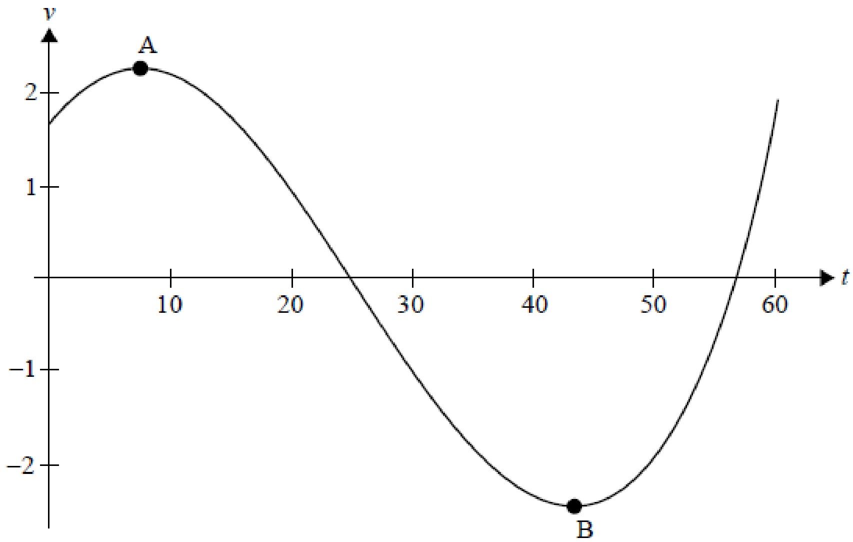
HL / velocity *[52 marks]*

A body moves in a straight line such that its velocity, $v\,\mathrm{ms}^{-1}$, after t seconds is given by $v = 2\sin\left(\frac{t}{10} + \frac{\pi}{5}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right)$ for $0 \leqslant t \leqslant 60$.

The following diagram shows the graph of v against t. Point A is a local maximum and point B is a local minimum.



1a. Determine the coordinates of point A and the coordinates of point B. *[4 marks]*

Markscheme

$\mathrm{A}\,(7.47,\,2.28)$ and $\mathrm{B}\,(43.4,\,-2.45)$ ***A1A1A1A1***

[4 marks]

1b. Hence, write down the maximum speed of the body. *[1 mark]*

Markscheme

maximum speed is $2.45\ \left(\mathrm{ms}^{-1}\right)$ ***A1***

[1 mark]

The body first comes to rest at time $t = t_1$. Find

1c. the value of t_1. *[2 marks]*

Markscheme

$v = 0 \Rightarrow t_1 = 25.1 \text{ (s)}$ ***(M1)A1***

[2 marks]

1d. the distance travelled between $t = 0$ and $t = t_1$. *[2 marks]*

Markscheme

$\int_0^{t_1} v \, \mathrm{d}t$ ***(M1)***

$= 41.0 \text{ (m)}$ ***A1***

[2 marks]

1e. the acceleration when $t = t_1$. *[2 marks]*

Markscheme

$a = \frac{\mathrm{d}v}{\mathrm{d}t}$ at $t = t_1 = 25.1$ ***(M1)***

$a = -0.200 \ \left(\mathrm{ms}^{-2}\right)$ ***A1***

Note: Accept $a = -0.2$.

[2 marks]

1f. Find the distance travelled in the first 30 seconds. *[3 marks]*

Markscheme

attempt to integrate between 0 and 30 ***(M1)***

Note: An unsupported answer of 38.6 can imply integrating from 0 to 30.

EITHER

$\int_0^{30} |v| \mathrm{d}t$ ***(A1)***

OR

$41.0 - \int_{t_1}^{30} v \, \mathrm{d}t$ ***(A1)***

THEN

$= 43.3 \text{ (m)}$ ***A1***

[3 marks]

A point P moves in a straight line with velocity v ms^{-1} given by $v(t) = \mathrm{e}^{-t} - 8t^2\mathrm{e}^{-2t}$ at time t seconds, where $t \geq 0$.

2a. Determine the first time t_1 at which P has zero velocity. *[2 marks]*

Markscheme

attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441$(s) ***A1***

[2 marks]

2b. Find an expression for the acceleration of P at time t. *[2 marks]*

Markscheme

$a\left(t\right) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\mathrm{e}^{-t} - 16t\mathrm{e}^{-2t} + 16t^2\mathrm{e}^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

[2 marks]

2c. Find the value of the acceleration of P at time t_1. *[1 mark]*

Markscheme

$a\left(t_1\right) = -2.28$ (ms^{-2}) ***A1***

[1 mark]

3. A particle moves in a straight line such that at time t seconds $\left(t \geqslant 0\right)$, its velocity v, in ms^{-1}, is given by $v = 10t\mathrm{e}^{-2t}$. Find the exact distance travelled by the particle in the first half-second. *[5 marks]*

Markscheme

$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t$

attempt at integration by parts ***M1***

$= \left[-5t\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} -5\mathrm{e}^{-2t}\mathrm{d}t$ ***A1***

$= \left[-5t\mathrm{e}^{-2t} - \frac{5}{2}\mathrm{e}^{-2t}\right]_0^{\frac{1}{2}}$ ***(A1)***

Note: Condone absence of limits (or incorrect limits) and missing factor of 10 up to this point.

$s = \int_0^{\frac{1}{2}} 10t\mathrm{e}^{-2t}\mathrm{d}t$ ***(M1)***

$= -5\mathrm{e}^{-1} + \frac{5}{2} \left(= \frac{-5}{\mathrm{e}} + \frac{5}{2}\right) \left(= \frac{5\mathrm{e}-10}{2\mathrm{e}}\right)$ ***A1***

[5 marks]

The displacement, s, in metres, of a particle t seconds after it passes through the origin is given by the expression $s = \ln(2 - e^{-t}), t \geqslant 0$.

4a. Find an expression for the velocity, v, of the particle at time t. *[2 marks]*

Markscheme

$v - \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{e}^{-t}}{2-\mathrm{e}^{-t}} \left(= \frac{1}{2\mathrm{e}^{t}-1} \text{ or } -1 + \frac{2}{2-\mathrm{e}^{-t}}\right)$ ***M1A1***

[2 marks]

4b. Find an expression for the acceleration, a, of the particle at time t. *[2 marks]*

Markscheme

$a=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{-\mathrm{e}^{-t}(2-\mathrm{e}^{-t})-\mathrm{e}^{-t}\times\mathrm{e}^{-t}}{(2-\mathrm{e}^{-t})^2}\quad\left(=\frac{-2\mathrm{e}^{-t}}{(2-\mathrm{e}^{-t})^2}\right)$ ***M1A1***

Note: If simplified in part (a) award ***(M1)A1*** for $a=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{-2\mathrm{e}^{t}}{(2\mathrm{e}^{t}-1)^2}$.

Note: Award ***M1A1*** for $a=-\mathrm{e}^{-t}(2-\mathrm{e}^{-t})^{-2}(\mathrm{e}^{-t})-\mathrm{e}^{-t}(2-\mathrm{e}^{-t})^{-1}$.

[2 marks]

4c. Find the acceleration of the particle at time $t=0$. *[1 mark]*

Markscheme

$a=-2\ (\mathrm{ms}^{-2})$ ***A1***

[1 mark]

A particle can move along a straight line from a point O. The velocity v, in ms^{-1}, is given by the function $v(t)=1-\mathrm{e}^{-\sin t^2}$ where time $t\geq 0$ is measured in seconds.

5a. Write down the first two times $t_1,\ t_2>0$, when the particle changes direction. *[2 marks]*

Markscheme

$t_1=1.77\ (\mathrm{s})\quad\left(=\sqrt{\pi}\ (\mathrm{s})\right)$ and $t_2=2.51\ (\mathrm{s})\quad\left(=\sqrt{2\pi}\ (\mathrm{s})\right)$ ***A1A1***

[2 marks]

5b. (i) Find the time $t<t_2$ when the particle has a maximum velocity. *[4 marks]*

(ii) Find the time $t<t_2$ when the particle has a minimum velocity.

Markscheme

(i) attempting to find (graphically or analytically) the first $t_{\max}$ ***(M1)***

$t = 1.25\ (\mathrm{s})\quad \left(= \sqrt{\frac{\pi}{2}}\ (\mathrm{s})\right)$ ***A1***

(ii) attempting to find (graphically or analytically) the first $t_{\min}$ ***(M1)***

$t = 2.17\ (\mathrm{s})\quad \left(= \sqrt{\frac{3\pi}{2}}\ (\mathrm{s})\right)$ ***A1***

[4 marks]

5c. Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$. *[2 marks]*

Markscheme

distance travelled $= \left|\int_{1.772\ldots}^{2.506\ldots} 1 - \mathrm{e}^{-\sin t^2}\mathrm{d}t\right|$ (or equivalent) ***(M1)***

$= 0.711\ (\mathrm{m})$ ***A1***

Note: Award ***M1*** for attempting to form a definite integral involving $1 - \mathrm{e}^{-\sin t^2}$. To award the ***A1***, correct limits leading to 0.711 must include the use of absolute value or a statement such as “distance must be positive”.

In part (c), award ***A1FT*** for a candidate working in degree mode $(5.39\ (\mathrm{m}))$.

[2 marks]

Total [8 marks]

A particle moves in a straight line, its velocity $v\mathrm{ms}^{-1}$ at time t seconds is given by $v = 9t - 3t^2, 0 \le t \le 5$.

At time $t = 0$, the displacement s of the particle from an origin O is 3 m.

6a. Find the displacement of the particle when $t = 4$. *[3 marks]*

Markscheme

METHOD 1

$s = \int (9t - 3t^2)\mathrm{d}t = \frac{9}{2}t^2 - t^3(+c)$ ***(M1)***

$t = 0,\ s = 3 \Rightarrow c = 3$ ***(A1)***

$t = 4 \Rightarrow s = 11$ ***A1***

METHOD 2

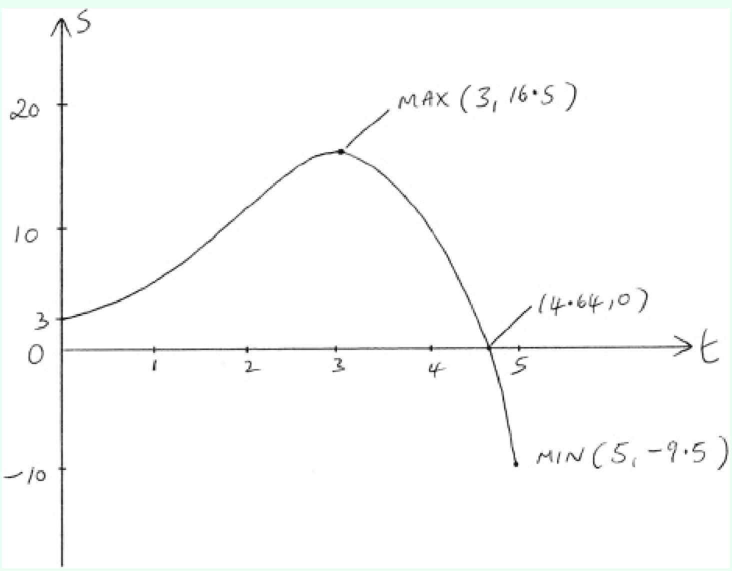
$s = 3 + \int_0^4 (9t - 3t^2)\mathrm{d}t$ ***(M1)(A1)***

$s = 11$ ***A1***

[3 marks]

6b. Sketch a displacement/time graph for the particle, $0 \le t \le 5$, showing clearly where the curve meets the axes and the coordinates of the points where the displacement takes greatest and least values. *[5 marks]*

Markscheme



correct shape over correct domain ***A1***

maximum at $(3,\ 16.5)$ ***A1***

t intercept at 4.64, s intercept at 3 ***A1***

minimum at $(5,\ -9.5)$ ***A1***

[5 marks]

6c. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ *[3 marks]*
such that s is continuous for all $t \geq 0$.

Given further that $s = 16.5$ when $t = 7.5$, find the values of a and b.

Markscheme

$-9.5 = a + b\cos 2\pi$

$16.5 = a + b\cos 3\pi$ ***(M1)***

Note: Only award ***M1*** if two simultaneous equations are formed over the correct domain.

$a = \frac{7}{2}$ ***A1***

$b = -13$ ***A1***

[3 marks]

6d. For $t > 5$, the displacement of the particle is given by $s = a + b\cos\frac{2\pi t}{5}$ *[4 marks]*
such that s is continuous for all $t \geq 0$.

Find the times t_1 and $t_2 (0 < t_1 < t_2 < 8)$ when the particle returns to its starting point.

Markscheme

at t_1:

$3 + \frac{9}{2}t^2 - t^3 = 3$ ***(M1)***

$t^2\left(\frac{9}{2} - t\right) = 0$

$t_1 = \frac{9}{2}$ ***A1***

solving $\frac{7}{2} - 13\cos\frac{2\pi t}{5} = 3$ ***(M1)***

$\text{GDC} \Rightarrow t_2 = 6.22$ ***A1***

Note: Accept graphical approaches.

[4 marks]

Total [15 marks]