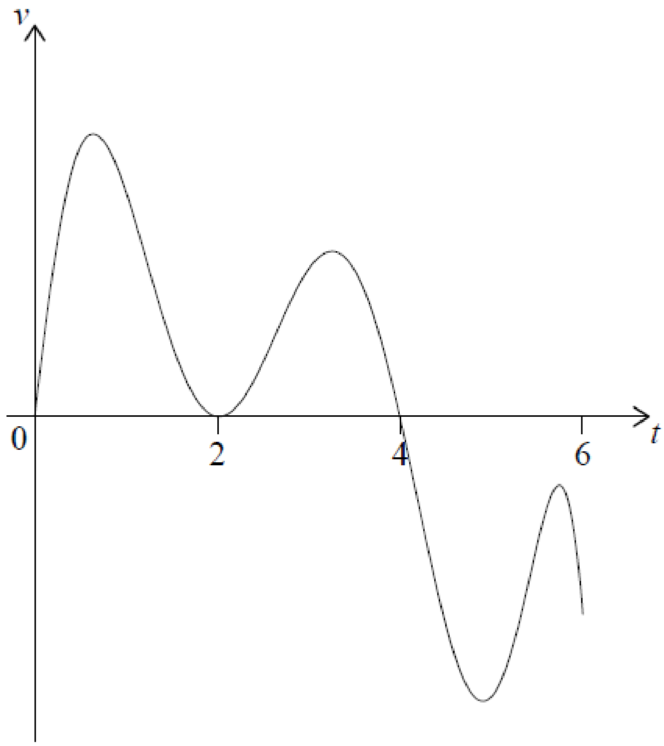
SL / velocity and acceleration

[77 marks]

A particle P starts from point O and moves along a straight line. The graph of its velocity, v ms^{-1} after t seconds, for 0 ≤ t ≤ 6 , is shown in the following diagram.



The graph of v has t-intercepts when t = 0, 2 and 4.

The function $s(t)$ represents the displacement of P from O after t seconds.

It is known that P travels a distance of 15 metres in the first 2 seconds. It is also known that $s(2) = s(5)$ and $\int_2^4 v \, \mathrm{d}t = 9$.

1a. Find the value of $s(4) - s(2)$. *[2 marks]*

Markscheme

recognizing relationship between v and s ***(M1)***

eg $\int v = s, \ s' = v$

$s(4) - s(2) = 9$ ***A1 N2***

[2 marks]

1b. Find the total distance travelled in the first 5 seconds. *[5 marks]*

Markscheme

correctly interpreting distance travelled in first 2 seconds (seen anywhere, including part (a) or the area of 15 indicated on diagram) ***(A1)***

eg $\int_0^2 v = 15,\ s(2) = 15$

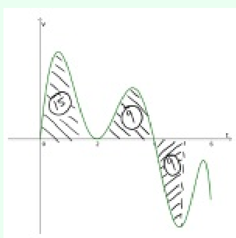
valid approach to find total distance travelled ***(M1)***

eg sum of 3 areas, $\int_0^4 v + \int_4^5 v$, shaded areas in diagram between 0 and 5

Note: Award ***M0*** if only $\int_0^5 |v|$ is seen.

correct working towards finding distance travelled between 2 and 5 (seen anywhere including within total area expression or on diagram) ***(A1)***

eg $\int_2^4 v - \int_4^5 v,\ \int_2^4 v = \int_4^5 |v|,\ \int_4^5 v\,\mathrm{d}t = -9,\ s(4) - s(2) - [s(5) - s(4)]$,

equal areas 

correct working using $s(5) = s(2)$ ***(A1)***

eg $15 + 9 - (-9),\ 15 + 2[s(4) - s(2)],\ 15 + 2(9),\ 2 \times s(4) - s(2)$,
$48 - 15$

total distance travelled = 33 (m) ***A1 N2***

[5 marks]

The population of fish in a lake is modelled by the function

$f(t) = \frac{1000}{1+24\mathrm{e}^{-0.2t}}$, $0 \leq t \leq 30$, where t is measured in months.

2a. Find the population of fish at $t = 10$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $f(10)$

235.402

235 (fish) (must be an integer) ***A1 N2***

[2 marks]

2b. Find the rate at which the population of fish is increasing at t = 10. *[2 marks]*

Markscheme

recognizing rate of change is derivative ***(M1)***

eg rate = f' , $f'(10)$, sketch of f' , 35 (fish per month)

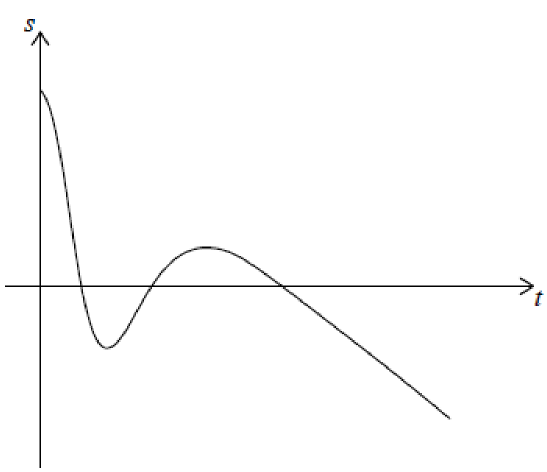
35.9976

36.0 (fish per month) ***A1 N2***

[2 marks]

In this question distance is in centimetres and time is in seconds.

Particle A is moving along a straight line such that its displacement from a point P, after t seconds, is given by $s_{\mathrm{A}} = 15 - t - 6t^3\mathrm{e}^{-0.8t}$, 0 ≤ t ≤ 25. This is shown in the following diagram.



3a. Find the initial displacement of particle A from point P. *[2 marks]*

Markscheme

valid approach **(M1)**

eg $s_{\mathrm{A}}(0)\,,\ s(0)\,,\ t=0$

15 (cm) **A1 N2**

[2 marks]

3b. Find the value of t when particle A first reaches point P. *[2 marks]*

Markscheme

valid approach **(M1)**

eg $s_{\mathrm{A}}=0,\ s=0,\ 6.79321,\ 14.8651$

2.46941

t = 2.47 (seconds) **A1 N2**

[2 marks]

3c. Find the value of t when particle A first changes direction. *[2 marks]*

Markscheme

recognizing when change in direction occurs **(M1)**

eg slope of s changes sign, $s'=0$, minimum point, 10.0144, (4.08, −4.66)

4.07702

t = 4.08 (seconds) **A1 N2**

[2 marks]

3d. Find the total distance travelled by particle A in the first 3 seconds. *[3 marks]*

Markscheme

METHOD 1 (using displacement)

correct displacement or distance from P at $t = 3$ (seen anywhere) ***(A1)***

eg −2.69630, 2.69630

valid approach ***(M1)***

eg 15 + 2.69630, $s(3) - s(0)$, −17.6963

17.6963

17.7 (cm) ***A1 N2***

METHOD 2 (using velocity)

attempt to substitute either limits or the velocity function into distance formula involving $|v|$ ***(M1)***

eg $\int_0^3 |v| \, \mathrm{d}t$, $\int \left|-1 - 18t^2\mathrm{e}^{-0.8t} + 4.8t^3\mathrm{e}^{-0.8t}\right|$

17.6963

17.7 (cm) ***A1 N2***

[3 marks]

Another particle, B, moves along the same line, starting at the same time as particle A. The velocity of particle B is given by $v_{\mathrm{B}} = 8 - 2t$, 0 ≤ t ≤ 25.

3e. Given that particles A and B start at the same point, find the displacement function s_{B} for particle B. *[5 marks]*

Markscheme

recognize the need to integrate velocity ***(M1)***

eg $\int v(t)$

$8t - \frac{2t^2}{2} + c$ (accept x instead of t and missing c) ***(A2)***

substituting initial condition into their integrated expression (must have c) ***(M1)***

eg $15 = 8(0) - \frac{2(0)^2}{2} + c, \quad c = 15$

$s_{\mathrm{B}}(t) = 8t - t^2 + 15$ ***A1 N3***

[5 marks]

3f. Find the other value of t when particles A and B meet. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $s_{\mathrm{A}} = s_{\mathrm{B}}$, sketch, (9.30404, 2.86710)

9.30404

$t = 9.30$ (seconds) ***A1 N2***

Note: If candidates obtain $s_{\mathrm{B}}(t) = 8t - t^2$ in part (e)(i), there are 2 solutions for part (e)(ii), 1.32463 and 7.79009. Award the last ***A1*** in part (e)(ii) only if both solutions are given.

[2 marks]

A particle moves along a straight line so that its velocity, v m s^{-1}, after t seconds is given by $v(t) = 1.4^t - 2.7$, for $0 \le t \le 5$.

4a. Find when the particle is at rest. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $v(t)=0$, sketch of graph

2.95195

$t=\log_{1.4}2.7$ (exact), $t=2.95$ (s) ***A1 N2***

[2 marks]

4b. Find the acceleration of the particle when $t=2$. *[2 marks]*

Markscheme

valid approach ***(M1)***

eg $a(t)=v'(t)$, $v'(2)$

0.659485

$a(2)$ = 1.96 ln 1.4 (exact), $a(2)$ = 0.659 (m s^{-2}) ***A1 N2***

[2 marks]

4c. Find the total distance travelled by the particle. *[3 marks]*

Markscheme

correct approach ***(A1)***

eg $\int_0^5 |v(t)|\ \mathrm{d}t$, $\int_0^{2.95}(-v(t))\ \mathrm{d}t+\int_{295}^{5}v(t)\ \mathrm{d}t$

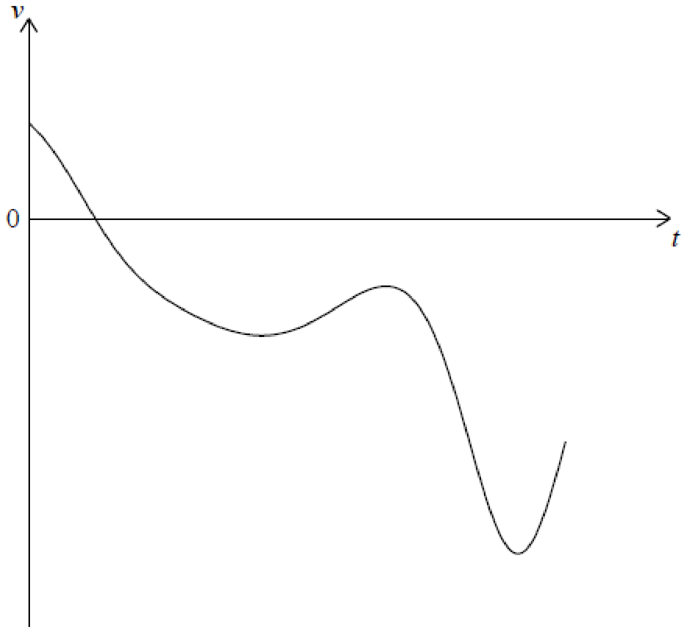
5.3479

distance = 5.35 (m) ***A2 N3***

[3 marks]

A particle P moves along a straight line. The velocity v m s^{-1} of P after t seconds is given by $v(t) = 7\cos t - 5t^{\cos t}$, for $0 \le t \le 7$.

The following diagram shows the graph of v.



5a. Find the initial velocity of P. *[2 marks]*

Markscheme

initial velocity when $t = 0$ ***(M1)***

eg $v(0)$

$v = 17$ (m s^{-1}) ***A1 N2***

[2 marks]

5b. Find the maximum speed of P. *[3 marks]*

Markscheme

recognizing maximum speed when $|v|$ is greatest ***(M1)***

eg minimum, maximum, $v' = 0$

one correct coordinate for minimum ***(A1)***

eg 6.37896, −24.6571

24.7 (ms^{-1}) ***A1 N2***

[3 marks]

5c. Write down the number of times that the acceleration of P is 0 m s^{-2} . *[3 marks]*

Markscheme

recognizing a = *v*′ ***(M1)***

eg $a = \frac{\mathrm{d}v}{\mathrm{d}t}$, correct derivative of first term

identifying when a = 0 ***(M1)***

eg turning points of *v*, *t*-intercepts of *v*′

3 ***A1 N3***

[3 marks]

5d. Find the acceleration of P when it changes direction. *[4 marks]*

Markscheme

recognizing P changes direction when *v* = 0 ***(M1)***

t = 0.863851 ***(A1)***

−9.24689

a = −9.25 (ms^{-2}) ***A2 N3***

[4 marks]

5e. Find the total distance travelled by P. *[3 marks]*

Markscheme

correct substitution of limits or function into formula ***(A1)***

eg $\int_0^7 |\, v \,|\,,\ \int_0^{0.8638} v\mathrm{d}t - \int_{0.8638}^7 v\mathrm{d}t,\ \int |\, 7\cos x - 5x^{\cos x} \,|\, dx,\ 3.32 = 60.6$

63.8874

63.9 (metres) ***A2 N3***

[3 marks]

Note: In this question, distance is in metres and time is in seconds.

A particle P moves in a straight line for five seconds. Its acceleration at time t is given by $a = 3t^2 - 14t + 8$, for $0 \leqslant t \leqslant 5$.

6a. Write down the values of t when $a = 0$. *[2 marks]*

Markscheme

$t = \frac{2}{3}\ (\text{exact}), 0.667,\ t = 4$ ***A1A1*** ***N2***

[2 marks]

6b. Hence or otherwise, find all possible values of t for which the velocity of P is decreasing. *[2 marks]*

Markscheme

recognizing that v is decreasing when a is negative ***(M1)***

eg $a < 0,\ 3t^2 - 14t + 8 \leqslant 0$, sketch of a

correct interval ***A1*** ***N2***

eg $\frac{2}{3} < t < 4$

[2 marks]

When $t = 0$, the velocity of P is $3\,\mathrm{m\,s^{-1}}$.

6c. Find an expression for the velocity of P at time t. *[6 marks]*

Markscheme

valid approach (do not accept a definite integral) ***(M1)***

eg $v\int a$

correct integration (accept missing c) ***(A1)(A1)(A1)***

$t^3 - 7t^2 + 8t + c$

substituting $t = 0,\ v = 3$, (must have c) ***(M1)***

eg $3 = 0^3 - 7(0^2) + 8(0) + c,\ c = 3$

$v = t^3 - 7t^2 + 8t + 3$ ***A1*** ***N6***

[6 marks]

6d. Find the total distance travelled by P when its velocity is increasing. *[4 marks]*

Markscheme

recognizing that v increases outside the interval found in part (b) ***(M1)***

eg $0 < t < \frac{2}{3},\ 4 < t < 5$, diagram

one correct substitution into distance formula ***(A1)***

eg $\int_0^{\frac{2}{3}} |v|,\ \int_4^5 |v|,\ \int_{\frac{2}{3}}^4 |v|,\ \int_0^5 |v|$

one correct pair ***(A1)***

eg 3.13580 and 11.0833, 20.9906 and 35.2097

14.2191 ***A1*** ***N2***

$d = 14.2\ (\mathrm{m})$

[4 marks]

A particle P moves along a straight line so that its velocity, $v\,\mathrm{ms}^{-1}$, after t seconds, is given by $v = \cos 3t - 2\sin t - 0.5$, for $0 \leqslant t \leqslant 5$. The initial displacement of P from a fixed point O is 4 metres.

7a. Find the displacement of P from O after 5 seconds. *[5 marks]*

Markscheme

METHOD 1

recognizing $s = \int v$ ***(M1)***

recognizing displacement of P in first 5 seconds (seen anywhere) ***A1***

(accept missing $\mathrm{d}t$)

eg $\int_0^5 v\mathrm{d}t,\ -3.71591$

valid approach to find total displacement ***(M1)***

eg $4 + (-3.7159),\ s = 4 + \int_0^5 v$

0.284086

0.284 (m) ***A2*** ***N3***

METHOD 2

recognizing $s = \int v$ ***(M1)***

correct integration ***A1***

eg $\frac{1}{3}\sin 3t + 2\cos t - \frac{t}{2} + c$ (do not penalize missing "c")

attempt to find c ***(M1)***

eg

$4 = \frac{1}{3}\sin(0) + 2\cos(0) - -\frac{0}{2} + c,\ 4 = \frac{1}{3}\sin 3t + 2\cos t - \frac{t}{2} + c,\ 2 + c = 4$

attempt to substitute $t = 5$ into their expression with c ***(M1)***

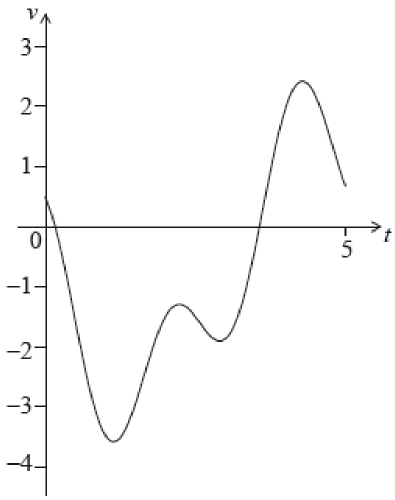
eg $s(5),\ \frac{1}{3}\sin(15) + 2\cos(5)5 - -\frac{5}{2} + 2$

0.284086

0.284 (m) ***A1*** ***N3***

[5 marks]

The following sketch shows the graph of v.



7b. Find when P is first at rest. *[2 marks]*

Markscheme

recognizing that at rest, $v = 0$ ***(M1)***

$t = 0.179900$

$t = 0.180 \text{ (secs)}$ ***A1*** ***N2***

[2 marks]

7c. Write down the number of times P changes direction. *[2 marks]*

Markscheme

recognizing when change of direction occurs ***(M1)***

eg v crosses t axis

2 (times) ***A1*** ***N2***

[2 marks]

7d. Find the acceleration of P after 3 seconds. *[2 marks]*

Markscheme

acceleration is v' (seen anywhere) ***(M1)***

eg$v'(3)$

0.743631

$0.744\ (\mathrm{ms}^{-2})$ ***A1*** ***N2***

[2 marks]

7e. Find the maximum speed of P. *[3 marks]*

Markscheme

valid approach involving max or min of v ***(M1)***

eg$v' = 0,\ a = 0$, graph

one correct co-ordinate for min ***(A1)***

eg$1.14102,\ -3.27876$

$3.28\ (\mathrm{ms}^{-1})$ ***A1*** ***N2***

[3 marks]