4 Kertaus

1. a) $F'(x) = D(\sqrt{1-x^2}) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}} \neq f(x)$

Eli ei ole.

b) $F'(x) = \frac{e^x \cdot x - 1 \cdot e^x}{x^2} = \frac{x-1}{x^2} e^x \neq f(x)$

Eli ei ole.

Vastaus: a) Ei b) Ei

2. $f'(x) = 2(e^x + e^{-x}) \cdot (e^x - e^{-x}) = 2\left((e^x)^2 - (e^{-x})^2\right) = 2(e^{2x} - e^{-2x})$

$g'(x) = 2(e^x - e^{-x}) \cdot (e^x + e^{-x}) = 2(e^{2x} - e^{-2x}) = f'(x) \quad \square$

3. $F(x) = \frac{A}{x} - \frac{B}{x-1}$ ja $f(x) = \frac{x^2 + 2x - 1}{x^2(x-1)^2}$

$$F'(x) = {}^{(x-1)^2)}\frac{-A}{x^2} - {}^{x^2)}\frac{-B}{(x-1)^2} = \frac{-A(x-1)^2 + Bx^2}{x^2(x-1)^2}$$

$$= \frac{-A(x^2 - 2x + 1) + Bx^2}{x^2(x-1)^2}$$

$$= \frac{(-A+B)x^2 + 2Ax - A}{x^2(x-1)^2}$$

Täten $F'(x) = f(x)$, jos

$$\begin{cases} -A + B = 1 \\ 2A = 2 \\ -A = -1 \end{cases} \Leftrightarrow \begin{cases} B = A + 1 = 2 \\ A = 1 \\ A = 1 \end{cases}$$

Vastaus: $A = 1$ ja $B = 2$

4. $$D\left(\ln\frac{2x^2+1}{3+x^3} + C\right) = D\left(\ln(2x^2+1) - \ln(3+x^3) + C\right) = \frac{4x}{2x^2+1} - \frac{3x^2}{3+x^3}$$

$\frac{4x}{2x^2+1} - \frac{3x^2}{3+x^3} = \frac{bx}{2x^2+1} - \frac{ax^2}{x^3+3}$, kun $a = 3$ ja $b = 4$

Vastaus: $a = 3$ ja $b = 4$

5. $F'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 \neq f(x)$ eli ei ole.

Funktioon F on lisättävä termi, jonka derivaatta hävittää 1:n. Tällaiseksi sopii termi $-x$.

Siis $\int \ln x \, dx = x \ln x - x + C$

Vastaus: Ei ole, esimerkiksi $x \ln x - x$ on

6. a) $\int 3x^3 \, dx = 3 \cdot \frac{1}{4} x^4 + C = \frac{3}{4} x^4 + C$

b)

$$\int \left(\frac{1}{x^2} + \frac{2}{x^3}\right) dx = \int (x^{-2} + 2x^{-3}) \, dx = -x^{-1} + 2 \cdot \left(-\frac{1}{2}\right) x^{-2} + C$$

$$= -\frac{1}{x} - \frac{1}{x^2} + C$$

c)

$$\int \left(x - \frac{1}{x}\right)^2 dx = \int \left(x^2 - 2 \cdot x \cdot \frac{1}{x} + \left(\frac{1}{x}\right)^2\right) dx = \int (x^2 - 2 + x^{-2}) \, dx$$

$$= \frac{1}{3} x^3 - 2x - x^{-1} + C$$

$$= \frac{1}{3} x^3 - 2x - \frac{1}{x} + C$$

Vastaus: a) $\frac{3}{4} x^4 + C$ b) $-\frac{1}{x} - \frac{1}{x^2} + C$ c) $\frac{1}{3} x^3 - 2x - \frac{1}{x} + C$

7. $F(x)=\int (2x-4x^3+2)\,dx = x^2 - x^4 + 2x + C$

$$\begin{aligned} F(-2) &= -2 \\ (-2)^2-(-2)^4+2(-2)+C &= -2 \\ C &= 14 \end{aligned}$$

Vastaus: $F(x)=x^2-x^4+2x+14$

8. a)

$$\int (\sqrt{x}+x)\,dx = \int (x^{\frac{1}{2}}+x)\,dx = \frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}x^2+C = \frac{2}{3}x\sqrt{x}+\frac{1}{2}x^2+C$$

b)

$$\int \left(\frac{1}{2\sqrt{x}}+\frac{x}{\sqrt[3]{x}}\right)dx = \int \left(\frac{1}{2}x^{-\frac{1}{2}}+x^{\frac{2}{3}}\right)dx = \frac{1}{2}\cdot\frac{2}{1}x^{\frac{1}{2}}+\frac{3}{5}x^{\frac{5}{3}}+C$$

$$= \sqrt{x}+\frac{3}{5}x\sqrt[3]{x^2}+C$$

Vastaus: a) $\frac{2}{3}x\sqrt{x}+\frac{1}{2}x^2+C$ b) $\sqrt{x}+\frac{3}{5}x\sqrt[3]{x^2}+C$

9. $F(x) = \int f(x)\,dx = \int (2x - 4x^3)\,dx = x^2 - x^4 + C$

$F'(x) = f(x) = 2x - 4x^3 = 2x(1 - 2x^2)$

$F'(x) = 0$

$x = 0$ tai $1 - 2x^2 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

	$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$	
$F'(x)$	+		–		+	–
$F(x)$	↗		↘		↗	↘

Testipisteet:

$F'(-2) = 28 > 0$

$F'(-0{,}5) = -0{,}5 < 0$

$F'(0{,}5) = 0{,}5 > 0$

$F'(2) = -28 < 0$

$F(\pm\frac{1}{\sqrt{2}}) = (\pm\frac{1}{\sqrt{2}})^2 - (\pm\frac{1}{\sqrt{2}})^4 + C = \frac{1}{2} - \frac{1}{4} + C = \frac{1}{4} + C$

Suurin arvo $\frac{1}{4} + C$ pitää olla -1, joten

$\frac{1}{4} + C = -1$

$C = -\frac{5}{4}$

Vastaus: $F(x) = x^2 - x^4 - \frac{5}{4}$

10. $F(x)=\int (x^{-\frac{1}{2}}-x^{-2})\,dx=\frac{2}{1}x^{\frac{1}{2}}-(-x^{-1})+C=2\sqrt{x}+\frac{1}{x}+C$

$$7x-16y+56=0$$
$$16y=7x+56$$
$$y=\frac{7}{16}x+\frac{7}{2}$$

Suoran kulmakerroin on $\frac{7}{16}$. Sivuamiskohdassa $F'(x)=\frac{7}{16}$, joten

$$f(x)=\frac{7}{16}$$
$$\frac{1}{\sqrt{x}}-\frac{1}{x^2}=\frac{7}{16}$$
$x=4$ tai $x=1{,}8058...$ (laskimella)

Jotta F:n kuvaaja sivuaa suoraa kohdassa x_0, pitää olla
$F(x_0)=y(x_0)$:

$x=4$:
$$F(4)=\frac{7}{16}\cdot 4+\frac{7}{2}$$
$$2\sqrt{4}+\frac{1}{4}+C=\frac{21}{4}$$
$$C=1$$

$x=1{,}8058...$
$$F(1{,}805...)=\frac{7}{16}\cdot 1{,}805...+\frac{7}{2}$$
$$2\sqrt{1{,}805...}+\frac{1}{1{,}805...}+C=4{,}2900...$$
$$C=1{,}0486...$$

Vastaus: $F(x)=2\sqrt{x}+\frac{1}{x}+1$ tai $F(x)\approx 2\sqrt{x}+\frac{1}{x}+1{,}049$

Huomautus: Kirjan 2. painokseen tehtävää helpotetaan siten, että kysytään integraalifunktiota, jonka kuvaaja sivuaa suoraa **kohdassa $x = 4$**. Tällöin riittää todeta, että $F'(4)=f(4)=\frac{7}{16}$, ja selvittää C samalla tavalla kuin edellä. Vastaus on siis $F(x)=2\sqrt{x}+\frac{1}{x}+1$.

11. a)

$$\int (2x+1)^{11}\,dx = \tfrac{1}{2}\int 2(2x+1)^{11}\,dx = \tfrac{1}{2}\cdot\tfrac{1}{12}(2x+1)^{12} + C$$

$$= \tfrac{1}{24}(2x+1)^{12} + C$$

b)

$$\int \frac{x}{(x^2+2)^5}\,dx = \tfrac{1}{2}\int 2x(x^2+2)^{-5}\,dx = \tfrac{1}{2}\cdot(-\tfrac{1}{4})(x^2+2)^{-4} + C$$

$$= -\frac{1}{8(x^2+2)^4} + C$$

Vastaus: a) $\tfrac{1}{24}(2x+1)^{12} + C$ b) $-\dfrac{1}{8(x^2+2)^4} + C$

12. a)

$$\int x^2(2x^3-2)^{\frac{1}{3}}\,dx = \tfrac{1}{6}\int 6x^2(2x^3-2)^{\frac{1}{3}}\,dx = \tfrac{1}{6}\cdot\tfrac{3}{4}(2x^3-2)^{\frac{4}{3}} + C$$

$$= \tfrac{1}{8}(2x^3-2)\sqrt[3]{2x^3-2} + C$$

b)

$$\int \frac{2x}{2x^2+3}\,dx = \tfrac{1}{2}\int \frac{4x}{2x^2+3}\,dx = \tfrac{1}{2}\ln\left|\underbrace{2x^2+3}_{>0}\right| + C$$

$$= \tfrac{1}{2}\ln(2x^2+3) + C$$

c)

$$\int \frac{2x}{\sqrt{2x^2+3}}\,dx = \tfrac{1}{2}\int 4x(2x^2+3)^{-\frac{1}{2}} = \tfrac{1}{2}\cdot\tfrac{2}{1}(2x^2+3)^{\frac{1}{2}} + C$$

$$= \sqrt{2x^2+3} + C$$

Vastaus:

a) $\tfrac{1}{8}(2x^3-2)\sqrt[3]{2x^3-2} + C$

b) $\tfrac{1}{2}\ln(2x^2+3) + C$

c) $\sqrt{2x^2+3} + C$

13. a)

$$\int \sqrt{x^2+x^4}\,dx = \int \sqrt{x^2(1+x^2)}\,dx = \int x\sqrt{1+x^2}\,dx$$

$$= \tfrac{1}{2}\int 2x(1+x^2)^{\frac{1}{2}}\,dx$$

$$= \tfrac{1}{2}\cdot\tfrac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

$$= \tfrac{1}{3}(1+x^2)\sqrt{1+x^2} + C$$

b)

$$\int \left(\frac{1}{x^2-1} - {}^{x+1)}\,\frac{1}{x-1}\right)dx = \int \frac{1-x-1}{x^2-1}\,dx = \int \frac{-x}{x^2-1}\,dx$$

$$= -\tfrac{1}{2}\int \frac{2x}{x^2-1}\,dx$$

$$= -\tfrac{1}{2}\ln\left|x^2-1\right| + C \quad (x>1)$$

$$= -\tfrac{1}{2}\ln(x^2-1) + C$$

Vastaus: a) $\tfrac{1}{3}(1+x^2)\sqrt{1+x^2} + C$ b) $-\tfrac{1}{2}\ln(x^2-1) + C$

14. a)

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx = \ln|\ln x| + C \quad (x > 0,\ x \neq 1)$$

b)

$$\int \frac{\ln x}{x}\,dx = \int \frac{1}{x}\cdot \ln x\,dx = \tfrac{1}{2}(\ln x)^2 + C$$

Vastaus: a) $\ln|\ln x| + C$ b) $\frac{1}{2}(\ln x)^2 + C$

15. $f'(x) = \dfrac{1}{2-x} - x - \frac{1}{2}$, missä $0 < x < 2$

$$f(x) = \int \left(\tfrac{1}{2-x} - x - \tfrac{1}{2}\right) dx = -\ln|2-x| - \tfrac{1}{2}x^2 - \tfrac{1}{2}x + C$$

$$= -\tfrac{1}{2}x^2 - \tfrac{1}{2}x - \ln(2-x) + C$$

$$f'(x) = 0$$

$$\tfrac{1}{2-x} - x - \tfrac{1}{2} = 0 \quad |\cdot 2(2-x)$$

$$2 - x \cdot 2(2-x) - (2-x) = 0$$

$$2x^2 - 3x = 0$$

$$x(2x-3) = 0$$

$$x = 0 \quad \text{tai} \quad x = \tfrac{3}{2}$$

	$0 \ldots \frac{3}{2}$	$\frac{3}{2} \ldots 2$	
$f'(x)$	–	+	
$f(x)$	↘	↗	

Testipisteet:
$f'(1) = -\frac{1}{2} < 0$
$f'(\frac{7}{4}) = 1\frac{3}{4} > 0$

Pienin arvo
$$f(\tfrac{3}{2}) = -\tfrac{1}{2}(\tfrac{3}{2})^2 - \tfrac{1}{2}\cdot\tfrac{3}{2} - \ln(2-\tfrac{3}{2}) + C = -\tfrac{15}{8} - \ln\tfrac{1}{2} + C = \ln 2 - \tfrac{15}{8} + C$$

$$\ln 2 - \tfrac{15}{8} + C = \tfrac{5}{8}$$

$$C = \tfrac{5}{2} - \ln 2$$

Vastaus: $f(x) = -\frac{1}{2}x^2 - \frac{1}{2}x - \ln(2-x) + \frac{5}{2} - \ln 2$

16. a)

$$F(x)=\int \frac{e^{3x}}{2}\,dx=\tfrac{1}{2}\int e^{3x}\,dx=\tfrac{1}{2}\cdot\tfrac{1}{3}\int 3e^{3x}\,dx=\tfrac{1}{6}e^{3x}+C$$

$$F(-1)=0$$

$$\tfrac{1}{6}e^{-3}+C=0$$

$$C=-\tfrac{1}{6e^{3}}$$

$$F(x)=\tfrac{1}{6}e^{3x}-\tfrac{1}{6e^{3}}$$

b)

$$F(x)=\int (x-\tfrac{1}{e^{2x}})\,dx=\int (x-e^{-2x})\,dx$$

$$=\tfrac{1}{2}x^{2}+\tfrac{1}{2}e^{-2x}+C$$

$$F(-1)=0$$

$$\tfrac{1}{2}(-1)^{2}+\tfrac{1}{2}e^{-2(-1)}+C=0$$

$$\tfrac{1}{2}+\tfrac{1}{2}e^{2}+C=0$$

$$C=-\tfrac{1}{2}-\tfrac{1}{2}e^{2}$$

$$F(x)=\tfrac{1}{2}x^{2}+\tfrac{1}{2}e^{-2x}-\tfrac{1}{2}-\tfrac{1}{2}e^{2}$$

Vastaus: a) $\tfrac{1}{6}e^{3x}-\tfrac{1}{6e^{3}}$ b) $\tfrac{1}{2}x^{2}+\tfrac{1}{2}e^{-2x}-\tfrac{1}{2}-\tfrac{1}{2}e^{2}$

17. a)

$$\int x^2 e^{x^3}\,dx = \tfrac{1}{3}\int 3x^2 e^{x^3}\,dx = \tfrac{1}{3}e^{x^3} + C$$

b)

$$\int (e^x + e^{-x})^3\,dx = \int (e^x + e^{-x})(e^{2x} + 2 + e^{-2x})\,dx$$

$$= \int (e^{3x} + 2e^x + e^{-x} + e^x + 2e^{-x} + e^{-3x})\,dx$$

$$= \int (e^{3x} + 3e^x + 3e^{-x} + e^{-3x})\,dx$$

$$= \tfrac{1}{3}e^{3x} + 3e^x - 3e^{-x} - \tfrac{1}{3}e^{-3x} + C$$

c)

$$\int -\frac{1}{1+e^x}\,dx = -\int \frac{1+e^x-e^x}{1+e^x}\,dx = -\int \left(1 - \frac{e^x}{1+e^x}\right)dx$$

$$= -\left(x - \ln\left|1+e^x\right|\right) + C$$

$$= -x + \ln(1+e^x) + C$$

Vastaus:

a) $\tfrac{1}{3}e^{x^3} + C$

b) $\tfrac{1}{3}e^{3x} + 3e^x - 3e^{-x} - \tfrac{1}{3}e^{-3x} + C$

c) $-x + \ln(1+e^x) + C$

18. a)

$$\int \cos 2x\,dx = \tfrac{1}{2}\int 2\cos 2x\,dx = \tfrac{1}{2}\sin 2x + C$$

b)

$$\int \sin x \cos 2x\,dx = \int \sin x(2\cos^2 x - 1)\,dx$$

$$= \int (2\sin x\cos^2 x - \sin x)\,dx$$

$$= 2\cdot(-\tfrac{1}{3}\cos^3 x) - (-\cos x) + C$$

$$= -\tfrac{2}{3}\cos^3 x + \cos x + C$$

c)

$$\int \frac{\tan x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\tan x\,dx = \tfrac{1}{2}\tan^2 x + C$$

Vastaus: a) $\tfrac{1}{2}\sin 2x + C$ b) $-\tfrac{2}{3}\cos^3 x + \cos x + C$ c) $\tfrac{1}{2}\tan^2 x + C$

19. $f''(x) = e^{-x}$

$$f'(x) = \int e^{-x}\,dx = -e^{-x} + C$$

$$f(x) = \int (-e^{-x} + C)\,dx = e^{-x} + Cx + D$$

$$\begin{cases} f(0) = 2 \\ f(1) = 1 \end{cases} \quad \begin{cases} e^{0} + D = 2 \\ e^{-1} + C + D = 1 \end{cases} \quad \begin{cases} D = 1 \\ C = 1 - D - \frac{1}{e} = -\frac{1}{e} \end{cases}$$

$$f(x) = e^{-x} - \tfrac{1}{e}x + 1$$

Vastaus: $f(x) = e^{-x} - \frac{1}{e}x + 1$

20.

$$F(x) = \int 4\cos^2 x\,dx = \int 4(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)\,dx = \int (2 + 2\cos 2x)\,dx$$

$$= 2x + \sin 2x + C$$

$$\begin{aligned} F(1) &= 0 \\ 2 + \sin 2 + C &= 0 \\ C &= -2 - \sin 2 \\ F(x) &= 2x + \sin 2x - 2 - \sin 2 \end{aligned}$$

Vastaus: $2x + \sin 2x - 2 - \sin 2$

21.

$$f''(x) = e^x + e^{-x}$$

$$f'(x) = \int (e^x + e^{-x})\,dx = e^x - e^{-x} + C$$

$f'(0) = \frac{3}{2}$

$$e^0 - e^0 + C = \frac{3}{2}$$

$$C = \frac{3}{2}$$

$f'(x) = e^x - e^{-x} + \frac{3}{2}$

$$f'(x) = 0$$

$$e^x - e^{-x} + \frac{3}{2} = 0 \quad \Big| \cdot 2e^x$$

$$2(e^x)^2 - 2 + 3e^x = 0$$

$$e^x = t$$

$$2t^2 + 3t - 2 = 0$$

$t = \frac{1}{2}$ tai $t = -2$

$e^x = \frac{1}{2}$ tai $e^x = -2$

$x = \ln\frac{1}{2}$ Ei ratkaisua

$x = -\ln 2$

		$-\ln 2$	
$f'(x)$	–	+	
$f(x)$	↘	↗	

Testipisteet:
$f'(-1) \approx -0{,}85 < 0$
$f'(0) = 1{,}5 > 0$

f on kasvava, kun $x \geq \ln 2$

Vastaus: $x \geq \ln 2$

22. $f(x) = 8x - x^4$

Nollakohdat:

$$8x - x^4 = 0$$
$$x(8 - x^3) = 0$$
$$x = 0 \quad \text{tai} \quad 8 - x^3 = 0$$
$$x^3 = 8 \quad \Big|\sqrt[3]{}$$
$$x = 2$$

Neljä osaväliä, osavälin pituus $\Delta x = \frac{1}{2}$, joten osavälit ovat:

$[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$ ja $[\frac{3}{2}, 2]$

Osavälien keskipisteet ovat:

$x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$, $x_3 = \frac{5}{4}$ ja $x_4 = \frac{7}{4}$

$$A \approx \sum_{i=1}^{4} \Delta x \cdot f(x_i) = \tfrac{1}{2} \cdot f(\tfrac{1}{4}) + \tfrac{1}{2} \cdot f(\tfrac{3}{4}) + \tfrac{1}{2} \cdot f(\tfrac{5}{4}) + \tfrac{1}{2} \cdot f(\tfrac{7}{4})$$
$$= \tfrac{1}{2}(8 \cdot \tfrac{1}{4} - (\tfrac{1}{4})^4 + 8 \cdot \tfrac{3}{4} - (\tfrac{3}{4})^4 + 8 \cdot \tfrac{5}{4} - (\tfrac{5}{4})^4 + 8 \cdot \tfrac{7}{4} - (\tfrac{7}{4})^4$$
$$= \tfrac{1}{2} \cdot \tfrac{5084}{256}$$
$$= 9\tfrac{119}{128} \approx 9{,}93$$

Vastaus: $9\frac{119}{128} \approx 9{,}93$

23. a)

$$\int_0^{\frac{\pi}{18}} \sin 6x\,dx = \tfrac{1}{6}\int_0^{\frac{\pi}{18}} 6\sin 6x\,dx = \tfrac{1}{6}\ \Big/_{0}^{\frac{\pi}{18}} (-\cos 6x)$$

$$= -\tfrac{1}{6}\left(\cos(6\cdot\tfrac{\pi}{18}) - \cos 0\right)$$

$$= -\tfrac{1}{6}\left(\cos\tfrac{\pi}{3} - 1\right)$$

$$= -\tfrac{1}{6}(\tfrac{1}{2} - 1)$$

$$= \tfrac{1}{12}$$

b)

$$\int_1^{\ln 2} e^{4x}\,dx = \tfrac{1}{4}\int_1^{\ln 2} 4e^{4x}\,dx = \tfrac{1}{4}\ \Big/_{1}^{\ln 2} e^{4x} = \tfrac{1}{4}(e^{4\ln 2} - e^4)$$

$$= \tfrac{1}{4}(e^{\ln 2^4} - e^4)$$

$$= \tfrac{1}{4}(16 - e^4)$$

$$= 4 - \tfrac{1}{4}e^4$$

Vastaus: a) $\frac{1}{12}$ b) $4 - \frac{1}{4}e^4$

24. a)

$$\int_0^2 2x(1-x)^2\,dx = \int_0^2 2x(1-2x+x^2)\,dx = \int_0^2 (2x-4x^2+2x^3)\,dx$$

$$= \mathop{/}\limits_0^2 (x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{2}x^4)$$

$$= 2^2 - \tfrac{4}{3}\cdot 2^3 + \tfrac{1}{2}\cdot 2^4 - 0$$

$$= 1\tfrac{1}{3}$$

b)

$$\int_{\frac{\pi}{2}}^{2\pi} \cos^4 x\,dx = \tfrac{1}{8}\int_{\frac{\pi}{2}}^{2\pi} (\cos 4x + 4\cos 2x + 3)\,dx$$

$$= \tfrac{1}{8} \mathop{/}\limits_{\frac{\pi}{2}}^{2\pi} (\tfrac{1}{4}\sin 4x + 2\sin 2x + 3x)$$

$$= \tfrac{1}{8}\left(\tfrac{1}{4}\sin 8\pi + 2\sin 4\pi + 6\pi - (\tfrac{1}{4}\sin 2\pi + 2\sin \pi + \tfrac{3\pi}{2})\right)$$

$$= \tfrac{1}{8}(0 + 0 + 6\pi - (0 + 0 + \tfrac{3\pi}{2})$$

$$= \tfrac{1}{8}(6\pi - \tfrac{3\pi}{2}) = \tfrac{9\pi}{16}$$

c)

$$\int_1^3 \frac{3x}{2x+1}\,dx = 3\int_1^3 \frac{x}{2x+1}\,dx = \tfrac{3}{2}\int_1^3 \frac{2x}{2x+1}\,dx = \tfrac{3}{2}\int_1^3 \frac{2x+1-1}{2x+1}\,dx$$

$$= \tfrac{3}{2}\int_1^3 \left(1 - \frac{1}{2x+1}\right)dx = \tfrac{3}{2}\int_1^3 \left(1 - \tfrac{1}{2}\cdot\frac{2}{2x+1}\right)dx$$

$$= \tfrac{3}{2} \mathop{/}\limits_1^3 \left(x - \tfrac{1}{2}\ln(2x+1)\right)$$

$$= \tfrac{3}{2}\left(3 - \tfrac{1}{2}\ln 7 - (1 - \tfrac{1}{2}\ln 3)\right)$$

$$= \tfrac{3}{2}(2 + \tfrac{1}{2}\ln\tfrac{3}{7})$$

$$= 3 + \tfrac{3}{4}\ln\tfrac{3}{7} \approx 2{,}36$$

Vastaus: a) $1\tfrac{1}{3}$ b) $\tfrac{9\pi}{16}$ c) $3 + \tfrac{3}{4}\ln\tfrac{3}{7}$

25.

$$\int_a^{a+1} (2x - 3ax^2)\, dx = 0$$

$$\underset{a}{/}^{a+1} (x^2 - ax^3) = 0$$

$$(a+1)^2 - a(a+1)^3 - (a^2 - a \cdot a^3) = 0$$

$$a^2 + 2a + 1 - a(a^3 + 3a^2 + 3a + 1) - a^2 + a^4 = 0$$

$$a^2 + 2a + 1 - a^4 - 3a^3 - 3a^2 - a - a^2 + a^4 = 0$$

$$-3a^3 - 3a^2 + a + 1 = 0$$

$$-3a^2(a+1) + (a+1) = 0$$

$$(a+1)(-3a^2 + 1) = 0$$

$a + 1 = 0$ tai $-3a^2 + 1 = 0$

$a = -1$ tai $a^2 = \frac{1}{3}$

$a = \pm\frac{1}{\sqrt{3}}$

Vastaus: $a = \pm\frac{1}{\sqrt{3}}$ tai $a = -1$

26. $F(x)=\int_0^x (ae^t+b)\,dt = {}_0/^x (ae^t+bt) = ae^x+bx-a$

$\begin{cases} F(1)=0 \\ F(2)=1 \end{cases}$ $\begin{cases} ae+b-a=0 \\ ae^2+2b-a=1 \end{cases}$ $\begin{cases} b=a-ae \\ ae^2+2b-a=1 \end{cases}$

$$ae^2+2(a-ae)-a=1$$
$$ae^2+2a-2ae-a=1$$
$$a(e^2-2e+1)=1$$
$$a=\frac{1}{e^2-2e+1}=\frac{1}{(e-1)^2}$$

$$b=a-ae=\frac{1}{(e-1)^2}-\frac{e}{(e-1)^2}=\frac{1-e}{(e-1)^2}=-\frac{e-1}{(e-1)^2}=-\frac{1}{e-1}$$

Vastaus: $a=\frac{1}{(e-1)^2}$ ja $b=-\frac{1}{e-1}$

27. $x \geq 2$

$$f(x) = \int_2^x (1 - \frac{9}{t^2})\,dt = \int_2^x (1 - 9t^{-2})\,dt = \mathop{/}_2^x (t + 9t^{-1}) = \mathop{/}_2^x (t + \frac{9}{t})$$

$$f(x) = x + \frac{9}{x} - (2 + \frac{9}{2})$$

$$f(x) = x + \frac{9}{x} - \frac{13}{2}$$

$$f'(x) = 1 - \frac{9}{x^2}$$

$$f'(x) = 0$$

$$\frac{9}{x^2} = 1$$

$$x^2 = 9 \quad \big|\sqrt{\ }$$

$$x = \pm 3$$

Vain $x = 3$ kelpaa

	2	3
$f'(x)$	–	+
$f(x)$	↘	↗

Testipisteet:
$f'(2{,}5) = -0{,}44 < 0$
$f'(4) = 0{,}4375 > 0$

Pienin arvo on $f(3) = 3 + \frac{9}{3} - \frac{13}{2} = -\frac{1}{2}$

Vastaus: $-\frac{1}{2}$

28. a)

$$\int_3^5 (e^{x^2} - 2x)\,dx + \int_3^5 (3 - e^{x^2})\,dx = \int_3^5 (e^{x^2} - 2x + 3 - e^{x^2})\,dx$$

$$= \int_3^5 (3 - 2x)\,dx$$

$$= \mathop{/}_{3}^{5} (3x - x^2)$$

$$= 3 \cdot 5 - 5^2 - (3 \cdot 3 - 3^2)$$

$$= -10$$

b)

$$\int_0^3 \sin^2 x\,dx - \int_3^0 (\cos^2 x + 1)\,dx = \int_0^3 \sin^2 x\,dx + \int_0^3 (\cos^2 x + 1)\,dx$$

$$= \int_0^3 (\underbrace{\sin^2 x + \cos^2 x}_{1} + 1)\,dx = \int_0^3 2\,dx$$

$$= \mathop{/}_{0}^{3} 2x$$

$$= 2 \cdot 3 - 0 = 6$$

Vastaus: a) -10 b) 6

29. $f(x) = \int_0^x (|t| - |t-1|)\,dt$

Merkitään:

$$g(t) = |t| - |t-1| = \begin{cases} -t - (-(t-1))\,, & t < 0 \\ t - (-(t-1))\,, & 0 \le t < 1 \\ t - (t-1)\,, & t \ge 1 \end{cases}$$

$$g(t) = \begin{cases} -1\,, & t < 0 \\ 2t - 1\,, & 0 \le t < 1 \\ 1\,, & t \ge 1 \end{cases}$$

Kun $-1 \le x < 0$, niin

$$f(x) = \int_0^x -1\,dt = \mathop{/}_0^x -t = -x$$

Kun $0 \le x < 1$, niin

$$f(x) = \int_0^x (2t-1)\,dt = \mathop{/}_0^x (t^2 - t) = x^2 - x$$

Kun $1 \le x \le 3$, niin

$$f(x) = \int_0^1 (2t-1)\,dt + \int_1^x 1\,dt = \mathop{/}_0^1 (t^2 - t) + \mathop{/}_0^x t = 1^2 - 1 - 0 + x - 1 = x - 1$$

$$\text{Siis } f(x) = \begin{cases} -x\,, & -1 \le x < 0 \\ x^2 - x\,, & 0 \le x < 1 \\ x - 1\,, & 1 \le x \le 3 \end{cases}$$

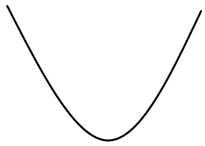
f on jatkuva funktio.

Kun $-1 \le x \le 0$, on *f* vähenevä ja pienin arvo tällä välillä on $f(0) = 0$

Kun $1 \le x \le 3$, on *f* kasvava ja pienin arvo tällä välillä on $f(1) = 0$.

Välillä $0 < x < 1$

$f'(x) = 2x - 1$
$f'(x) = 0$
$x = \frac{1}{2}$



$f(x) = x^2 - x$

Pienin arvo tällä välillä on huippukohdassa
$f(\frac{1}{2}) = (\frac{1}{2})^2 - \frac{1}{2} = -\frac{1}{4}$

Koska $-\frac{1}{4} < 0$, niin funktion *f* pienin arvo on $-\frac{1}{4}$

Koska $f(-1) = 1$ ja $f(3) = 2$, on funktion *f* suurin arvo 2

Vastaus: pienin arvo $-\frac{1}{4}$ ja suurin arvo 2

30. a) $y = \sin x + 2$ ja x-akseli rajaavat alueen.

$\sin x + 2 > 0$ kaikilla x, joten

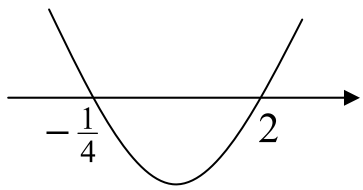
$$A = \int_0^{2\pi} (\sin x + 2)\,dx = \mathop{/}_{0}^{2\pi} (-\cos x + 2x)$$
$$= -\cos 2\pi + 2 \cdot 2\pi - (-\cos 0 + 0)$$
$$= -1 + 4\pi + 1$$
$$= 4\pi$$

b) $y = 4x^2 - 7x - 2$ ja x-akseli rajaavat alueen.

Nollakohdat:

$4x^2 - 7x - 2 = 0$

$x = -\frac{1}{4}$ tai $x = 2$ (laskimella)



$$A = -\int_{-\frac{1}{4}}^{2} (4x^2 - 7x - 2)\,dx = -\mathop{/}_{-\frac{1}{4}}^{2} (\tfrac{4}{3}x^3 - \tfrac{7}{2}x^2 - 2x)$$
$$= -\left(\tfrac{4}{3} \cdot 2^3 - \tfrac{7}{2} \cdot 2^2 - 2 \cdot 2 - (\tfrac{3}{3}(-\tfrac{1}{4})^3 - \tfrac{7}{2}(-\tfrac{1}{4})^2 - 2(-\tfrac{1}{4}))\right)$$
$$= 7\tfrac{19}{32}$$

Vastaus: a) 4π b) $7\frac{19}{32}$

31. $y=\dfrac{1+x}{x^3}$, x-akseli ja suorat $x=\frac{1}{3}$ ja $x=\frac{1}{2}$ rajoittavat alueen.

$\dfrac{1+x}{x^3}>0$, kun $x>0$, joten

$$A=\int_{\frac{1}{3}}^{\frac{1}{2}}\frac{1+x}{x^3}\,dx=\int_{\frac{1}{3}}^{\frac{1}{2}}\left(\frac{1}{x^3}+\frac{1}{x^2}\right)dx=\int_{\frac{1}{3}}^{\frac{1}{2}}\left(x^{-3}+x^{-2}\right)dx$$

$$=\ {}_{\frac{1}{3}}\!\Big/^{\frac{1}{2}}\left(-\tfrac{1}{2}x^{-2}-x^{-1}\right)$$

$$=-\tfrac{1}{2}\cdot 2^2-2-\left(-\tfrac{1}{2}\cdot 3^2-3\right)$$

$$=3\tfrac{1}{2}$$

Vastaus: $3\frac{1}{2}$

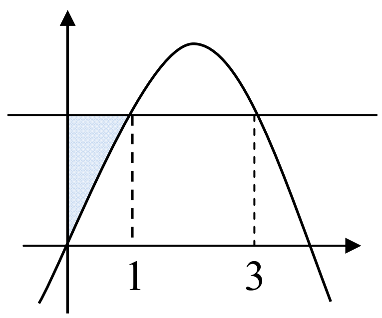
32. y-akseli, $y=8x-2x^2$ ja $y=6$ rajoittavat alueen.

Leikkauskohdat:

$8x-2x^2=6$

$-2x^2+8x-6=0$

$x=1$ tai $x=3$ (laskimella)



$$A=\int_0^1\left(6-(8x-2x^2)\right)dx=\int_0^1(2x^2-8x+6)\,dx=\ {}_0\!\Big/^{1}\left(\tfrac{2}{3}x^3-4x^2+6x\right)$$

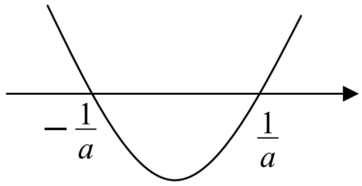
$$=\tfrac{2}{3}-4+6-0$$

$$=2\tfrac{2}{3}$$

Vastaus: $2\frac{2}{3}$

33. $y = a^3x^2 - a$, missä $a > 0$, ja x-akseli rajaavat alueen.

Nollakohdat:



$$a^3x^2 - a = 0$$
$$x^2 = \frac{1}{a^2}$$
$$x = \pm\frac{1}{a}$$

$$A = -\int_{-\frac{1}{a}}^{\frac{1}{a}} (a^3x^2 - a)\,dx = -\underset{-\frac{1}{a}}{\overset{\frac{1}{a}}{/}} \left(\frac{a^3}{3}x^3 - ax\right)$$
$$= -\left(\frac{a^3}{3}\left(\frac{1}{a}\right)^3 - a\cdot\frac{1}{a} - \left(\frac{a^3}{3}\left(-\frac{1}{a}\right)^3 - a\left(-\frac{1}{a}\right)\right)\right)$$
$$= -\left(\frac{1}{3} - 1 - \left(-\frac{1}{3} + 1\right)\right)$$
$$= 1\frac{1}{3}$$

Vastaus: $1\frac{1}{3}$

34. Suorakulmion paraabelilla oleva piste on muotoa $(a,\ a^2)$.

Suorakulmion ala on $A_1 = a \cdot a^2 = a^3$

Paraabelin $y = x^2$ alle jäävä pinta-ala:

$$A_2 = \int_0^a x^2\,dx = \underset{0}{\overset{a}{/}} \frac{1}{3}x^3 = \frac{1}{3}(a^3 - 0) = \frac{1}{3}a^3$$

Jakosuhde on siten:

$$\frac{A_2}{A_1 - A} = \frac{\frac{1}{3}a^3}{\frac{2}{3}a^3} = \frac{1}{3} : \frac{2}{3} = \frac{1}{3}\cdot\frac{3}{2} = 1:2$$

Vastaus: $1:2$

35. x-akseli, $y=\dfrac{x}{x^2+1}$ ja suorat $x=a$ ja $x=a+1$ ($a>0$) rajaavat alueen.

$$A=\int_a^{a+1}\frac{x}{x^2+1}dx=\tfrac{1}{2}\int_a^{a+1}\frac{2x}{x^2+1}dx=\tfrac{1}{2}\ \Big/_{a}^{a+1}\ln(x^2+1)$$

$$=\tfrac{1}{2}\left(\ln\left((a+1)^2+1\right)-\ln(a^2+1)\right)$$

$$=\tfrac{1}{2}\ln\frac{(a+1)^2+1}{a^2+1}$$

Merkitään $g(a)=\dfrac{(a+1)^2+1}{a^2+1}$. Ala on suurin, kun $g(a)$ on suurin.

$$g'(a)=\frac{2(a+1)\cdot(a^2+1)-\left((a+1)^2+1\right)\cdot 2a}{(a^2+1)^2}$$

$$=\frac{(2a+2)(a^2+1)-2a(a^2+2a+2)}{(a^2+1)^2}$$

$$=\frac{2a^3+2a+2a^2+2-2a^3-4a^2-4a}{(a^2+1)^2}=\frac{-2a^2-2a+2}{(a^2+1)^2}$$

$g'(a)=0$

$-2a^2-2a+2=0$

$a=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}\approx 0{,}62$ tai $\left(a=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}\approx -1{,}62\right)$

	0	$-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}$
$g'(a)$	$+$	$-$
$g(a)$	↗	↘

Testipisteet:
$g'(0{,}5)\approx 0{,}32>0$
$g'(1)=-0{,}5<0$

Suurin arvo, kun $a=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$. Tällöin

$$A=\frac{1}{2}\ln\frac{(\frac{1}{2}+\frac{1}{2}\sqrt{5})^2+1}{(-\frac{1}{2}+\frac{1}{2}\sqrt{5})^2+1}=\frac{1}{2}\ln\frac{\frac{1}{4}+\frac{1}{2}\sqrt{5}+\frac{5}{4}+1}{\frac{1}{4}-\frac{1}{2}\sqrt{5}+\frac{5}{4}+1}=\frac{1}{2}\ln\frac{\frac{5}{2}+\frac{1}{2}\sqrt{5}}{\frac{5}{2}-\frac{1}{2}\sqrt{5}}$$

$$A=\frac{1}{2}\ln\frac{5+\sqrt{5}}{5-\sqrt{5}}\approx 0{,}48$$

Vastaus: $\frac{1}{2}\ln\frac{5+\sqrt{5}}{5-\sqrt{5}}\approx 0{,}48$

36. $y=2x^2-x-3$ ja $y=x^2+x$ rajaavat alueen

Leikkauskohdat:

$2x^2-x-3=x^2+x$

$x^2-2x-3=0$

$x=-1$ tai $x=3$ (laskimella)

Kun $x=0$, niin $x^2+x>2x^2-x-3$, joten

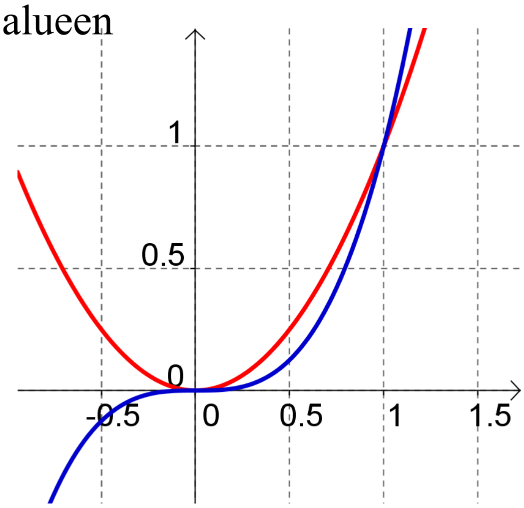
$$A=\int_{-1}^{3}(x^2+x-(2x^2-x-3))\,dx=\int_{-1}^{3}(-x^2+2x+3)\,dx$$

$$=\Big/_{-1}^{3}(-\tfrac{1}{3}x^3+x^2+3x)$$

$$=-\tfrac{1}{3}\cdot 3^3+3^2+3\cdot 3-\left(-\tfrac{1}{3}(-1)^2+(-1)^2+3(-1)\right)=10\tfrac{2}{3}$$

Vastaus: $10\frac{2}{3}$

37. Käyrät $y = x^2$ ja $y = x^3$ rajaavat alueen



Leikkauskohdat:

$$x^2 = x^3$$

$$x^3 - x^2 = 0$$

$$x^2(x-1) = 0$$

$$x = 0 \text{ tai } x = 1$$

Kun $x = \frac{1}{2}$, niin $x^2 > x^3$, joten

$$A = \int_0^1 (x^2 - x^3)\,dx = \Big/_0^1 \left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right) = \tfrac{1}{3} - \tfrac{1}{4} - 0 = \tfrac{1}{12}$$

Vastaus: $\frac{1}{12}$

38. Käyrät $y = e^{2x}$ ja $y = e^{-x}$ ja suora $y = e$ rajoittavat alueen.

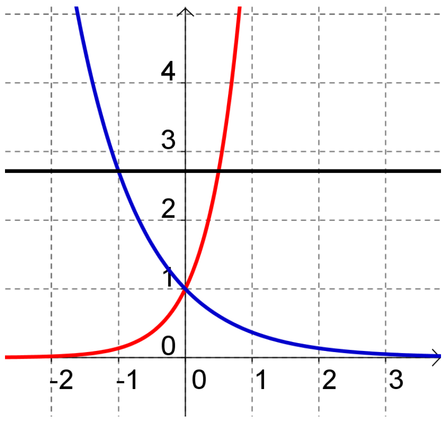
Leikkauskohdat:

$$e^{2x} = e^{-x}$$
$$2x = -x$$
$$3x = 0$$
$$x = 0$$

$$e^{2x} = e$$
$$e^{2x} = e^1$$
$$2x = 1$$
$$x = \tfrac{1}{2}$$

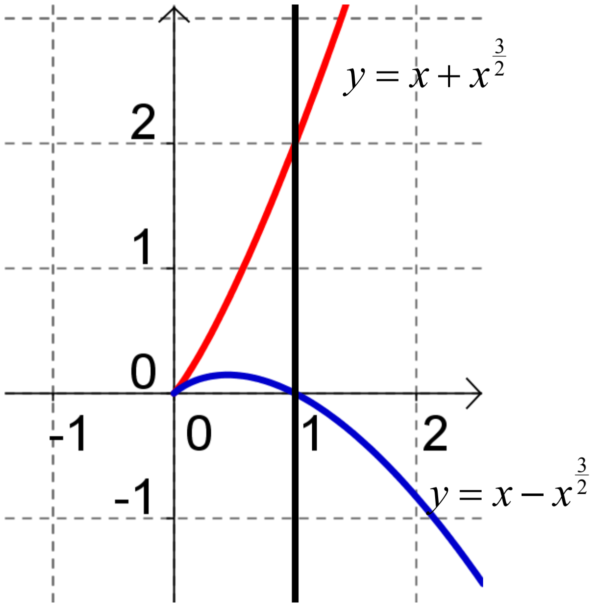
$$e^{-x} = e$$
$$e^{-x} = e^1$$
$$-x = 1$$
$$x = -1$$

$$A = \int_{-1}^{0} (e - e^{-x})\,dx + \int_{0}^{\frac{1}{2}} (e - e^{2x})\,dx$$
$$= \mathop{/}_{-1}^{0} (ex + e^{-x}) + \mathop{/}_{0}^{\frac{1}{2}} (ex - \tfrac{1}{2}e^{2x})$$
$$= 0 + e^0 - (-e + e^1) + \tfrac{1}{2}e - \tfrac{1}{2}e^1 - (0 - \tfrac{1}{2}e^0)$$
$$= 1\tfrac{1}{2}$$



Vastaus: $1\tfrac{1}{2}$

39. Käyrä $(y-x)^2 = x^3$ ja suora $x=1$ rajaavat alueen



$$(y-x)^2 = x^3$$

$$y - x = \pm x^{\frac{3}{2}}$$

$$y = x \pm x^{\frac{3}{2}} \quad , \quad x \geq 0$$

$$A = \int_0^1 (x + x^{\frac{3}{2}} - (x - x^{\frac{3}{2}}))\, dx$$

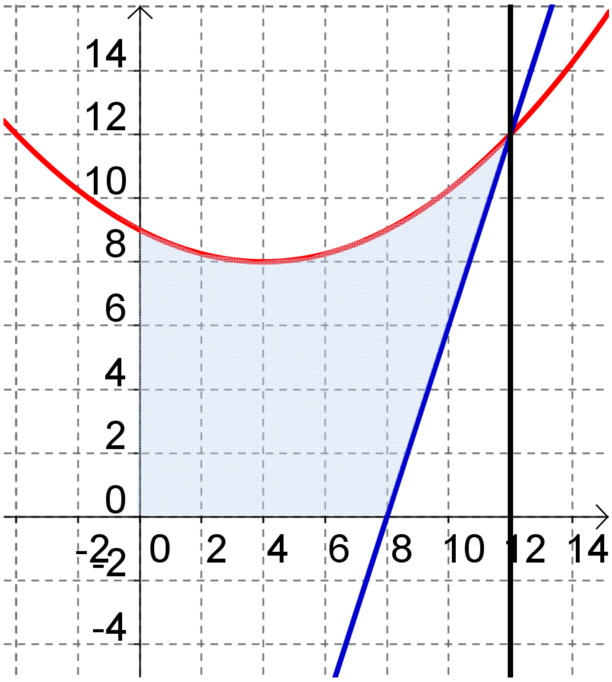
$$= \int_0^1 2x^{\frac{3}{2}}\, dx$$

$$= 2 \mathop{/}_0^1 \tfrac{2}{5} x^{\frac{5}{2}}$$

$$= 2(\tfrac{2}{5} - 0) = \tfrac{4}{5}$$

Vastaus: $\frac{4}{5}$

40. Epäyhtälöt $0 \le x \le 12$, $y \ge 0$, $y \le \frac{1}{16}(x-4)^2 + 8$ ja $y \ge 3x - 24$ rajaavat alueen.



$$A = \int_0^{12} (\tfrac{1}{16}(x-4)^2 + 8)\,dx - \int_8^{12} (3x-24)\,dx$$

$$= \Big/_0^{12} (\tfrac{1}{16} \cdot \tfrac{1}{3}(x-4)^3 + 8x) - \Big/_8^{12} (\tfrac{3}{2}x^2 - 24x)$$

$$= \tfrac{1}{48} \cdot 8^3 + 8 \cdot 12 - (\tfrac{1}{48}(-4)^3 + 0) - \left(\tfrac{3}{2} \cdot 12^2 - 24 \cdot 12 - (\tfrac{3}{2} \cdot 8^2 - 24 \cdot 8)\right)$$

$$= 84$$

Vastaus: 84

41. Käyrät $y = \frac{x^2}{a} = f(x)$ ja $y = 1 - ax^2 = g(x)$ $(a > 0)$ rajoittavat alueen.

Leikkauskohdat:

$$\frac{x^2}{a} = 1 - ax^2$$

$$ax^2 + \frac{x^2}{a} = 1$$

$$x^2(a + \tfrac{1}{a}) = 1$$

$$x^2 \cdot \frac{a^2+1}{a} = 1$$

$$x^2 = \frac{a}{a^2+1}$$

$$x = \pm\sqrt{\frac{a}{a^2+1}}$$

Merkitään $x_1 = \sqrt{\frac{a}{a^2+1}}$ ja $x_2 = -x_1 = -\sqrt{\frac{a}{a^2+1}}$

Kun $x = 0$, niin $f(0) = 0$ ja $g(0) = 1$ eli $g(x) \geq f(x)$ välillä $[x_2, x_1]$, joten

$$A = \int_{-x_1}^{x_1} \left(1 - ax^2 - \frac{x^2}{a}\right)dx = \mathop{/}_{x_2}^{x_1} (x - \tfrac{1}{3}ax^3 - \tfrac{1}{3a}x^3) = \mathop{/}_{x_2}^{x_1} (x - \tfrac{1}{3}(a + \tfrac{1}{a})x^3)$$

$$= \mathop{/}_{-x_1}^{x_1} (x - \tfrac{1}{3}\tfrac{a^2+1}{a}x^3)$$

$$= x_1 - \tfrac{1}{3}\tfrac{a^2+1}{a}x_1{}^3 - (-x_1 - \tfrac{1}{3}\tfrac{a^2+1}{a}(-x_1)^3)$$

$$= 2x_1 - \tfrac{1}{3}\tfrac{a^2+1}{a}x_1{}^3 - \tfrac{1}{3}\tfrac{a^2+1}{a}x_1{}^3$$

$$= 2x_1 - \tfrac{2}{3}\tfrac{a^2+1}{a}x_1{}^3$$

$$= 2\sqrt{\frac{a}{a^2+1}} - \frac{2}{3}\frac{a^2+1}{a}\frac{a}{a^2+1}\sqrt{\frac{a}{a^2+1}} = 2\sqrt{\frac{a}{a^2+1}} - \frac{2}{3}\sqrt{\frac{a}{a^2+1}} = \frac{4}{3}\sqrt{\frac{a}{a^2+1}}$$

Suurin arvo, kun $g(a)=\frac{a}{a^2+1}$ on suurin.

$$g'(a)=\frac{1\cdot(a^2+1)-a\cdot 2a}{(a^2+1)^2}=\frac{1-a^2}{(a^2+1)^2}$$

$$g'(a)=0$$
$$1-a^2=0$$
$$a^2=1$$
$$a=\pm 1$$

	0	1
$g'(a)$	+	–
$g(a)$	↗	↘

Testipisteet:
$g'(0{,}5)\approx 0{,}48>0$
$g'(2)\approx -0{,}12<0$

Alan suurin arvo, kun $a=1$

Vastaus: $a=1$

42. Merkitään: kolmion sivun pituus = a
kolmion korkeus = h

Korkeudella x on $a = -\frac{1}{4}x^2 + 2x + 1$

$$h^2 + (\tfrac{1}{2}a)^2 = a^2$$
$$h^2 = \tfrac{3}{4}a^2$$
$$h = \tfrac{\sqrt{3}}{2}a$$
$$h = \tfrac{\sqrt{3}}{2}(-\tfrac{1}{4}x^2 + 2x + 1)$$

Korkeudella x poikkileikkauksen ala on

$$A = \tfrac{1}{2}ah = \tfrac{1}{2}(-\tfrac{1}{4}x^2 + 2x + 1)\cdot\tfrac{\sqrt{3}}{2}(-\tfrac{1}{4}x^2 + 2x + 1)$$
$$= \tfrac{\sqrt{3}}{4}(\tfrac{1}{16}x^4 - \tfrac{1}{2}x^3 - \tfrac{1}{4}x^2 - \tfrac{1}{2}x^3 + 4x^2 + 2x - \tfrac{1}{4}x^2 + 2x + 1)$$
$$= \tfrac{\sqrt{3}}{4}(\tfrac{1}{16}x^4 - x^3 + \tfrac{7}{2}x^2 + 4x + 1)$$
$$= \tfrac{\sqrt{3}}{64}(x^4 - 16x^3 + 56x^2 + 64x + 16)$$

Tilavuus:

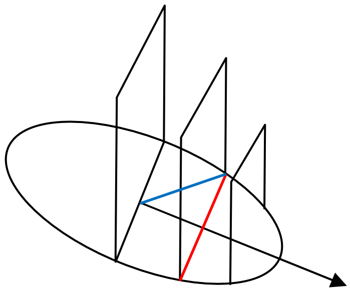
$$A = \tfrac{\sqrt{3}}{64}\int_0^8 (x^4 - 16x^3 + 56x^2 + 64x + 16)\,dx$$
$$= \tfrac{\sqrt{3}}{64}\ \underset{0}{\overset{8}{/}}\ (\tfrac{1}{5}x^5 - 4x^4 + \tfrac{56}{3}x^3 + 32x^2 + 16x)$$
$$= \tfrac{\sqrt{3}}{64}\left(\tfrac{1}{5}\cdot 8^5 - 4\cdot 8^4 + \tfrac{56}{3}\cdot 8^3 + 32\cdot 8^2 + 16\cdot 8 - 0\right)$$
$$= \tfrac{1784\sqrt{3}}{15}$$

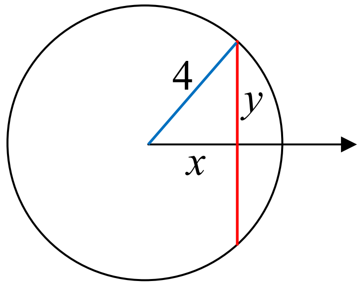
1 pituusyksikkö vastaa 5 cm luonnossa, joten 1 tilavuusyksikkö on $(5\,\text{cm})^3 = 125\ \text{cm}^3$

Tilavuus on siis $\frac{1784\sqrt{3}}{15}\cdot 125\ \text{cm}^3 \approx 26\,000\ \text{cm}^3 = 26\ \text{l}$

43. Pohjan säde 4. Poikkileikkaukset suorakulmioita, joissa etäisyydellä x ympyrän keskipisteestä, kanta on $2y$ ja korkeus $4y$.

$$x^2 + y^2 = 4^2$$
$$y^2 = 16 - x^2$$
$$y = \sqrt{16 - x^2}$$





Poikkileikkauksen ala:

$$A = 2y \cdot 4y = 8y^2$$
$$A(x) = 8(16 - x^2) = 128 - 8x^2$$

Tilavuus:

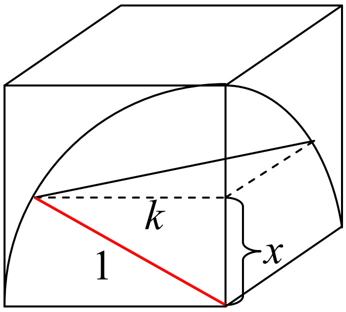
$$V = \int_{-4}^{4} (128 - 8x^2)\,dx = 2 \cdot \int_{0}^{4} (128 - 8x^2)\,dx = 2 \,\Big/_{0}^{4} (128x - \tfrac{8}{3}x^3)$$
$$= 2(128 \cdot 4 - \tfrac{8}{3} \cdot 4^3 - 0)$$
$$= 682\tfrac{2}{3}$$

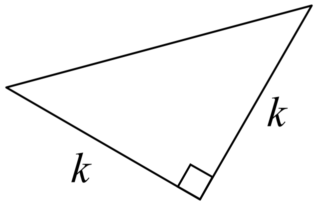
Vastaus: $682\frac{2}{3}$

44. Vaakasuuntaiset poikkileikkaukset ovat tasakylkisiä, suorakulmaisia kolmioita.

$k^2 + x^2 = 1^1$

$k^2 = 1 - x^2$





Poikkileikkauksen ala:

$A = \frac{1}{2} k \cdot k = \frac{1}{2} k^2$

$A(x) = \frac{1}{2}(1 - x^2)$

Tilavuus:

$$V = \int_0^1 \frac{1}{2}(1 - x^2)\, dx = \frac{1}{2} \,\Big/_{0}^{1} \left(x - \frac{1}{3}x^3\right) = \frac{1}{2}\left(1 - \frac{1}{3} - 0\right) = \frac{1}{3}$$

Vastaus: $\frac{1}{3}$

45. Käyrän $xy = 2$, $1 \le x \le 4$, ja x-akselin välinen alue pyörähtää x-akselin ympäri.

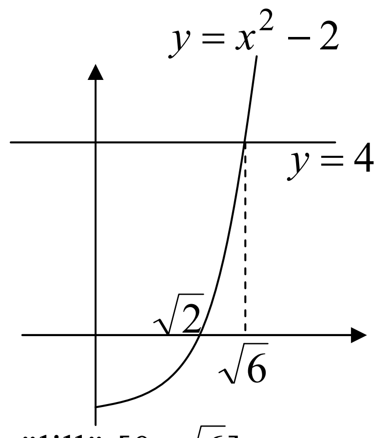
$$xy = 2$$
$$y = \tfrac{2}{x}$$
$$f(x) = \tfrac{2}{x}$$

$$V = \pi \int_1^4 f(x)^2\,dx = \pi \int_1^4 (\tfrac{2}{x})^2\,dx = \pi \int_1^4 \tfrac{4}{x^2}\,dx = 4\pi \int_1^4 x^{-2}\,dx$$
$$= 4\pi \mathop{/}_1^4 (-x^{-1})$$
$$= -4\pi \mathop{/}_1^4 \tfrac{1}{x}$$
$$= -4\pi(\tfrac{1}{4} - 1)$$
$$= 3\pi$$

Vastaus: 3π

46. Positiiviset koordinaattiakselit, suora $y = 4$ ja paraabeli $y = x^2 - 2$ rajoittavat alueen, joka pyörähtää

a) x-akselin ympäri:



Leikkauskohdat:	Nollakohdat:
$x^2 - 2 = 4$	$x^2 - 2 = 0$
$x^2 = 6$	$x^2 = 2$
$x = \pm\sqrt{6}$	$x = \pm\sqrt{2}$

Kysytty tilavuus V saadaan seuraavasti:

V_1 = suora $y = 4$ pyörähtää x-akselin ympäri välillä $[0, \sqrt{6}]$,

V_2 = paraabeli pyörähtää x-akselin ympäri välillä $[\sqrt{2}, \sqrt{6}]$

$V = V_1 - V_2$

$$V = \pi \int_0^{\sqrt{6}} 4^2 \, dx - \pi \int_{\sqrt{2}}^{\sqrt{6}} (x^2 - 2)^2 \, dx$$

$$= \pi \int_0^{\sqrt{6}} 16 \, dx - \pi \int_{\sqrt{2}}^{\sqrt{6}} (x^2 - 4x + 4) \, dx$$

$$= \pi \mathop{/}_{0}^{\sqrt{6}} 16x - \pi \mathop{/}_{\sqrt{2}}^{\sqrt{6}} (\tfrac{1}{3}x^3 - 2x^2 + 4x)$$

$$= \pi(16\sqrt{6} - 0) - \pi\left(\tfrac{1}{3} \cdot \sqrt{6}^3 - 2 \cdot \sqrt{6}^2 + 4 \cdot \sqrt{6} - (\tfrac{1}{3} \cdot \sqrt{2}^3 - 2 \cdot \sqrt{2}^2 + 4 \cdot \sqrt{2})\right)$$

$$= 16\sqrt{6}\pi - (\tfrac{16}{5}\sqrt{6} - \tfrac{32}{15}\sqrt{2})\pi$$

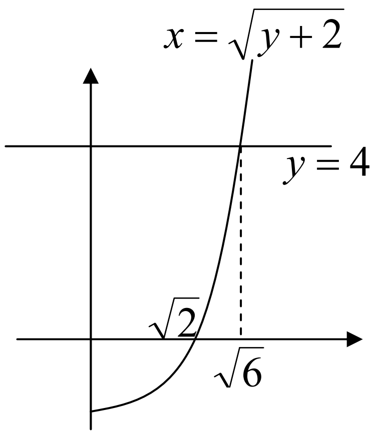
$$= \tfrac{64}{5}\sqrt{6}\pi - \tfrac{32}{15}\sqrt{2}\pi \approx 89{,}0$$

b) y-akselin ympäri
Käänteisfunktion lauseke:

$$y = x^2 - 2$$

$$x^2 = y + 2$$

$$x = \sqrt{y + 2}$$



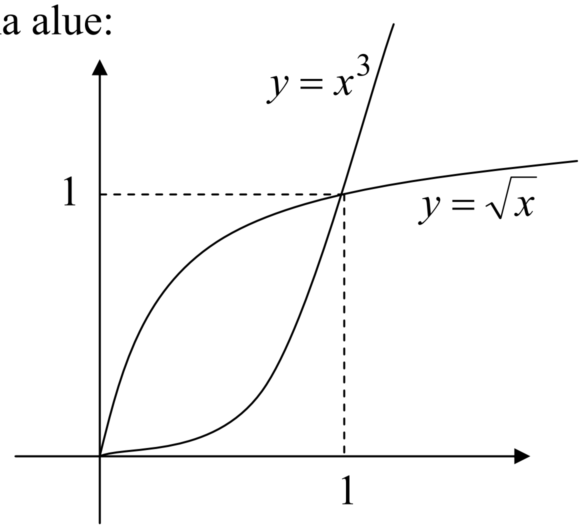
$$V = \pi \int_0^4 (\sqrt{y+2})^2 \, dy = \pi \int_0^4 (y + 2) \, dy = \pi \mathop{/}_{0}^{4} (\tfrac{1}{2}y^2 + 2y)$$

$$= \pi(\tfrac{1}{2} \cdot 4^2 + 2 \cdot 4 - 0)$$

$$= 16\pi \approx 50{,}3$$

Vastaus: a) $\frac{64}{5}\sqrt{6}\pi - \frac{32}{15}\sqrt{2}\pi \approx 89{,}0$ b) $16\pi \approx 50{,}3$

47. Käyrien $y = x^3$ ja $y = \sqrt{x}$ rajoittama alue:



Leikkauskohdat:

$$x^3 = \sqrt{x} \quad \Big|()^2$$

$$x^6 = x$$

$$x^6 - x = 0$$

$$x(x^5 - 1) = 0$$

$$x = 0 \text{ tai } x = 1$$

a) Pyörähtää x-akselin ympäri:

$$V = \pi\int_0^1 (\sqrt{x})^2\,dx - \pi\int_0^1 (x^3)^2\,dx = \pi\int_0^1 x\,dx - \pi\int_0^1 x^6\,dx$$

$$= \pi \mathop{/}_0^1 \tfrac{1}{2}x^2 - \pi \mathop{/}_0^1 \tfrac{1}{7}x^7$$

$$= \tfrac{\pi}{2}(1-0) - \tfrac{\pi}{7}(1-0)$$

$$= \tfrac{5\pi}{14}$$

b) Pyörähtää y-akselin ympäri

Käänteisfunktioiden lausekkeet:

$$y = \sqrt{x} \quad \Big|()^2 \qquad y = x^3$$

$$x = y^2 \qquad x = \sqrt[3]{y} = y^{\frac{1}{3}}$$

$$V = \pi\int_0^1 (y^{\frac{1}{3}})^2\,dy - \pi\int_0^1 (y^2)^2\,dy = \pi\int_0^1 y^{\frac{2}{3}}\,dy - \pi\int_0^1 y^4\,dy$$

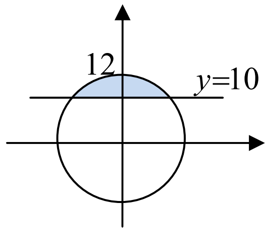
$$= \pi \mathop{/}_0^1 \tfrac{3}{5}y^{\frac{5}{3}} - \pi \mathop{/}_0^1 \tfrac{1}{5}y^5$$

$$= \tfrac{3\pi}{5}(1-0) - \tfrac{\pi}{5}(1-0)$$

$$= \tfrac{2\pi}{5}$$

Vastaus: a) $\frac{5\pi}{14}$ b) $\frac{2\pi}{5}$

48. Sormus syntyy tummennetun alueen pyörähtäessä x-akselin ympäri.



Eli suoran $y = 10$ ja ympyrän $x^2 + y^2 = 12^2$ rajaaman pienemmän segmentin pyörähtäessä x-akselin ympäri.

$$x^2 + y^2 = 12^2$$
$$y^2 = 144 - x^2$$
$$y = \sqrt{144 - x^2}$$

Leikkauskohdat:

$$\sqrt{144 - x^2} = 10 \quad \Big|\,()^2$$
$$144 - x^2 = 100$$
$$x^2 = 44 \quad \Big|\sqrt{\ }$$
$$x = \pm 2\sqrt{11}$$

$$V = \pi \int_{-2\sqrt{11}}^{2\sqrt{11}} y^2\,dx - \pi \int_{-2\sqrt{11}}^{2\sqrt{11}} 10^2\,dx = 2\pi \int_0^{2\sqrt{11}} \Big((144 - x^2) - 100\Big) dx$$
$$= 2\pi \int_0^{2\sqrt{11}} (44 - x^2)\,dx$$
$$= 2\pi \mathop{/}_{0}^{2\sqrt{11}} (44x - \tfrac{1}{3}x^3)$$
$$= 2\pi(44 \cdot 2\sqrt{11} - \tfrac{1}{3} \cdot (2\sqrt{11})^3 - 0)$$
$$= \frac{352\sqrt{11}}{3}\pi = 1222{,}55...\ \text{mm}^3 = 1{,}2225...\ \text{cm}^3$$

Tiheys $\rho = 19{,}3\ \frac{\text{g}}{\text{cm}^3}$ ja hinta 42 400 €/kg = 42,4 €/g

Sormuksen hinta on

$$19{,}3\ \tfrac{\text{g}}{\text{cm}^3} \cdot 1{,}22...\ \text{cm}^3 \cdot 42{,}4\ \tfrac{€}{\text{g}} = 1000{,}43...€ \approx 1000\,€$$

Vastaus: 1000 €

Testi 1

1. a) $h(x) = 2xe^{2x} - \sin x^2 + 3$ ja $g(x) = (4x+2)e^{2x} - 2x\cos x^2$

h on g:n integraalifunktio, jos ja vain jos $h' = g$

$$\begin{aligned} h'(x) &= 2 \cdot e^{2x} + 2x \cdot e^{2x} \cdot 2 - \cos x^2 \cdot 2x \\ &= 2e^{2x} + 4xe^{2x} - 2x\cos x^2 \\ &= (4x+2)e^{2x} - 2x\cos^2 x \\ &= g(x) \end{aligned}$$

b)

$$\begin{aligned} \int (2x^2 - 3x - \tfrac{1}{x})\,dx &= 2 \cdot \tfrac{1}{3}x^3 - 3 \cdot \tfrac{1}{2}x^2 - \ln|x| + C \\ &= \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2 - \ln x + C \end{aligned}$$

c)

$$\begin{aligned} \int_0^4 (2-\sqrt{x})^2\,dx &= \int_0^4 (4 - 4\sqrt{x} + x)\,dx = \int_0^4 (4 - 4x^{\frac{1}{2}} + x)\,dx \\ &= \mathop{/}_0^4 (4x - 4 \cdot \tfrac{2}{3}x^{\frac{3}{2}} + \tfrac{1}{2}x^2) \\ &= \mathop{/}_0^4 (4x - \tfrac{8}{3}x\sqrt{x} + \tfrac{1}{2}x^2) \\ &= 4 \cdot 4 - \tfrac{8}{3} \cdot 4 \cdot \sqrt{4} + \tfrac{1}{2} \cdot 4^2 - 0 \\ &= 16 - \tfrac{64}{3} + 8 \\ &= \tfrac{8}{3} \end{aligned}$$

Vastaus: a) b) $\frac{2}{3}x^3 - \frac{3}{2}x^2 - \ln x + C$ c) $\frac{8}{3}$

2. a)

$$\int_1^3 (3a - 3x^2)\,dx = 12$$

$$\Big/_1^3 (3ax - x^3) = 12$$

$$3a \cdot 3 - 3^3 - (3a - 1^3) = 12$$

$$6a - 26 = 12$$

$$6a = 38$$

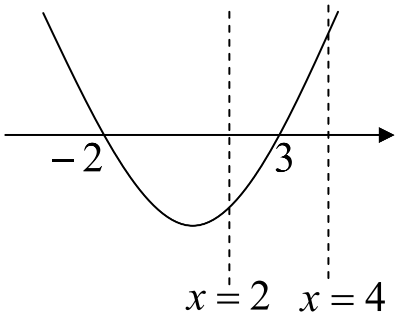
$$a = \tfrac{19}{3}$$

b) Paraabeli $y = x^2 - x - 6$ ja suorat $x = 2$ ja $x = 4$ rajaavat alueen.

Nollakohdat:

$x^2 - x - 6 = 0$

$x = 3$ tai $x = -2$ (laskimella)



$$A = -\int_2^3 (x^2 - x - 6)\,dx + \int_3^4 (x^2 - x - 6)\,dx$$

$$= -\Big/_2^3 (\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 6x) + \Big/_3^4 (\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 - 6x)$$

$$= -\Big(\tfrac{1}{3}\cdot 3^3 - \tfrac{1}{2}\cdot 3^2 - 6\cdot 3 - (\tfrac{1}{3}\cdot 2^3 - \tfrac{1}{2}\cdot 2^2 - 6\cdot 2)\Big)$$

$$+ \tfrac{1}{3}\cdot 4^3 - \tfrac{1}{2}\cdot 4^2 - 6\cdot 4 - (\tfrac{1}{3}\cdot 3^3 - \tfrac{1}{2}\cdot 3^2 - 6\cdot 3)$$

$$= \tfrac{13}{6} + \tfrac{17}{6}$$

$$= 5$$

Vastaus: a) $\tfrac{19}{3}$ b) 5

3. a)

$$\int_0^{\frac{\pi}{3}} \sin 3x\,dx = \frac{1}{3}\int_0^{\frac{\pi}{3}} 3\sin 3x\,dx = \frac{1}{3}\,\Big/_0^{\frac{\pi}{3}}(-\cos 3x)$$

$$= -\frac{1}{3}(\cos\pi - \cos 0)$$

$$= -\frac{1}{3}(-1-1)$$

$$= \frac{2}{3}$$

b)

$$\int_1^2 xe^{1-x^2}\,dx = -\frac{1}{2}\int_1^2 -2xe^{1-x^2}\,dx = -\frac{1}{2}\,\Big/_1^2 e^{1-x^2}$$

$$= -\frac{1}{2}(e^{1-2^2} - e^{1-1^2})$$

$$= -\frac{1}{2}(e^{-3} - e^0)$$

$$= -\frac{1}{2}(\frac{1}{e^3} - 1)$$

$$= \frac{1}{2} - \frac{1}{2e^3}$$

c)

$$\int_0^1 6x^2(1-x^3)^6\,dx = -2\int_0^1 -3x^2(1-x^3)^6\,dx = -2\,\Big/_0^1 \frac{1}{7}(1-x^3)^7$$

$$= -\frac{2}{7}\left((1-1^3)^7 - (1-0)^7\right)$$

$$= -\frac{2}{7}(0-1)$$

$$= \frac{2}{7}$$

Vastaus: a) $\frac{2}{3}$ b) $\frac{1}{2} - \frac{1}{2e^3}$ c) $\frac{2}{7}$

4. $h(x)=\dfrac{1}{\sqrt{2x+6}}=(2x+6)^{-\frac{1}{2}}$

$$H(x)=\int (2x+6)^{-\frac{1}{2}}\,dx=\tfrac{1}{2}\int 2(2x+6)^{-\frac{1}{2}}\,dx=\tfrac{1}{2}\cdot\tfrac{2}{1}(2x+6)^{\frac{1}{2}}+C$$
$$=\sqrt{2x+6}+C$$

Kulkee pisteen $(15,\,29)$ kautta, eli

$$H(15)=29$$
$$\sqrt{2\cdot 15+6}+C=29$$
$$6+C=29$$
$$C=23$$

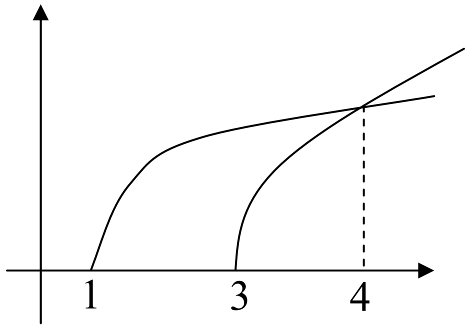
Siis $H(x)=\sqrt{2x+6}+23$

Vastaus: $H(x)=\sqrt{2x+6}+23$

5. x-akseli ja käyrät $y=\sqrt{x-1}$ ja $y=\sqrt{3x-9}$ rajoittavat alueen.

Leikkauskohdat:

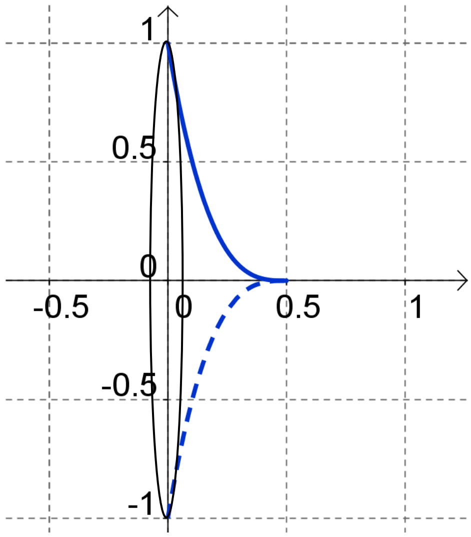
$$\sqrt{x-1}=\sqrt{3x-9} \quad \Big|()^2$$
$$x-1=3x-9$$
$$2x=8$$
$$x=4$$



$$A=\int_1^4 \sqrt{x-1}\,dx-\int_3^4 \sqrt{3x-9}\,dx$$
$$=\int_1^4 (x-1)^{\frac{1}{2}}\,dx-\tfrac{1}{3}\int_3^4 3(3x-9)^{\frac{1}{2}}\,dx$$
$$=\mathop{/}_1^4 \tfrac{2}{3}(x-1)^{\frac{3}{2}}-\tfrac{1}{3}\mathop{/}_3^4 \tfrac{2}{3}(3x-9)^{\frac{3}{2}}$$
$$=\tfrac{2}{3}\mathop{/}_1^4 (x-1)\sqrt{x-1}-\tfrac{2}{9}\mathop{/}_3^4 (3x-9)\sqrt{3x-9}$$
$$=\tfrac{2}{3}\left(3\sqrt{3}-0\right)-\tfrac{2}{9}\left(3\sqrt{3}-0\right)$$
$$=\tfrac{4\sqrt{3}}{3}$$

Vastaus: $\frac{4\sqrt{3}}{3}$

6. Koordinaattiakselit ja käyrä $y=(1-2x)^3$ rajaavat alueen, joka pyörähtää x-akselin ympäri.



Nollakohta:

$$(1-2x)^3=0$$
$$1-2x=0$$
$$x=\tfrac{1}{2}$$

$$V=\pi\int_0^{\frac{1}{2}}((1-2x)^3)^2\,dx$$
$$=-\tfrac{\pi}{2}\int_0^{\frac{1}{2}}-2(1-2x)^6\,dx$$
$$=-\tfrac{\pi}{2}\,\Big/_0^{\frac{1}{2}}\tfrac{1}{7}(1-2x)^7$$
$$=-\tfrac{\pi}{14}\left(0^7-1^7\right)$$
$$=\tfrac{\pi}{14}$$

Vastaus: $\frac{\pi}{14}$

7. Käyrä $y = 2x^3 - 1$, missä $0 \le x \le 3$, pyörähtää y-akselin ympäri

Käänteisfunktion lauseke:

$$y = 2x^3 - 1$$
$$x^3 = \frac{y+1}{2}$$
$$x = \sqrt[3]{\frac{y+1}{2}}$$

Siis $g(y) = \sqrt[3]{\frac{y+1}{2}} = \frac{1}{\sqrt[3]{2}}(y+1)^{\frac{1}{3}}$

Rajat:

$x = 0 \rightarrow y = -1$

$x = 3 \rightarrow y = 53$

$$V = \pi \int_{-1}^{53} \left(\frac{1}{\sqrt[3]{2}}(y+1)^{\frac{1}{3}} \right)^2 dy = \pi \int_{-1}^{53} \frac{1}{\sqrt[3]{4}}(y+1)^{\frac{2}{3}}\, dy$$

$$= \frac{\pi}{\sqrt[3]{4}} \mathop{/}_{-1}^{53} \frac{3}{5}(y+1)^{\frac{5}{3}}$$

$$= \frac{3\pi}{5\sqrt[3]{4}} \mathop{/}_{-1}^{53} (y+1)\sqrt[3]{(y+1)^2}$$

$$= \frac{3\pi}{5\sqrt[3]{4}} \left(54 \cdot \sqrt[3]{54^2} - 0 \right)$$

$$= \frac{1458\pi}{2} \approx 916{,}1$$

Vastaus: 916,1

8. Käyrä $y = \dfrac{1}{1+e^{-x}}$ sekä suorat $y=0$, $x=0$ ja $x=-a$ $\quad (a>0)$ rajoittavat alueen.

$$A = \int_{-a}^{0} \frac{1}{1+e^{-x}}\,dx = \int_{-a}^{0} \frac{1+e^{-x}-e^{-x}}{1+e^{-x}}\,dx = \int_{-a}^{0} \left(1 + \frac{-e^{-x}}{1+e^{-x}}\right)dx$$

$$= \underset{-a}{\overset{0}{/}} (x + \ln(1+e^{-x})) = 0 + \ln 2 - (-a + \ln(1+e^{a}))$$

$$= \ln 2 + a - \ln(1+e^{a}) = \ln\frac{2}{1+e^{a}} + a$$

Ala = $\ln\frac{3}{2}$

$$\ln\tfrac{2}{1+e^a} + a = \ln\tfrac{3}{2}$$

$$\ln\tfrac{2}{1+e^a} - \ln\tfrac{3}{2} = -a$$

$$\ln(\tfrac{2}{1+e^a} : \tfrac{3}{2}) = -a$$

$$\ln\tfrac{4}{3+3e^a} = -a$$

$$\tfrac{4}{3+3e^a} = e^{-a}$$

$$4 = e^{-a}(3+3e^{a})$$

$$4 = 3e^{-a} + 3$$

$$e^{-a} = \tfrac{1}{3}$$

$$-a = \ln\tfrac{1}{3}$$

$$-a = \ln 1 - \ln 3$$

$$a = \ln 3$$

Vastaus: Ala on $\ln\dfrac{2}{1+e^{a}} + a$, $a = \ln 3$

Testi 2

1. a)

$$\int_1^3 \frac{2x^2-3x}{x^3}\,dx = \int_1^3 \left(\tfrac{2}{x}-3x^{-2}\right)dx = \underset{1}{\overset{3}{/}}\left(2\ln x+\tfrac{3}{x}\right)$$

$$= 2\ln 3+1-(2\ln 1+3)$$

$$= 2\ln 3-2$$

b)

$$\int_0^1 2e^{3x}\,dx = \tfrac{2}{3}\int_0^1 3e^{3x}\,dx = \tfrac{2}{3}\underset{0}{\overset{1}{/}} e^{3x} = \tfrac{2}{3}(e^3-e^0) = \tfrac{2}{3}(e^3-1)$$

c)

$$\int_1^2 x\sqrt{3x^2+1}\,dx = \tfrac{1}{6}\int_1^2 6x(3x^2+1)^{\frac{1}{2}}\,dx = \tfrac{1}{6}\underset{1}{\overset{2}{/}}\tfrac{2}{3}(3x^2+1)^{\frac{3}{2}}$$

$$= \tfrac{1}{9}\underset{1}{\overset{2}{/}}(3x^2+1)\sqrt{3x^2+1}$$

$$= \tfrac{1}{9}\left(13\sqrt{13}-4\sqrt{4}\right)$$

$$= \tfrac{1}{9}\left(13\sqrt{13}-8\right)$$

Vastaus: a) $2\ln 3-2$ b) $\tfrac{2}{3}(e^3-1)$ c) $\tfrac{1}{9}\left(13\sqrt{13}-8\right)$

2. $f(x) = 3\cos 4x$

$$F(x) = \int 3\cos 4x\,dx = \tfrac{3}{4}\int 4\cos 4x\,dx = \tfrac{3}{4}\sin 4x + C$$

Kulkee pisteen $(\tfrac{5}{16}\pi,\ \sqrt{2})$ kautta, joten

$$F(\tfrac{5}{16}\pi) = \sqrt{2}$$

$$\tfrac{3}{4}\sin(\tfrac{5\pi}{4}) + C = \sqrt{2}$$

$$\tfrac{3}{4}(-\tfrac{1}{\sqrt{2}}) + C = \sqrt{2}$$

$$-\tfrac{3}{4\sqrt{2}} + C = \sqrt{2}$$

$$C = \sqrt{2} + \tfrac{3}{4\sqrt{2}} = \tfrac{11}{4\sqrt{2}} = \tfrac{11\sqrt{2}}{8}$$

Siis $F(x) = \tfrac{3}{4}\sin 4x + \tfrac{11\sqrt{2}}{8}$

Vastaus: $\tfrac{3}{4}\sin 4x + \tfrac{11\sqrt{2}}{8}$

3. x-akseli ja käyrä $y = e^{-x} - 3$ rajaavat alueen välillä $-\ln 4 \le x \le 0$.

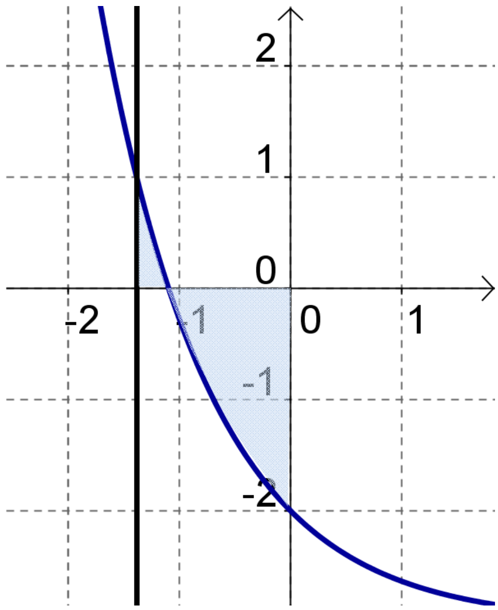
Nollakohta:

$$e^{-x} - 3 = 0$$

$$e^{-x} = 3$$

$$-x = \ln 3$$

$$x = -\ln 3$$



$$A = \int_{-\ln 4}^{-\ln 3} (e^{-x} - 3)\,dx - \int_{-\ln 3}^{0} (e^{-x} - 3)\,dx$$

$$= \Big/_{-\ln 4}^{-\ln 3} (-e^{-x} - 3x) - \Big/_{-\ln 3}^{0} (-e^{-x} - 3x)$$

$$= -\Big/_{-\ln 4}^{-\ln 3} (e^{-x} + 3x) + \Big/_{-\ln 3}^{0} (e^{-x} + 3x)$$

$$= -(e^{\ln 3} - 3\ln 3 - (e^{\ln 4} - 3\ln 4)) + e^{0} + 0 - (e^{\ln 3} - 3\ln 3)$$

$$= -(3 - 3\ln 3 - 4 + 3\ln 4) + 1 - 3 + 3\ln 3$$

$$= -3 + 3\ln 3 + 4 - 3\ln 2^2 + 1 - 3 + 3\ln 3$$

$$= 6\ln 3 - 6\ln 2 - 1$$

$$= 6(\ln 3 - \ln 2) - 1$$

$$= 6\ln \tfrac{3}{2} - 1$$

Vastaus: $6\ln \tfrac{3}{2} - 1$

4. Suora $y = 4x - 3$ ja käyrä $y = x^3 - 3$ rajaavat alueen

Merkitään: $f(x) = 4x - 3$ ja $g(x) = x^3 - 3$

Leikkauskohdat:

$4x - 3 = x^3 - 3$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x = 0$ tai $x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

$f(-1) = -7$

$g(-1) = -4$ joten välillä $[-2, 0]$ $g(x) \geq f(x)$

$f(1) = 1$

$g(1) = -2$ joten välillä $[0, 2]$ $f(x) \geq g(x)$

$$A = \int_{-2}^{0} (g(x) - f(x))\,dx + \int_{0}^{2} (f(x) - g(x))\,dx$$

$$= \int_{-2}^{0} (x^3 - 3 - (4x - 3))\,dx + \int_{0}^{2} (4x - 3 - (x^3 - 3))\,dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx + \int_{0}^{2} (4x - x^3)\,dx$$

$$= \mathop{/}_{-2}^{0} (\tfrac{1}{4}x^4 - 2x^2) + \mathop{/}_{0}^{2} (2x^2 - \tfrac{1}{4}x^4)$$

$$= 0 - (\tfrac{1}{4}(-2)^4 - 2(-2)^2) + 2 \cdot 2^2 - \tfrac{1}{4} \cdot 2^4 - 0$$

$$= 8$$

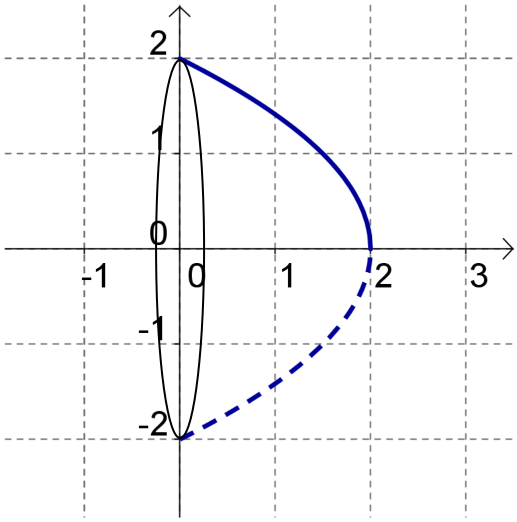
Vastaus: 8

5. Koordinaattiakselit ja käyrä $y = \sqrt{4-2x}$ rajoittavat alueen, joka pyörähtää

a) x-akselin ympäri

Nollakohdat:

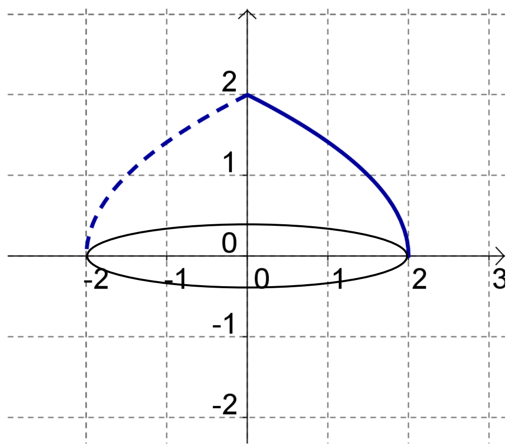
$$\sqrt{4-2x} = 0$$
$$4 - 2x = 0$$
$$x = 2$$



$$V = \pi \int_0^2 \sqrt{4-2x}^{\,2}\,dx$$
$$= \pi \int_0^2 (4-2x)\,dx$$
$$= \pi \mathop{/}_{0}^{2} (4x - x^2) = \pi(4\cdot 2 - 2^2 - 0) = 4\pi$$

b) y-akselin ympäri

Kääntеisfunktion lauseke:

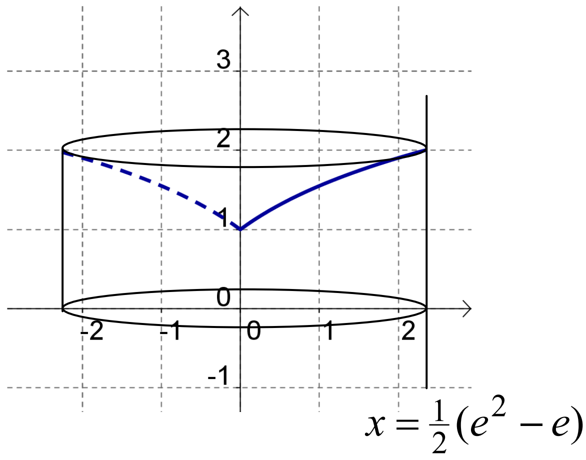


$$y = \sqrt{4-2x}$$
$$4 - 2x = y^2$$
$$x = 2 - \tfrac{1}{2}y^2$$

$$V = \pi \int_0^2 (2 - \tfrac{1}{2}y^2)^2\,dy = \pi \int_0^2 (4 - 2y^2 + \tfrac{1}{4}y^4)\,dy$$
$$= \pi \mathop{/}_{0}^{2} (4y - \tfrac{2}{3}y^3 + \tfrac{1}{20}y^5)$$
$$= \pi\left(4\cdot 2 - \tfrac{2}{3}\cdot 2^3 + \tfrac{1}{20}\cdot 2^5 - 0\right)$$
$$= \tfrac{64\pi}{15}$$

Vastaus: a) 4π b) $\frac{64\pi}{15}$

6. Koordinaattiakselien, funktion $f(x) = \ln(2x + e)$ ja suoran $x = \frac{1}{2}(e^2 - e)$ rajaama alue pyörähtää y-akselin ympäri.



Leikkauskohta:

$$\begin{cases} y = \ln(2x + e) \\ x = \frac{1}{2}(e^2 - e) \end{cases}$$

$$y = \ln(2 \cdot \tfrac{1}{2}(e^2 - e) + e)$$

$$y = \ln(e^2 - e + e)$$

$$y = \ln e^2 = 2$$

Käänteisfunktion lauseke:

$$y = \ln(2x + e)$$

$$2x + e = e^y$$

$$2x = e^y - e$$

$$x = \tfrac{1}{2}(e^y - e)$$

$$V = \pi \cdot \left(\tfrac{1}{2}(e^2 - e)\right)^2 \cdot 2 - \pi \int_1^2 (\tfrac{1}{2}(e^y - e))^2 \, dy$$

$$= \tfrac{\pi}{2}(e^2 - e)^2 - \pi \int_1^2 \tfrac{1}{4}(e^{2y} - 2ee^y + e^2) \, dy$$

$$= \tfrac{\pi}{2}(e^2 - e)^2 - \tfrac{\pi}{4} \mathop{/}_1^2 (\tfrac{1}{2}e^{2y} - 2e^{y+1} + e^2 y)$$

$$= \tfrac{\pi}{2}(e^2 - e)^2 - \tfrac{\pi}{4}\left(\tfrac{1}{2}e^4 - 2e^3 + 2e^2 - (\tfrac{1}{2}e^2 - 2e^2 + e^2)\right)$$

$$= \tfrac{\pi}{2}(e^4 - 2e^3 + e^2) - \tfrac{\pi}{4}(\tfrac{1}{2}e^4 - 2e^3 + 2\tfrac{1}{2}e^2)$$

$$= \pi(\tfrac{1}{2}e^4 - e^3 + \tfrac{1}{2}e^2 - \tfrac{1}{8}e^4 + \tfrac{1}{2}e^3 - \tfrac{5}{8}e^2)$$

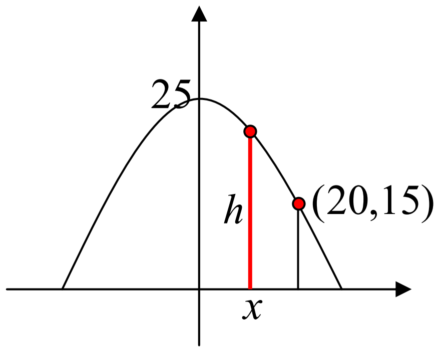
$$= \pi(\tfrac{3}{8}e^4 - \tfrac{1}{2}e^3 - \tfrac{1}{8}e^2) \ (\text{dm}^3)$$

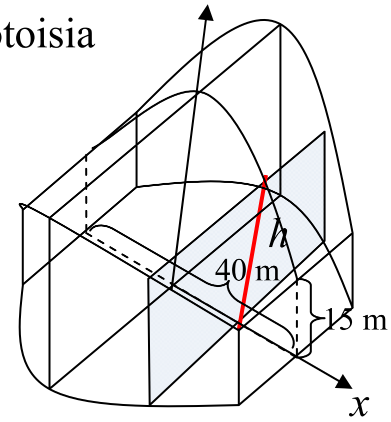
tiheys $\rho = 2500\ \frac{\text{kg}}{\text{m}^3} = 2500\ \frac{\text{g}}{\text{dm}^3}$

massa =
$\rho \cdot V = 2500 \cdot \pi(\frac{3}{8}e^4 - \frac{1}{2}e^3 - \frac{1}{8}e^2)\text{dm}^3 = 74\ 674{,}91...g \approx 75\ \text{kg}$

Vastaus: 75 kg

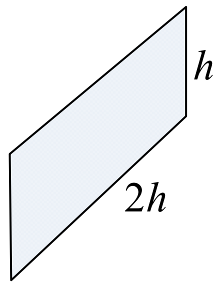
7. Poikkileikkaukset paraabelin $y = ax^2 + c$ muotoisia





Poikkileikkauksen korkeus kohdalla x on h ja kanta $2h$.

Poikkileikkauksen ala: $A = 2h \cdot h = 2h^2$



Paraabelin yhtälö:

$y = ax^2 + c$

Piste (0, 25) on paraabelilla, joten $c = 25$.

$y = ax^2 + 25$

Piste (20, 15) on paraabelilla, joten

$$a \cdot 20^2 + 25 = 15$$
$$400a = -10$$
$$a = -\tfrac{1}{40}$$

Siis paraabelin yhtälö on $y = -\frac{1}{40}x^2 + 25$

Kohdassa x on siis $h = -\frac{1}{40}x^2 + 25$

Poikkileikkauksen ala:

$A = 2h^2$

$$A(x) = 2(-\tfrac{1}{40}x^2 + 25)^2 = 2(\tfrac{1}{1600}x^4 - \tfrac{5}{4}x^2 + 625) = \tfrac{1}{800}x^4 - \tfrac{5}{2}x^2 + 1250$$

Tilavuus:

$$V = 2\int_0^{20} A(x)\,dx = 2\int_0^{20} (\tfrac{1}{800}x^4 - \tfrac{5}{2}x^2 + 1250)\,dx$$

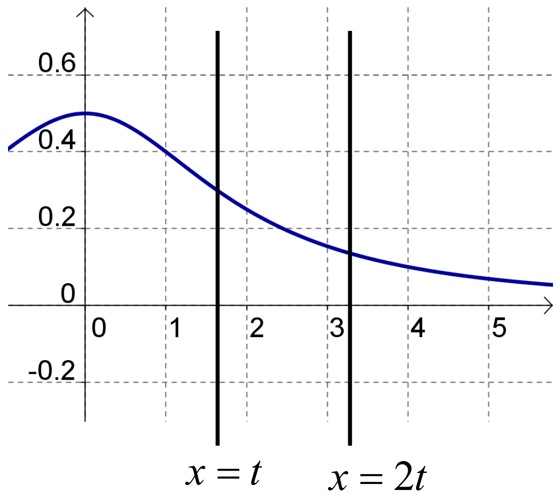
$$= 2 \mathop{/}_0^{20} (\tfrac{1}{4000}x^5 - \tfrac{5}{6}x^3 + 1250x)$$

$$= 2\left(\tfrac{1}{4000}\cdot 20^5 - \tfrac{5}{6}\cdot 20^3 + 1250\cdot 20 - 0\right)$$

$$= \tfrac{114\,800}{3} \approx 38\,300\ (\mathrm{m}^3)$$

Vastaus: 38 300 m^3

8. Funktion $f(x)=\dfrac{2}{x^2+4}$ kuvaaja ja x-akseli sekä suorat $x=t$ ja $x=2t$ $(t>0)$ rajoittavat alueen.



Pinta-ala on

$$A=\int_t^{2t}\frac{2}{x^2+4}\,dx$$

Merkitään $f(x)=\dfrac{2}{x^2+4}$

ja $F(x)=\int f(x)\,dx$

Tällöin

$$A(t)=\Big/_{t}^{2t}F(x)=F(2t)-F(t)$$

$$A'(t)=F'(2t)\cdot D(2t)-F'(t)=f(2t)\cdot 2-f(t)=2\cdot\frac{2}{(2t)^2+4}-\frac{2}{t^2+4}$$

$$A'(t)=\frac{4}{4t^2+4}-\frac{2}{t^2+4}$$

$$A'(t)=0$$

$$\frac{4}{4t^2+4}=\frac{2}{t^2+4}$$

$$4t^2+16=8t^2+8$$

$$4t^2=8$$

$$t^2=2$$

$$t=\pm\sqrt{2}$$

	0	$\sqrt{2}$
$A'(t)$	+	–
$A(t)$	↗	↘

Testipisteet:
$A'(1)=0{,}1>0$
$A'(2)=-0{,}05<0$

Suurin arvo, kun $t=\sqrt{2}$

Vastaus: $t = \sqrt{2}$