SL / Integration *[82 marks]*

Let
$f(x) = \frac{2x}{x^2+5}$.

1a. Use the quotient rule to show that
$f'(x) = \frac{10-2x^2}{(x^2+5)^2}$. *[4 marks]*

Markscheme

derivative of
$2x$ is
2 (must be seen in quotient rule) ***(A1)***

derivative of
x^2+5 is
$2x$ (must be seen in quotient rule) ***(A1)***

correct substitution into quotient rule ***A1***

eg
$\frac{(x^2+5)(2)-(2x)(2x)}{(x^2+5)^2}$, $\frac{2(x^2+5)-4x^2}{(x^2+5)^2}$

correct working which clearly leads to given answer ***A1***

eg
$\frac{2x^2+10-4x^2}{(x^2+5)^2}$, $\frac{2x^2+10-4x^2}{x^4+10x^2+25}$

$f'(x) = \frac{10-2x^2}{(x^2+5)^2}$ ***AG*** ***N0***

[4 marks]

1b. Find
$\int \frac{2x}{x^2+5} \mathrm{d}x$. *[4 marks]*

Markscheme

valid approach using substitution or inspection ***(M1)***

eg
$u = x^2+5,\ \mathrm{d}u = 2x\mathrm{d}x,\ \frac{1}{2}\ln(x^2+5)$

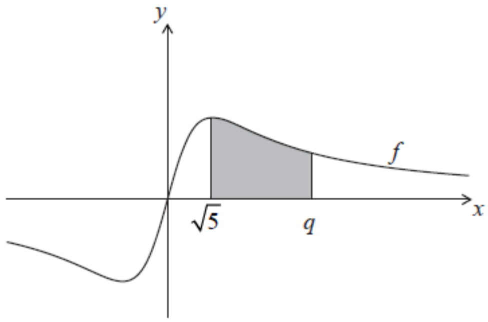
$\int \frac{2x}{x^2+5}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u$ ***(A1)***

$\int \frac{1}{u}\mathrm{d}u = \ln u + c$ ***(A1)***

$\ln(x^2+5)+c$ ***A1*** ***N4***

[4 marks]

1c. The following diagram shows part of the graph of f. *[7 marks]*



The shaded region is enclosed by the graph of f, the x-axis, and the lines $x = \sqrt{5}$ and $x = q$. This region has an area of $\ln 7$. Find the value of q.

Markscheme

correct expression for area *(A1)*

eg

$\left[\ln\left(x^2+5\right)\right]_{\sqrt{5}}^{q},\ \int\limits_{\sqrt{5}}^{q}\frac{2x}{x^2+5}\mathrm{d}x$

substituting limits into **their** integrated function and subtracting (in either order) *(M1)*

eg

$\ln(q^2+5)-\ln\left(\sqrt{5}^2+5\right)$

correct working *(A1)*

eg

$\ln\left(q^2+5\right)-\ln 10,\ \ln\frac{q^2+5}{10}$

equating **their** expression to $\ln 7$ (seen anywhere) *(M1)*

eg

$\ln\left(q^2+5\right)-\ln 10=\ln 7,\ \ln\frac{q^2+5}{10}=\ln 7,\ \ln(q^2+5)=\ln 7+\ln 10$

correct equation without logs *(A1)*

eg

$\frac{q^2+5}{10}=7,\ q^2+5=70$

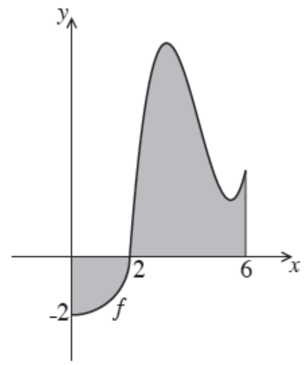
$q^2=65$ *(A1)*

$q=\sqrt{65}$ ***A1*** ***N3***

Note: Award ***A0*** for $q=\pm\sqrt{65}$.

[7 marks]

The following is the graph of a function f, for $0 \le x \le 6$.



The first part of the graph is a quarter circle of radius 2 with centre at the origin.

2a. (a) Find $\int_0^2 f(x)\mathrm{d}x$. *[7 marks]*

(b) The shaded region is enclosed by the graph of f, the x-axis, the y-axis and the line $x = 6$. The area of this region is 3π.

Find $\int_2^6 f(x)\mathrm{d}x$.

Markscheme

(a) attempt to find quarter circle area ***(M1)***

eg $\frac{1}{4}(4\pi)$, $\frac{\pi r^2}{4}$, $\int_0^2 \sqrt{4 - x^2}\mathrm{d}x$

area of region $= \pi$ ***(A1)***

$\int_0^2 f(x)\mathrm{d}x = -\pi$ ***A2 N3***

[4 marks]

(b) attempted set up with both regions ***(M1)***

eg $\text{shaded area} - \text{quarter circle}$, $3\pi - \pi$, $3\pi - \int_0^2 f = \int_2^6 f$

$\int_2^6 f(x)\mathrm{d}x = 2\pi$ ***A2 N2***

[3 marks]

Total [7 marks]

2b. Find $\int_0^2 f(x)\mathrm{d}x$. *[4 marks]*

Markscheme

attempt to find quarter circle area ***(M1)***

eg
$\frac{1}{4}(4\pi)$,
$\frac{\pi r^2}{4}$,
$\int_0^2 \sqrt{4-x^2 \mathrm{d}x}$

area of region
$= \pi$ ***(A1)***

$\int_0^2 f(x)\mathrm{d}x = -\pi$ ***A2 N3***

[4 marks]

2c. The shaded region is enclosed by the graph of f , the x-axis, the y-axis and the line $x = 6$. The area of this region is 3π . *[3 marks]*

Find
$\int_2^6 f(x)\mathrm{d}x$.

Markscheme

attempted set up with both regions ***(M1)***

eg
$\text{shaded area} - \text{quarter circle}$,
$3\pi - \pi$,
$3\pi - \int_0^2 f = \int_2^6 f$

$\int_2^6 f(x)\mathrm{d}x = 2\pi$ ***A2 N2***

[3 marks]

Total [7 marks]

Let
$f(x) = \mathrm{e}^{\frac{x}{4}}$ and
$g(x) = mx$, where
$m \geq 0$, and
$-5 \leq x \leq 5$. Let
R be the region enclosed by the
y-axis, the graph of
f , and the graph of
g .

Let
$m = 1.$

3a. (i) Sketch the graphs of f and g on the same axes. *[7 marks]*

(ii) Find the area of R .

Markscheme

(i)

A1A1 ***N2***

Notes: Award ***A1*** for the graph of f positive, increasing and concave up.

Award ***A1*** for graph of g increasing and linear with y-intercept of 0.

Penalize one mark if domain is not [-5, 5] and/or if f and g do not intersect in the first quadrant.

[2 marks]

(ii)
attempt to find intersection of the graphs of f ***and*** g ***(M1)***

eg $\mathrm{e}^{\frac{x}{4}} = x$

$x = 1.42961\ldots$ ***A1***

valid attempt to find area of R ***(M1)***

eg
$\int (x - \mathrm{e}^{\frac{x}{4}})\mathrm{d}x$,
$\int_0^1 (g - f)$,
$\int (f - g)$

area $= 0.697$ ***A2*** ***N3***

[5 marks]

3b. Find the area of R . *[5 marks]*

Markscheme

attempt to find intersection of the graphs of f and g ***(M1)***

eg $\mathrm{e}^{\frac{x}{4}} = x$

$x = 1.42961\ldots$ ***A1***

valid attempt to find area of R ***(M1)***

eg $\int (x - \mathrm{e}^{\frac{x}{4}})\mathrm{d}x$, $\int_0^1 (g - f)$, $\int (f - g)$

area $= 0.697$ ***A2*** ***N3***

[5 marks]

3c. Consider all values of m such that the graphs of f and g intersect. Find the value of m that gives the greatest value for the area of R . *[8 marks]*

Markscheme

recognize that area of R is a maximum at point of tangency ***(R1)***

eg $m = f'(x)$

equating functions ***(M1)***

eg $f(x) = g(x)$, $\mathrm{e}^{\frac{x}{4}} = mx$

$f'(x) = \frac{1}{4}\mathrm{e}^{\frac{x}{4}}$ ***(A1)***

equating gradients ***(A1)***

eg $f'(x) = g'(x)$, $\frac{1}{4}\mathrm{e}^{\frac{x}{4}} = m$

attempt to solve system of two equations for x ***(M1)***

eg $\frac{1}{4}\mathrm{e}^{\frac{x}{4}} \times x = \mathrm{e}^{\frac{x}{4}}$

$x = 4$ ***(A1)***

attempt to find m ***(M1)***

eg $f'(4)$, $\frac{1}{4}\mathrm{e}^{\frac{4}{4}}$

$m = \frac{1}{4}e$ (exact), 0.680 ***A1*** ***N3***

[8 marks]

The following diagram shows part of the graph of the function
$f(x) = 2x^2$.

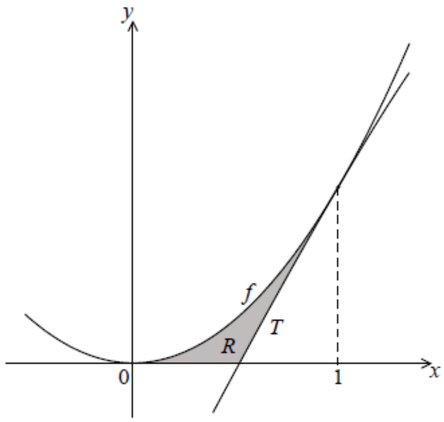


diagram
not to scale

The line *T* is the tangent to the graph of *f* at
$x = 1$.

4a. Show that the equation of *T* is
$y = 4x - 2$. *[5 marks]*

Markscheme

$f(1) = 2$ ***(A1)***

$f'(x) = 4x$ ***A1***

evidence of finding the gradient of *f* at
$x = 1$ ***M1***

e.g. substituting
$x = 1$ into
$f'(x)$

finding gradient of *f* at
$x = 1$ ***A1***

e.g.
$f'(1) = 4$

evidence of finding equation of the line ***M1***

e.g.
$y - 2 = 4(x - 1)$,
$2 = 4(1) + b$

$y = 4x - 2$ ***AG*** ***N0***

[5 marks]

4b. Find the *x*-intercept of *T* . *[2 marks]*

Markscheme

appropriate approach ***(M1)***

e.g.
$4x - 2 = 0$

$x = \frac{1}{2}$ ***A1*** ***N2***

[2 marks]

4c. The shaded region R is enclosed by the graph of f, the line T, and the x-axis. *[9 marks]*

(i) Write down an expression for the area of R.

(ii) Find the area of R.

Markscheme

(i) bottom limit
$x = 0$ (seen anywhere) ***(A1)***

approach involving subtraction of integrals/areas ***(M1)***

e.g.
$\int f(x) - \text{area of triangle}$,
$\int f - \int l$

correct expression ***A2 N4***

e.g.
$\int_0^1 2x^2 \mathrm{d}x - \int_{0.5}^1 (4x-2)\mathrm{d}x$,
$\int_0^1 f(x)\mathrm{d}x - \frac{1}{2}$,
$\int_0^{0.5} 2x^2 \mathrm{d}x + \int_{0.5}^1 (f(x) - (4x-2))\mathrm{d}x$

(ii) **METHOD 1 (using only integrals)**

correct integration ***(A1)(A1)(A1)***

$\int 2x^2 \mathrm{d}x = \frac{2x^3}{3}$,
$\int (4x-2)\mathrm{d}x = 2x^2 - 2x$

substitution of limits ***(M1)***

e.g.
$\frac{1}{12} + \frac{2}{3} - 2 + 2 - \left(\frac{1}{12} - \frac{1}{2} + 1\right)$

area =
$\frac{1}{6}$ ***A1 N4***

METHOD 2 (using integral and triangle)

area of triangle =
$\frac{1}{2}$ ***(A1)***

correct integration ***(A1)***

$\int 2x^2 \mathrm{d}x = \frac{2x^3}{3}$

substitution of limits ***(M1)***

e.g.
$\frac{2}{3}(1)^3 - \frac{2}{3}(0)^3$,
$\frac{2}{3} - 0$

correct simplification ***(A1)***

e.g.
$\frac{2}{3} - \frac{1}{2}$

area =
$\frac{1}{6}$ ***A1 N4***

[9 marks]

Let
$f(x) = \sqrt{x}$. Line L is the normal to the graph of f at the point (4, 2) .

5a. Show that the equation of L is
$y = -4x + 18$. *[4 marks]*

Markscheme

finding derivative *(A1)*

e.g.
$f'(x) = \frac{1}{2}x^{\frac{1}{2}}, \frac{1}{2\sqrt{x}}$

correct value of derivative or its negative reciprocal (seen anywhere) ***A1***

e.g.
$\frac{1}{2\sqrt{4}}$,
$\frac{1}{4}$

gradient of normal =
$\frac{1}{\text{gradient of tangent}}$ (seen anywhere) ***A1***

e.g.
$-\frac{1}{f'(4)} = -4$,
$-2\sqrt{x}$

substituting into equation of line (for normal) ***M1***

e.g.
$y - 2 = -4(x - 4)$

$y = -4x + 18$ ***AG N0***

[4 marks]

5b. Point A is the *x*-intercept of *L* . Find the *x*-coordinate of A. *[2 marks]*

Markscheme

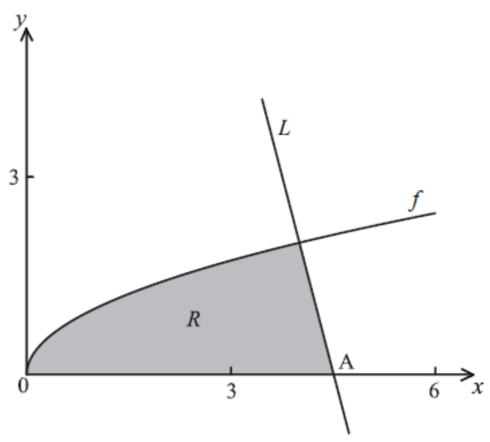
recognition that
$y = 0$ at A *(M1)*

e.g.
$-4x + 18 = 0$

$x = \frac{18}{4}$
$\left(= \frac{9}{2}\right)$ ***A1 N2***

[2 marks]

In the diagram below, the shaded region *R* is bounded by the *x*-axis, the graph of *f* and the line *L* .



5c. Find an expression for the area of *R* . *[3 marks]*

Markscheme

splitting into two appropriate parts (areas and/or integrals) ***(M1)***

correct expression for area of R ***A2*** ***N3***

e.g. area of R =
$\int_0^4 \sqrt{x} \mathrm{d}x + \int_4^{4.5} (-4x + 18) \mathrm{d}x$,
$\int_0^4 \sqrt{x} \mathrm{d}x + \frac{1}{2} \times 0.5 \times 2$ (triangle)

Note: Award ***A1*** if d*x* is missing.

[3 marks]

5d. The region R is rotated 360° about the *x*-axis. Find the volume of the solid formed, giving your answer in terms of π . *[8 marks]*

Markscheme

correct expression for the volume from
$x = 0$ to
$x = 4$ ***(A1)***

e.g.
$V = \int_0^4 \pi \left[f(x)^2 \right] \mathrm{d}x$,
$\int_0^4 \pi \sqrt{x}^2 \mathrm{d}x$,
$\int_0^4 \pi x \mathrm{d}x$

$V = \left[\frac{1}{2} \pi x^2 \right]_0^4$ ***A1***

$V = \pi \left(\frac{1}{2} \times 16 - \frac{1}{2} \times 0 \right)$ ***(A1)***

$V = 8\pi$ ***A1***

finding the volume from
$x = 4$ to
$x = 4.5$

EITHER

recognizing a cone ***(M1)***

e.g.
$V = \frac{1}{3} \pi r^2 h$

$V = \frac{1}{3} \pi (2)^2 \times \frac{1}{2}$ ***(A1)***

$= \frac{2\pi}{3}$ ***A1***

total volume is
$8\pi + \frac{2}{3}\pi$
$\left(= \frac{26}{3}\pi \right)$ ***A1*** ***N4***

OR

$V = \pi \int_4^{4.5} (-4x + 18)^2 \mathrm{d}x$ ***(M1)***

$= \int_4^{4.5} \pi (16x^2 - 144x + 324) \mathrm{d}x$

$= \pi \left[\frac{16}{3} x^3 - 72x^2 + 324x \right]_4^{4.5}$ ***A1***

$= \frac{2\pi}{3}$ ***A1***

total volume is
$8\pi + \frac{2}{3}\pi$
$\left(= \frac{26}{3}\pi \right)$ ***A1*** ***N4***

[8 marks]