases}18r + 14s = 14 & |:2 \\ 14r + 30s = -18 & |:2\end{cases}$$

$$\begin{cases}9r + 7s = 7 & |\cdot 7 \\ 7r + 15s = -9 & |\cdot(-9)\end{cases}$$

$$\begin{cases} 63r + 49s = 49 \\ -63r - 135s = 81 \end{cases}$$

$$-86s = 130$$

$$s = -\frac{130}{86} = -\frac{65}{43}$$

$$7r + 15 \cdot \left(-\frac{65}{43}\right) = -9$$

$$7r = \frac{588}{43}$$

$$r = \frac{84}{43}$$

Sijoitetaan saadut arvot vektorin $\bar{d}$ lausekkeeseen.

$$\bar{d} = (-4 + 4 \cdot \frac{84}{43} + 2 \cdot \left(-\frac{65}{43}\right))\bar{i} + (-6 - \frac{84}{43} - 5 \cdot \left(-\frac{65}{43}\right))\bar{j} + (-4 + \frac{84}{43} + \left(-\frac{65}{43}\right))\bar{k}$$

$$= \frac{34}{43}\bar{i} - \frac{17}{43}\bar{j} - \frac{153}{43}\bar{k}$$

Lasketaan vektorin pituus.

$$|\bar{d}| = \sqrt{\left(\frac{34}{43}\right)^2 + \left(-\frac{17}{43}\right)^2 + \left(-\frac{153}{43}\right)^2}$$

$$= \sqrt{\frac{578}{43}}$$

$$\approx 3{,}7$$

**63.** Pohjaympyrän ala $A_p = \pi \cdot 5^2$

Tason $x - 2y + 2z - 2 = 0$ normaalivektori:

$$\overline{n} = \overline{i} - 2\overline{j} + 2\overline{k}$$

Huippu pisteessä $P(2, -3, 3)$.

Merkitään kirjaimella $Q$ pistettä $P$ lähinnä olevaa tason pistettä.

Normaalin parametriesitys:

$$\begin{cases} x = 2 + t \\ y = -3 - 2t \\ z = 3 + 2t \end{cases}$$

Lasketaan normaalin ja tason leikkauspiste sijoittamalla koordinaattien $x$, $y$, $z$ lausekkeet tason yhtälöön.

$$\begin{aligned} (2 + t) - 2(-3 - 2t) + 2(3 + 2t) - 2 &= 0 \\ 2 + t + 6 + 4t + 6 + 4t - 2 &= 0 \\ 9t &= -12 \\ t &= -\frac{12}{9} = -\frac{4}{3} \end{aligned}$$

Leikkauspiste on siis $Q = (x, y, z) = (2 + t, -3 - 2t, 3 + 2t)$, jossa $t = -\frac{4}{3}$.

Tällöin

$$\begin{aligned}\overline{PQ} &= (x-2)\overline{i} + (y+3)\overline{j} + (z-3)\overline{k} \\ &= (2+t-2)\overline{i} + (-3-2t+3)\overline{j} + (3+2t-3)\overline{k} \\ &= t\,\overline{i} - 2t\,\overline{j} + 2t\,\overline{k} \\ &= t\left(\overline{i} - 2\overline{j} + 2\overline{k}\right) \\ &= -\frac{4}{3}\left(\overline{i} - 2\overline{j} + 2\overline{k}\right)\end{aligned}$$

Kartion korkeus on

$$h = \left|\overline{PQ}\right| = \frac{4}{3}\sqrt{1^2 + (-2)^2 + 2^2} = \frac{4}{3}\sqrt{9} = \frac{4}{3}\cdot 3 = 4$$

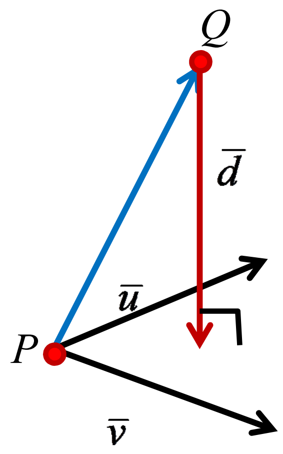
Kartion tilavuus

$$\begin{aligned}V &= \frac{1}{3}A_p \cdot h \\ &= \frac{1}{3}\pi \cdot 5^2 \cdot 4 \\ &= \frac{100\pi}{3}\end{aligned}$$

**64.** $\bar{u} = 4\bar{i} - \bar{j} - \bar{k}$

$\bar{v} = 2\bar{i} + 3\bar{j} + \bar{k}$

$\overline{PQ} = -3\bar{i} - 2\bar{j} + 4\bar{k}$




Etäisyysvektori

$$\begin{aligned}\bar{d} &= \overline{QP} + r\bar{u} + s\bar{v} \\ &= -\left(-3\bar{i} - 2\bar{j} + 4\bar{k}\right) + r(4\bar{i} - \bar{j} - \bar{k}) + s\left(2\bar{i} + 3\bar{j} + \bar{k}\right) \\ &= (3 + 4r + 2s)\bar{i} + (2 - r + 3s)\bar{j} + (-4 - r + s)\bar{k}\end{aligned}$$

Kohtisuoruusehdosta saadaan:

$$\begin{cases} \bar{d} \cdot \bar{u} = 0 \\ \bar{d} \cdot \bar{v} = 0 \end{cases}$$

$$\begin{cases} 4(3 + 4r + 2s) - (2 - r + 3s) - (-4 - r + s) = 0 \\ 2(3 + 4r + 2s) + 3(2 - r + 3s) + (-4 - r + s) = 0 \end{cases}$$

$$\begin{cases} 12 + 16r + 8s - 2 + r - 3s + 4 + r - s = 0 \\ 6 + 8r + 4s + 6 - 3r + 9s - 4 - r + s = 0 \end{cases}$$

$$\begin{cases} 18r + 4s + 14 = 0 & |\cdot(-7) \\ 4r + 14s + 8 = 0 & |\cdot 2 \end{cases}$$

$$\begin{cases} -126r - 28s - 98 = 0 \\ 8r + 28s + 16 = 0 \end{cases}$$

$$-118r - 82 = 0$$

$$r = -\frac{82}{118} = -\frac{41}{59}$$

Sijoitetaan ylempään yhtälöön.

$$18\cdot\left(-\frac{41}{59}\right)+4s+14=0$$

$$4s=-\frac{88}{59}$$

$$s=-\frac{22}{59}$$

Etäisyysvektori

$$\bar{d}=\left(3+4\cdot\left(-\frac{41}{59}\right)+2\cdot\left(-\frac{22}{59}\right)\right)\bar{i}+\left(2-\left(-\frac{41}{59}\right)+3\cdot\left(-\frac{22}{59}\right)\right)\bar{j}$$

$$+\left(-4-\left(-\frac{41}{59}\right)+\left(-\frac{22}{59}\right)\right)\bar{k}$$

$$=-\frac{31}{59}\bar{i}+\frac{93}{59}\bar{j}-\frac{217}{59}\bar{k}$$

Pisteen $Q$ etäisyys tasosta

$$\left|\bar{d}\right|=\sqrt{\left(-\frac{31}{59}\right)^2+\left(\frac{93}{59}\right)^2+\left(-\frac{217}{59}\right)^2}$$

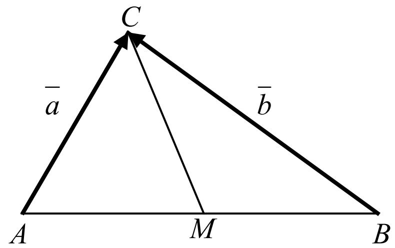
$$=\sqrt{\frac{961}{59}}$$

$$=4{,}035...$$

$$\approx 4{,}04$$

## Testi 1

1.




a) Vektori $\overline{AB}$ saadaan, kun kuljetaan ensin vektori $\overline{a}$ ja sitten vektorin $\overline{b}$ vastavektori $-\overline{b}$.

Siis $\overline{AB} = \overline{a} - \overline{b}$

b) Piste $M$ puolittaa vektorin $\overline{AB}$, joten $\overline{AM} = \frac{1}{2}\overline{AB}$.

$$\begin{aligned}
\overline{CM} &= \overline{CA} + \overline{AM} \\
&= -\overline{a} + \frac{1}{2}\overline{AB} \\
&= -\overline{a} + \frac{1}{2}(\overline{a} - \overline{b}) \\
&= -\overline{a} + \frac{1}{2}\overline{a} - \frac{1}{2}\overline{b} \\
&= -\frac{1}{2}\overline{a} - \frac{1}{2}\overline{b}
\end{aligned}$$

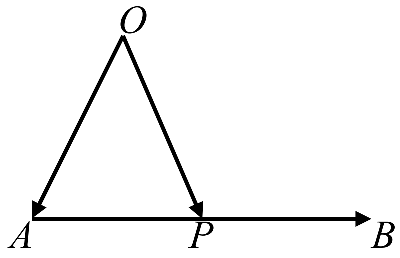
Vastaus a) $\overline{AB} = \overline{a} - \overline{b}$ b) $\overline{CM} = -\frac{1}{2}\overline{a} - \frac{1}{2}\overline{b}$

2. $A = (-1, 2, 4)$ ja $B = (3, -4, -8)$

a) $\overline{AB} = (3-(-1))\bar{i} + (-4-2)\bar{j} + (-8-4)\bar{k} = 4\bar{i} - 6\bar{j} - 12\bar{k}$

b) Keskipisteen $P$ paikkavektori $\overline{OP}$:

$$\begin{aligned}\overline{OP} &= \overline{OA} + \frac{1}{2}\overline{AB} \\ &= -\bar{i} + 2\bar{j} + 4\bar{k} + \frac{1}{2}(4\bar{i} - 6\bar{j} - 12\bar{k}) \\ &= -\bar{i} + 2\bar{j} + 4\bar{k} + 2\bar{i} - 3\bar{j} - 6\bar{k} \\ &= \bar{i} - \bar{j} - 2\bar{k}\end{aligned}$$




Keskipiste $P = (1, -1, -2)$

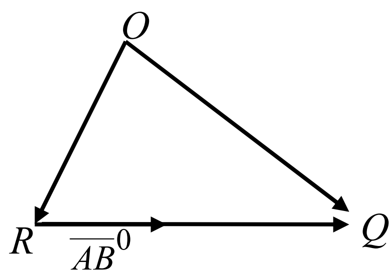
c) Vektorin $\overline{AB}$ pituus on

$$|\overline{AB}| = \sqrt{4^2 + (-6)^2 + (-12)^2} = \sqrt{196} = 14$$

$\overline{AB}$:n suuntainen yksikkövektori on

$$\overline{AB}^0 = \frac{\overline{AB}}{|\overline{AB}|} = \frac{4\bar{i} - 6\bar{j} - 12\bar{k}}{14} = \frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} - \frac{6}{7}\bar{k}$$

Lähdetään pisteestä $R = (2, 5, -3)$ ja kuljetaan 15 yksikköä vektorin $\overline{AB}$ suuntaan, jolloin päädytään kysyttyyn pisteeseen $Q$. Muodostetaan pisteen $Q$ paikkavektori:



$$\overline{OQ} = \overline{OR} + 15\overline{AB}^0 = 2\bar{i} + 5\bar{j} - 3\bar{k} + 15(\frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} - \frac{6}{7}\bar{k})$$

$$= 2\bar{i} + 5\bar{j} - 3\bar{k} + \frac{30}{7}\bar{i} - \frac{45}{7}\bar{j} - \frac{90}{7}\bar{k}$$

$$= 6\frac{2}{7}\bar{i} - 1\frac{3}{7}\bar{j} - 15\frac{6}{7}\bar{k}$$

Kysytty piste on siis $\left(6\frac{2}{7}, -1\frac{3}{7}, -15\frac{6}{7}\right)$

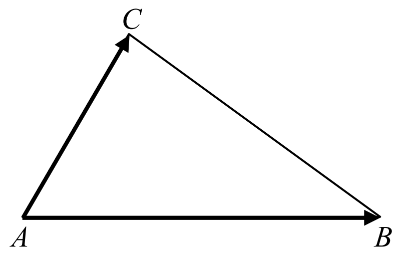
Vastaus a) $\overline{AB} = 4\bar{i} - 6\bar{j} - 12\bar{k}$

b) $\overline{OP} = \bar{i} - \bar{j} - 2\bar{k}$ ja keskipiste on (1, -1, -2)

c) $\overline{AB}^0 = \frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} - \frac{6}{7}\bar{k}$ ja $\left(6\frac{2}{7}, -1\frac{3}{7}, -15\frac{6}{7}\right)$

3. $\overline{AB} = 4\bar{i} - \bar{j} + 3\bar{k}$

$\overline{AC} = \bar{i} - 2\bar{j} + \bar{k}$




a)

$$\overline{BC} = \overline{BA} + \overline{AC}$$

$$= -(4\bar{i} - \bar{j} + 3\bar{k}) + \bar{i} - 2\bar{j} + \bar{k}$$

$$= -3\bar{i} - \bar{j} - 2\bar{k}$$

b) Vektorin $\overline{BC}$ pituus

$$\left|\overline{BC}\right| = \sqrt{(-3)^2 + (-1)^2 + (-2)^2} = \sqrt{14}$$

c) Kulma $ACB$ voidaan laskea vektorien $\overline{CA}$ ja $\overline{CB}$ välisenä kulmana eli yhtä hyvin vektorien $\overline{AC}$ ja $\overline{BC}$ välisenä kulmana.

$$\cos(\overline{AC},\overline{BC}) = \frac{\overline{AC}\cdot\overline{BC}}{|\overline{AC}||\overline{BC}|} = \frac{1\cdot(-3)+(-2)\cdot(-1)+1\cdot(-2)}{\sqrt{1^2+(-2)^2+1^2}\cdot\sqrt{14}}$$

$$\cos(\overline{AC},\overline{BC}) = \frac{-3}{\sqrt{6}\sqrt{14}} \quad \Big|\cos^{-1}$$

$$\angle(\overline{AC},\overline{BC}) = 109{,}10...^\circ \approx 109^\circ$$

Vastaus
a) $\overline{BC} = -3\overline{i} - \overline{j} - 2\overline{k}$
b) $|\overline{BC}| = \sqrt{14}$
c) $109°$

4.

a) Vektorin $\overline{u} = a\cdot\overline{i} + \overline{j} - 2\overline{k}$ pituus on

$$|\overline{u}| = \sqrt{a^2+1^2+(-2)^2} = \sqrt{a^2+5}$$

Pituus on $\sqrt{7}$, joten

$$\sqrt{a^2+5} = \sqrt{7} \quad \Big|()^2$$ Molemmat puolet ovat positiiviset

$$a^2+5=7$$
$$a^2=2$$
$$a=\pm\sqrt{2}$$

b) Jaetaan vektori $\overline{x} = 8\overline{i} - 5\overline{j}$ vektorien $\overline{a} = 4\overline{i} + 3\overline{j}$ ja $\overline{b} = 3\overline{i} - 2\overline{j}$ suuntaisiin komponentteihin etsimällä sellaiset luvut *r* ja *s*, että

$$\overline{x} = r\overline{a} + s\overline{b}$$
$$8\overline{i} - 5\overline{j} = r(4\overline{i} + 3\overline{j}) + s(3\overline{i} - 2\overline{j})$$
$$8\overline{i} - 5\overline{j} = 4r\,\overline{i} + 3r\,\overline{j} + 3s\,\overline{i} - 2s\,\overline{j}$$
$$8\overline{i} - 5\overline{j} = \overline{i}\,(4r + 3s) + \overline{j}(3r - 2s)$$

Komponentteihin jako on yksikäsitteinen, joten

$$\begin{cases} 4r + 3s = 8 & \Big| \cdot 2 \\ 3r - 2s = -5 & \Big| \cdot 3 \end{cases} \Leftrightarrow \begin{cases} 8r + 6s = 16 \\ 9r - 6s = -15 \end{cases}$$

$$+ \quad \overline{\phantom{9r - 6s = -15}}$$

$$17r = 1$$
$$r = \frac{1}{17}$$

$$4 \cdot \frac{1}{17} + 3s = 8$$
$$3s = \frac{132}{17} \quad \Big| :3$$
$$s = \frac{44}{17} = 2\frac{10}{17}$$

Siis $\overline{x} = \frac{1}{17}\overline{a} + 2\frac{10}{17}\overline{b}$

Vastaus a) $a = \pm\sqrt{2}$ b) $\overline{x} = \frac{1}{17}\overline{a} + 2\frac{10}{17}\overline{b}$

5.

a) Vektorit $\overline{a} = k\,\overline{u} - 5\overline{v}$ ja $\overline{b} = -\frac{1}{3}\overline{u} + k\,\overline{v}$ ovat samansuuntaiset, jos $\overline{a} = t\,\overline{b}$, missä $t > 0$.

$$k\,\overline{u} - 5\overline{v} = t\left(-\frac{1}{3}\overline{u} + k\,\overline{v}\right)$$

$$k\,\overline{u} - 5\overline{v} = -\frac{1}{3}t\,\overline{u} + kt\,\overline{v}$$

Komponentteihin jako on yksikäsitteinen, joten

$$\begin{cases} -\frac{1}{3}t = k \\ kt = -5 \end{cases}$$

Käytetään sijoitusmenetelmää.

$$-\frac{1}{3}t \cdot t = -5$$

$$-\frac{1}{3}t^2 = -5 \;\Big|\cdot(-3)$$

$$t^2 = 15$$

$$t = \pm\sqrt{15}$$

Vain negatiivinen $t$:n arvo kelpaa eli $t = -\sqrt{15}$.

$$k = -\frac{1}{3}(-\sqrt{15}) = \frac{1}{3}\sqrt{15}$$

b) Vektorit $\bar{x} = \bar{i} - t\,\bar{j} - \bar{k}$ ja $\bar{y} = t\,\bar{i} + 2t\,\bar{j} - \bar{k}$ ovat kohtisuorassa, jos $\bar{x} \cdot \bar{y} = 0$

$$1 \cdot t + (-t) \cdot 2t + (-1) \cdot (-1) = 0$$

$$-2t^2 + t + 1 = 0$$

$$t = \frac{-1 \pm \sqrt{1^2 - 4 \cdot (-2) \cdot 1}}{2 \cdot (-2)} = \frac{-1 \pm 3}{-4}$$

$$t = -\frac{1}{2} \text{ tai } t = 1$$

Vastaus a) $k = \frac{1}{3}\sqrt{15}$ b) $t = -\frac{1}{2}$ tai $t = 1$

6. Pisteiden (3, 7, -2) ja (-1, 4, -6) kautta kulkevan suoran suuntavektori $\bar{s}$ on
$\bar{s} = (-1-3)\bar{i} + (4-7)\bar{j} + (-6-(-2))\bar{k} = -4\bar{i} - 3\bar{j} - 4\bar{k}$

Suoran parametriesitys on

$$\begin{cases} x = x_0 + s_x t \\ y = y_0 + s_y t \ , (t \in \mathbf{R}) \\ z = z_0 + s_z t \end{cases} \qquad (3, 7, -2) = (x_0, y_0, z_0)$$

$$-4\bar{i} - 3\bar{j} - 4\bar{k} = s_x\bar{i} + s_y\bar{j} + s_z\bar{k}$$

$$\begin{cases} x = 3 - 4t \\ y = 7 - 3t \quad , (t \in \mathbf{R}) \\ z = -2 - 4t \end{cases}$$

Suora leikkaa *xy*-tason pisteessä, jossa *z* = 0.

$$-2 - 4t = 0$$
$$-4t = 2 \quad |:(-4)$$
$$t = -\frac{1}{2}$$

$$\begin{cases} x = 3 - 4 \cdot (-\frac{1}{2}) = 5 \\ y = 7 - 3 \cdot (-\frac{1}{2}) = 8\frac{1}{2} \\ z = 0 \end{cases}$$

Siis leikkauspiste on $(5,\ 8\frac{1}{2},\ 0)$.

Vastaus $\begin{cases} x = 3 - 4t \\ y = 7 - 3t \quad , (t \in \mathbf{R}) \\ z = -2 - 4t \end{cases}$ , leikkauspiste $(5,\ 8\frac{1}{2},\ 0)$

7. Pisteen ja suuntavektorin avulla saadaan suoran parametriesitys

$$\begin{cases} x = 1 + t \\ y = -2 - 2t \quad (t \in \mathbf{R}) \\ z = 5 + t \end{cases}$$

$$(1, -2, 5) = (x_0, y_0, z_0)$$

$$\bar{i} - 2\bar{j} + \bar{k} = s_x\bar{i} + s_y\bar{j} + s_z\bar{k}$$

Tason $2x + 3y - 4z = 0$ leikkauspisteessä

$$\begin{aligned} 2(1+t) + 3(-2-2t) - 4(5+t) &= 0 \\ 2 + 2t - 6 - 6t - 20 - 4t &= 0 \\ -8t &= 24 \quad \Big| : (-8) \\ t &= -3 \end{aligned}$$

Parametriesityksestä saadaan leikkauspiste

$$\begin{cases} x = 1 - 3 = -2 \\ y = -2 - 2(-3) = 4 \\ z = 5 - 3 = 2 \end{cases}$$

Eli leikkauspiste on $(-2, 4, 2)$

Vastaus $(-2, 4, 2)$

8. Muodostetaan tason vektorit
$\overline{AB} = (2-1)\bar{i} + (1-1)\bar{j} + (3-1)\bar{k} = \bar{i} + 2\bar{k}$
$\overline{AC} = (5-1)\bar{i} + (2-1)\bar{j} + (0-1)\bar{k} = 4\bar{i} + \bar{j} - \bar{k}$

Tason yhtälö normaalimuodossa voidaan muodostaa, jos tunnetaan jokin tason piste ja jokin normaalivektori.

Tason eräs normaalivektori olkoon $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$, joka on kohtisuorassa kaikkia tason vektoreita vastaan

$$\begin{cases} \overline{AB} \perp \bar{n} \\ \overline{AC} \perp \bar{n} \end{cases} \Leftrightarrow \begin{cases} \overline{AB} \cdot \bar{n} = 0 \\ \overline{AC} \cdot \bar{n} = 0 \end{cases} \Leftrightarrow \begin{cases} 1 \cdot a + 0 \cdot b + 2 \cdot c = 0 \\ 4 \cdot a + 1 \cdot b + (-1) \cdot c = 0 \end{cases}$$

$$\begin{cases} a + 2c = 0 \\ 4a + b - c = 0 \end{cases}$$

Koska kelpaa mikä tahansa normaalivektori, voidaan valita esimerkiksi $c = 1$, jolloin

$$\begin{cases} a + 2 = 0 \\ b = -4a + c \end{cases} \Leftrightarrow \begin{cases} a = -2 \\ b = -4(-2) + 1 = 9 \end{cases}$$

Eräs normaalivektori on siis $\bar{n} = -2\bar{i} + 9\bar{j} + \bar{k}$.

Tason yhtälö normaalimuodossa on

$$\begin{aligned} a(x-x_0) + b(y-y_0) + c(z-z_0) &= 0 \\ -2(x-1) + 9(y-1) + 1(z-1) &= 0 \\ -2x + 2 + 9y - 9 + z - 1 &= 0 \\ -2x + 9y + z - 8 &= 0 \end{aligned}$$

Valitaan tason pisteistä $(x_0, y_0, z_0) = (1, 1, 1)$

Koska $xz$-tasolla $y = 0$, saadaan leikkaussuoraksi

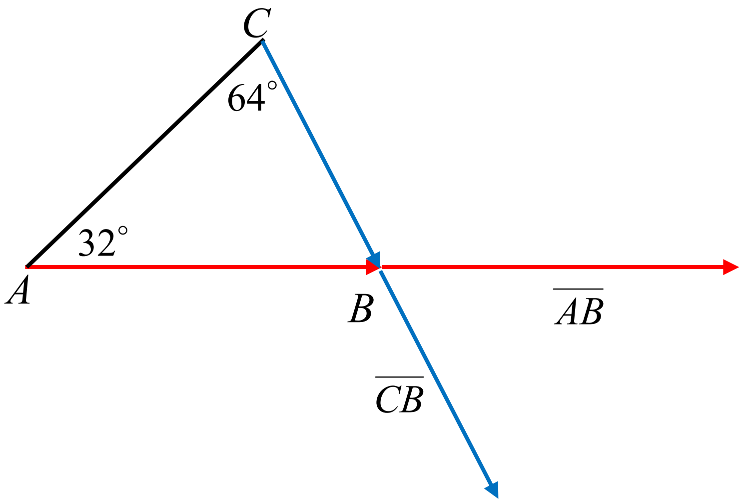
$$-2x + 9 \cdot 0 + z - 8 = 0$$
$$-2x + z - 8 = 0$$

Vastaus Tason yhtälö $-2x + 9y + z - 8 = 0$
Leikkaa $xz$-tason pitkin suoraa $-2x + z - 8 = 0$

**Testi 2**

1.

a) Vektorien $\overline{AB}$ ja $\overline{CB}$ välinen kulma on $180^\circ - 32^\circ - 64^\circ = 84^\circ$




b) $\overline{a} = 5\overline{i} + 3\overline{j}$ ja $\overline{b} = -2\overline{i} - 4\overline{j}$

Vektoreiden välinen kulma lasketaan kaavalla $\cos(\overline{a}, \overline{b}) = \dfrac{\overline{a} \cdot \overline{b}}{|\overline{a}||\overline{b}|}$

$$\overline{a} \cdot \overline{b} = 5 \cdot (-2) + 3 \cdot (-4) = -10 - 12 = -22$$

$$|\overline{a}| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

$$|\overline{b}| = \sqrt{(-2)^2 + (-4)^2} = \sqrt{20}$$

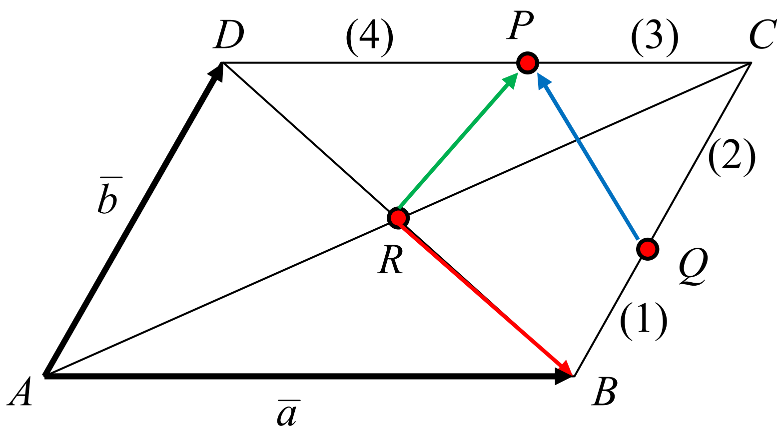
$$\cos(\overline{a}, \overline{b}) = \frac{-22}{\sqrt{34}\sqrt{20}}$$

$$\cos(\overline{a}, \overline{b}) = -0{,}843... \quad \Big| \cos^{-1}$$

$$\angle(\overline{a}, \overline{b}) = 147{,}52... \approx 148^\circ$$

Vastaus a) 84° b) 148°

2.




a)

$$\overline{QP} = \overline{QC} + \overline{CP}$$
$$= \frac{2}{3}\overline{b} - \frac{3}{7}\overline{a}$$

b)

$$\overline{RB} = \frac{1}{2}\overline{DB}$$
$$= \frac{1}{2}(-\overline{b} + \overline{a})$$
$$= \frac{1}{2}\overline{a} - \frac{1}{2}\overline{b}$$

c)

$$\overline{RP} = \overline{RB} + \overline{BC} + \overline{CP}$$
$$= \frac{1}{2}\overline{a} - \frac{1}{2}\overline{b} + \overline{b} - \frac{3}{7}\overline{a}$$
$$= \frac{1}{14}\overline{a} + \frac{1}{2}\overline{b}$$

Vastaus a) $\overline{QP} = \frac{2}{3}\overline{b} - \frac{3}{7}\overline{a}$ b) $\overline{RB} = \frac{1}{2}\overline{a} - \frac{1}{2}\overline{b}$

c) $\overline{RP} = \frac{1}{14}\overline{a} + \frac{1}{2}\overline{b}$

3. Vektori $-3\overline{a}-30\overline{b}$ jaetaan vektorien $2\overline{a}-\overline{b}$ ja $\overline{a}+4\overline{b}$ suuntaisiin komponentteihin etsimällä sellaiset luvut $r$ ja $s$, että

$$-3\overline{a}-30\overline{b}=r(2\overline{a}-\overline{b})+s(\overline{a}+4\overline{b})$$
$$-3\overline{a}-30\overline{b}=2r\overline{a}-r\overline{b}+s\overline{a}+4s\overline{b}$$
$$-3\overline{a}-30\overline{b}=\overline{a}(2r+s)+\overline{b}(-r+4s)$$

Komponentteihin jako on yksikäsitteinen, joten

$$\begin{cases} 2r+s=-3 \\ -r+4s=-30 \quad |\cdot 2 \end{cases} \Leftrightarrow + \begin{cases} 2r+s=-3 \\ -2r+8s=-60 \end{cases}$$

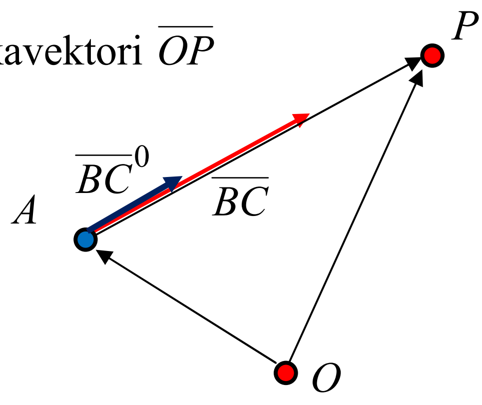
$$9s=-63 \quad |:9$$
$$s=-7$$

$$2r-7=-3$$
$$2r=4 \quad |:2$$
$$r=2$$

Siis $-3\overline{a}-30\overline{b}=2(2\overline{a}-\overline{b})-7(\overline{a}+4\overline{b})$

Vastaus $\quad -3\overline{a}-30\overline{b}=2(2\overline{a}-\overline{b})-7(\overline{a}+4\overline{b})$

4. Piste $P$ tunnetaan, kun tunnetaan sen paikkavektori $\overline{OP}$




$$\overline{OP} = \overline{OA} + \overline{AP}$$
$$= \overline{OA} + 8\overline{BC}^0$$

$$\overline{BC} = (-1-(-4))\overline{i} + (9-5)\overline{j} = 3\overline{i} + 4\overline{j}$$

$$\left|\overline{BC}\right| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$$\overline{BC}^0 = \frac{\overline{BC}}{\left|\overline{BC}\right|} = \frac{1}{5}(3\overline{i} + 4\overline{j}) = \frac{3}{5}\overline{i} + \frac{4}{5}\overline{j}$$

$$\overline{OP} = \overline{OA} + 8\overline{BC}^0$$
$$= -7\overline{i} - 5\overline{j} + 8\left(\frac{3}{5}\overline{i} + \frac{4}{5}\overline{j}\right)$$
$$= -7\overline{i} - 5\overline{j} + \frac{24}{5}\overline{i} + \frac{32}{5}\overline{j}$$
$$= -\frac{11}{5}\overline{i} + \frac{7}{5}\overline{j}$$

Siis piste $P = \left(-\frac{11}{5}, \frac{7}{5}\right)$

Vastaus $\qquad P = \left(-\frac{11}{5}, \frac{7}{5}\right)$

5.

a) $|\overline{a}| = 3$, $|\overline{b}| = 2$ ja $\overline{a} \cdot \overline{b} = -1$

$$\begin{aligned}(2\overline{a} - 3\overline{b}) \cdot (\overline{a} - 4\overline{b}) &= 2\overline{a} \cdot \overline{a} - 8\overline{a} \cdot \overline{b} - 3\overline{a} \cdot \overline{b} + 12\overline{b} \cdot \overline{b} \\ &= 2|\overline{a}|^2 - 11\overline{a} \cdot \overline{b} + 12|\overline{b}|^2 \\ &= 2 \cdot 3^2 - 11(-1) + 12 \cdot 2^2 \\ &= 77\end{aligned}$$

b) Vektorit $\overline{a} = (t-1)\overline{i} + 2\overline{j} + 3t\overline{k}$ ja $\overline{b} = 5\overline{i} + 3\overline{j} + 2t\overline{k}$ ovat toisiaan vastaan kohtisuorassa, kun

$$\overline{a} \cdot \overline{b} = 0$$

$$(t-1) \cdot 5 + 2 \cdot 3 + 3t \cdot 2t = 0$$

$$5t - 5 + 6 + 6t^2 = 0$$

$$6t^2 + 5t + 1 = 0$$

$$t = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot 1}}{2 \cdot 6} = \frac{-5 \pm 1}{12}$$

$$t = \frac{-6}{12} = -\frac{1}{2} \text{ tai } t = \frac{-4}{12} = -\frac{1}{3}$$

Vastaus a) 77 b) $t = -\frac{1}{2}$ tai $t = -\frac{1}{3}$

6.

a) Pisteiden (2, 4, -6) ja (1, -2, 3) kautta kulkevan suoran suuntavektori $\bar{s}$ on

$$\bar{s} = (1-2)\bar{i} + (-2-4)\bar{j} + (3-(-6))\bar{k} = -\bar{i} - 6\bar{j} + 9\bar{k}$$

Suoran parametriesitys on

$$\begin{cases} x = x_0 + s_x t \\ y = y_0 + s_y t \text{ , } (t \in \mathbf{R}) \\ z = z_0 + s_z t \end{cases} \qquad (2, 4, -6) = (x_0, y_0, z_0)$$

$$-\bar{i} - 6\bar{j} + 9\bar{k} = s_x\bar{i} + s_y\bar{j} + s_z\bar{k}$$

$$\begin{cases} x = 2 - t \\ y = 4 - 6t \quad (t \in \mathbf{R}) \\ z = -6 + 9t \end{cases}$$

b) *yz*-tasolla $x = 0$, joten

$$\begin{aligned} 2 - t &= 0 \\ t &= 2 \end{aligned} \quad \Rightarrow \quad \begin{cases} x = 0 \\ y = 4 - 6 \cdot 2 = -8 \\ z = -6 + 9 \cdot 2 = 12 \end{cases}$$

Siis leikkauspiste on (0, -8, 12)

c) Piste (5, 4, -8) on suoralla, jos löytyy yksikäsitteinen $t$, että

$$\begin{cases} 2 - t = 5 \\ 4 - 6t = 4 \\ -6 + 9t = -8 \end{cases} \Leftrightarrow \begin{cases} -t = 3 \\ -6t = 0 \\ 9t = -2 \end{cases} \Leftrightarrow \begin{cases} t = -3 \\ t = 0 \\ 9t = -2 \end{cases}$$ Eli piste ei ole suoralla.

Vastaus a) $\begin{cases} x = 2 - t \\ y = 4 - 6t \quad (t \in \mathbf{R}) \\ z = -6 + 9t \end{cases}$ b) (0, -8, 12) c) ei

7. Suoran $\begin{cases} x = 3 - 2t \\ y = 5 + 7t \quad (t \in \mathbf{R}) \\ z = -1 + t \end{cases}$

eräs suuntavektori on $\overline{a} = -2\overline{i} + 7\overline{j} + \overline{k}$.

a) Suoran $\dfrac{x-2}{3} = \dfrac{y+1}{5} = \dfrac{2-z}{2} \Leftrightarrow \dfrac{x-2}{3} = \dfrac{y+1}{5} = \dfrac{z-2}{-2}$

eräs suuntavektori on $\overline{b} = 3\overline{i} + 5\overline{j} - 2\overline{k}$

Suuntavektorien välinen kulma:

$$\cos(\overline{a}, \overline{b}) = \frac{\overline{a} \cdot \overline{b}}{|\overline{a}||\overline{b}|} = \frac{-2 \cdot 3 + 7 \cdot 5 + 1 \cdot (-2)}{\sqrt{(-2)^2 + 7^2 + 1^2}\sqrt{3^2 + 5^2 + (-2)^2}} = \frac{27}{\sqrt{54}\sqrt{38}}$$

$$\cos(\overline{a}, \overline{b}) = 0{,}596.. \quad \Big| \cos^{-1}$$

$$\angle(\overline{a}, \overline{b}) = 53{,}41...^\circ \approx 53{,}4^\circ$$

Koska suuntavektorien välinen kulma on terävä, on se samalla suorien välinen kulma.

b) Suoran ja tason $5x - 4y + 2z - 11 = 0$ leikkauspisteessä on voimassa

$$5(3-2t)-4(5+7t)+2(-1+t)-11=0$$
$$15-10t-20-28t-2+2t-11=0$$
$$-36t=18 \quad \Big| :(-36)$$
$$t=-\frac{1}{2}$$

Leikkauspiste saadaan suoran parametriesityksestä

$$\begin{cases} x = 3-2\cdot(-\frac{1}{2}) = 4 \\ y = 5+7\cdot(-\frac{1}{2}) = \frac{3}{2} \\ z = -1+(-\frac{1}{2}) = -\frac{3}{2} \end{cases}$$

Leikkauspiste on siis $\left(4, \frac{3}{2}, -\frac{3}{2}\right)$

Vastaus a) 53,4° b) $\left(4, \frac{3}{2}, -\frac{3}{2}\right)$

8. $\overline{u} = 3\overline{i} + 4\overline{j} + 3\overline{k}$ ja $\overline{v} = 7\overline{i} - 3\overline{j} + 44\overline{k}$

a) Tason eräs normaalivektori olkoon $\overline{n} = a\overline{i} + b\overline{j} + c\overline{k}$, joka on kohtisuorassa kaikkia tason vektoreita vastaan

$$\begin{cases} \overline{u} \perp \overline{n} \\ \overline{v} \perp \overline{n} \end{cases} \Leftrightarrow \begin{cases} \overline{u} \cdot \overline{n} = 0 \\ \overline{v} \cdot \overline{n} = 0 \end{cases} \Leftrightarrow \begin{cases} 3a + 4b + 3c = 0 \\ 7a - 3b + 44c = 0 \end{cases}$$

Koska kelpaa mikä tahansa normaalivektori, voidaan valita esimerkiksi $c = 1$, jolloin

$$\begin{cases} 3a + 4b + 3 = 0 \quad | \cdot 3 \\ 7a - 3b + 44 = 0 \quad | \cdot 4 \end{cases} \Leftrightarrow \begin{cases} 9a + 12b + 9 = 0 \\ 28a - 12b + 176 = 0 \end{cases} \quad +$$

$$37a + 185 = 0$$

$$37a = -185 \quad |:37$$

$$a = -5$$

$$9(-5) + 12b + 9 = 0$$

$$12b = 36 \quad |:12$$

$$b = 3$$

Eräs normaalivektori on siis $\overline{n} = -5\overline{i} + 3\overline{j} + \overline{k}$.

b) Tason yhtälö normaalimuodossa on

$$\begin{aligned} a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\ -5(x - 3) + 3(y - 8) + 1(z - 7) &= 0 \\ -5x + 15 + 3y - 24 + z - 7 &= 0 \\ -5x + 3y + z - 16 &= 0 \end{aligned}$$

Tason tunnettu piste on $(x_0, y_0, z_0) = (3, 8, 7)$

Vastaus a) $\overline{n} = -5\overline{i} + 3\overline{j} + \overline{k}$ b) $-5x + 3y + z - 16 = 0$