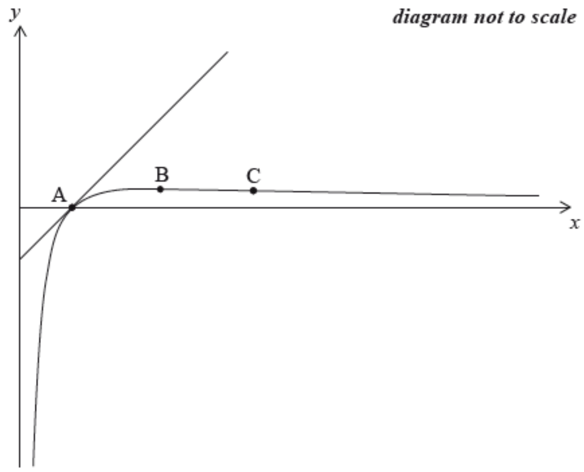
# HL / Integration *[94 marks]*

Consider the function
$f(x) = \frac{\ln x}{x},\ x > 0.$
The sketch below shows the graph of
$y = \ f(x)$ and its tangent at a point A.




1a. Show that
$f'(x) = \frac{1-\ln x}{x^2}$. *[2 marks]*

## Markscheme

$f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \frac{1-\ln x}{x^2}$ ***AG***

***[2 marks]***

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1b. Find the coordinates of B, at which the curve reaches its maximum value. *[3 marks]*

## Markscheme

$\frac{1-\ln x}{x^2} = 0$ has solution
$x = \mathrm{e}$ ***M1A1***

$y = \frac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point
$\left(\mathrm{e},\ \frac{1}{\mathrm{e}}\right)$

***[3 marks]***

1c. Find the coordinates of C, the point of inflexion on the curve. *[5 marks]*

## Markscheme

$f''(x) = \frac{x^2\left(-\frac{1}{x}\right)-2x(1-\ln x)}{x^4}$ ***M1A1***

$= \frac{2\ln x-3}{x^3}$

**Note:** The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where
$f''(x) = 0$ ***M1***

so
$x = \mathrm{e}^{\frac{3}{2}},\ y = \frac{3}{2}\mathrm{e}^{\frac{-3}{2}}$ ***A1A1***

C has coordinates
$\left(\mathrm{e}^{\frac{3}{2}},\ \frac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

***[5 marks]***

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1d. The graph of $y = f(x)$ crosses the $x$-axis at the point A. *[4 marks]*

Find the equation of the tangent to the graph of $f$ at the point A.

## Markscheme

$f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

$y = x + c$ ***(M1)***

through (1, 0)

equation is
$y = x - 1$ ***A1***

***[4 marks]***

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1e. The graph of $y = f(x)$ crosses the $x$-axis at the point A. *[7 marks]*

Find the area enclosed by the curve $y = f(x)$, the tangent at A, and the line $x = \mathrm{e}$.

# Markscheme

**METHOD 1**

area
$= \int_1^{\mathrm{e}} x - 1 - \frac{\ln x}{x} \mathrm{d}x$ ***M1A1A1***

**Note:** Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$\int \frac{\ln x}{x} \mathrm{d}x = \frac{(\ln x)^2}{2} (+c)$ ***(M1)A1***

$\int (x-1) \mathrm{d}x = \frac{x^2}{2} - x (+c)$ ***A1***

$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^{\mathrm{e}}$

$= \left(\frac{1}{2}\mathrm{e}^2 - \mathrm{e} - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}\mathrm{e}^2 - \mathrm{e}$ ***A1***

**METHOD 2**

area = area of triangle
$- \int_1^e \frac{\ln x}{x} \mathrm{d}x$ ***M1A1***

**Note:** ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$\int \frac{\ln x}{x} \mathrm{d}x = \frac{(\ln x)^2}{2} (+c)$ ***(M1)A1***

area of triangle
$= \frac{1}{2}(e-1)(e-1)$ ***M1A1***

$\frac{1}{2}(e-1)(e-1) - \left(\frac{1}{2}\right) = \frac{1}{2}\mathrm{e}^2 - \mathrm{e}$ ***A1***

***[7 marks]***

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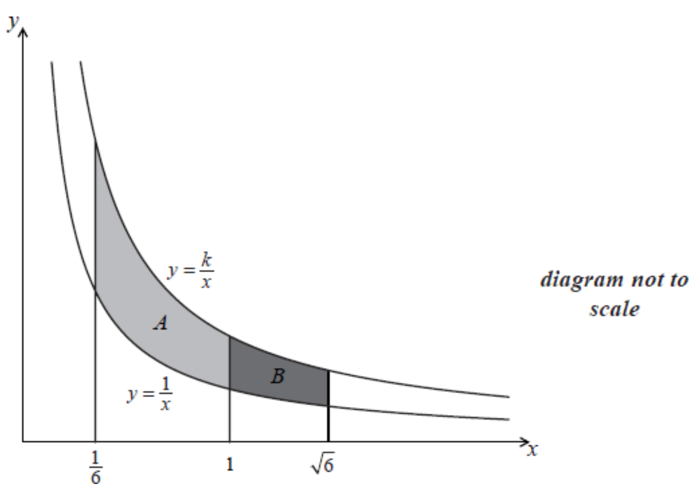
The graph below shows the two curves
$y = \frac{1}{x}$ and
$y = \frac{k}{x}$, where
$k > 1$.




2a. Find the area of region $A$ in terms of $k$ . *[3 marks]*

## Markscheme

$\int_{\frac{1}{6}}^{1} \frac{k}{x} - \frac{1}{x} \mathrm{d}x = (k-1)[\ln x]_{\frac{1}{6}}^{1}$ ***M1*** ***A1***

**Note:** Award ***M1*** for $\int \frac{k}{x} - \frac{1}{x} \mathrm{d}x$ or $\int \frac{1}{x} - \frac{k}{x} \mathrm{d}x$ and ***A1*** for $(k-1)\ln x$ seen in part (a) or later in part (b).

$= (1-k)\ln\frac{1}{6}$ ***A1***

***[3 marks]***

2b. Find the area of region $B$ in terms of $k$. *[2 marks]*

## Markscheme

$\int_{1}^{\sqrt{6}} \frac{k}{x} - \frac{1}{x} \mathrm{d}x = (k-1)[\ln x]_{1}^{\sqrt{6}}$ ***(A1)***

**Note:** Award ***A1*** for correct change of limits.

$= (k-1)\ln\sqrt{6}$ ***A1***

***[2 marks]***

2c. Find the ratio of the area of region $A$ to the area of region $B$. *[3 marks]*

## Markscheme

$(1-k)\ln\frac{1}{6} = (k-1)\ln 6$ ***A1***

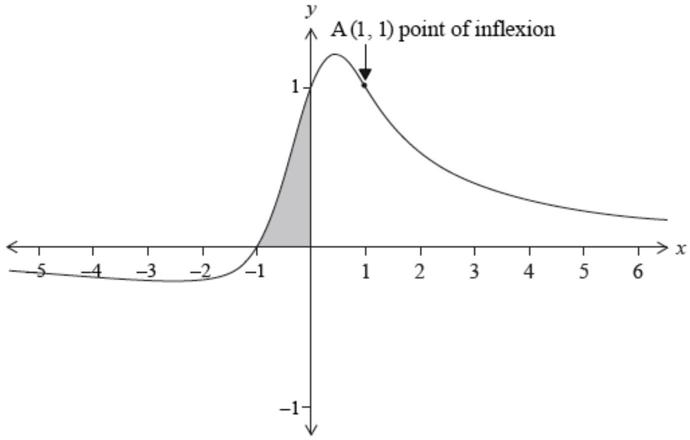
$(k-1)\ln\sqrt{6} = \frac{1}{2}(k-1)\ln 6$ ***A1***

***Note:*** This simplification could have occurred earlier, and marks should still be awarded.

ratio is 2 (or 2:1) ***A1***

***[3 marks]***

The graph of the function
$f(x)=\frac{x+1}{x^2+1}$ is shown below.




3a. Find
$f'(x)$. *[2 marks]*

## Markscheme

(a)
$f'(x)=\frac{(x^2+1)-2x(x+1)}{(x^2+1)^2}\left(=\frac{-x^2-2x+1}{(x^2+1)^2}\right)$ ***M1A1***

***[2 marks]***

3b. Hence find the
$x$-coordinates of the points where the gradient of the graph of
$f$ is zero. *[1 mark]*

## Markscheme

$\frac{-x^2-2x+1}{(x^2+1)^2}=0$

$x=-1\pm\sqrt{2}$ ***A1***

***[1 mark]***

3c. Find
$f''(x)$ expressing your answer in the form
$\frac{p(x)}{(x^2+1)^3}$, where
$p(x)$ is a polynomial of degree 3. *[3 marks]*

## Markscheme

$f''(x) = \frac{(-2x-2)(x^2+1)^2 - 2(2x)(x^2+1)(-x^2-2x+1)}{(x^2+1)^4}$ ***A1A1***

**Note:** Award ***A1*** for
$(-2x-2)(x^2+1)^2$ or equivalent.

**Note:** Award ***A1*** for
$-2(2x)(x^2+1)(-x^2-2x+1)$ or equivalent.

$= \frac{(-2x-2)(x^2+1) - 4x(-x^2-2x+1)}{(x^2+1)^3}$

$= \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$ ***A1***

$\left(= \frac{2(x^3+3x^2-3x-1)}{(x^2+1)^3}\right)$

***[3 marks]***

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The point (1, 1) is a point of inflexion. There are two other points of inflexion.

3d. Find the
$x$-coordinates of the other two points of inflexion. *[4 marks]*

## Markscheme

recognition that
$(x-1)$ is a factor ***(R1)***

$(x-1)(x^2+bx+c) = (x^3+3x^2-3x-1)$ ***M1***

$\Rightarrow x^2+4x+1=0$ ***A1***

$x = -2 \pm \sqrt{3}$ ***A1***

**Note:** Allow long division / synthetic division.

***[4 marks]***

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3e. Find the area of the shaded region. Express your answer in the form
$\frac{\pi}{a} - \ln\sqrt{b}$, where
$a$ and
$b$ are integers. *[6 marks]*

## Markscheme

$\int_{-1}^{0} \frac{x+1}{x^2+1} \mathrm{d}x$ ***M1***

$\int \frac{x+1}{x^2+1} \mathrm{d}x = \int \frac{x}{x^2+1} \mathrm{d}x + \int \frac{1}{x^2+1} \mathrm{d}x$ ***M1***

$= \frac{1}{2}\ln(x^2+1) + \arctan(x)$ ***A1A1***

$= \left[\frac{1}{2}\ln(x^2+1) + \arctan(x)\right]_{-1}^{0} = \frac{1}{2}\ln 1 + \arctan 0 - \frac{1}{2}\ln 2 - \arctan(-1)$ ***M1***

$= \frac{\pi}{4} - \ln\sqrt{2}$ ***A1***

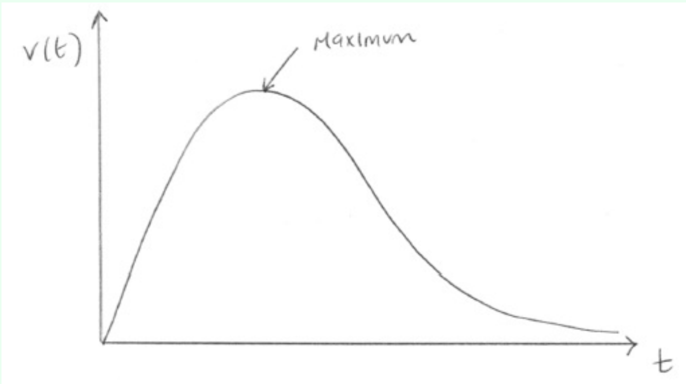
***[6 marks]***

Particle $A$ moves such that its velocity $v\ \mathrm{ms}^{-1}$, at time $t$ seconds, is given by $v(t) = \frac{t}{12+t^4}$, $t \geqslant 0$.

4a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2 marks]*

# Markscheme

(a)




*A1*

**A1** for correct shape and correct domain

$(1.41,\ 0.0884)$ $\left(\sqrt{2},\ \frac{\sqrt{2}}{16}\right)$ ***A1***

***[2 marks]***

4b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4}\mathrm{d}t$. *[4 marks]*

# Markscheme

**EITHER**

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ ***A1***

**OR**

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$ ***A1***

**THEN**

$\int \frac{t}{12+t^4}\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$ ***M1***

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ ***M1***

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent ***A1***

***[4 marks]***

4c. Find the exact distance travelled by particle $A$ between $t = 0$ and $t = 6$ seconds. *[3 marks]*

Give your answer in the form $k\arctan(b),\ k,\ b \in \mathbb{R}$.

## Markscheme

$\int_0^6 \frac{t}{12+t^4}\mathrm{d}t$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\ \left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)\ (\mathrm{m})$ ***A1***

**Note:** Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

***[3 marks]***

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Particle *B* moves such that its velocity $v\ \mathrm{ms}^{-1}$ is related to its displacement $s\ \mathrm{m}$, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

4d. Find the acceleration of particle B when $s = 0.1\ \mathrm{m}$. *[3 marks]*

## Markscheme

$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$

$a = \arcsin\left(\sqrt{s}\right) \times \frac{1}{2\sqrt{s(1-s)}}$ ***(M1)***

$a = \arcsin\left(\sqrt{0.1}\right) \times \frac{1}{2\sqrt{0.1 \times 0.9}}$

$a = 0.536\ (\mathrm{ms}^{-2})$ ***A1***

***[3 marks]***

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An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, $2\pi$ radians about the *y*-axis. Units on the coordinate axes are defined to be in centimetres.

5a. When the glass contains water to a height $h$ cm, find the volume $V$ of water in terms of $h$. *[3 marks]*

## Markscheme

volume

$= \pi \int_0^h x^2 \mathrm{d}y$ ***(M1)***

$\pi \int_0^h y \mathrm{d}y$ ***M1***

$\pi \left[ \frac{y^2}{2} \right]_0^h = \frac{\pi h^2}{2}$ ***A1***

***[3 marks]***

5b. If the water in the glass evaporates at the rate of 3 cm$^3$ per hour for each cm$^2$ of exposed surface area of the water, show that, *[6 marks]*

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where

$t$ is measured in hours.

## Markscheme

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3 \times$ surface area ***A1***

surface area

$= \pi x^2$ ***(M1)***

$= \pi h$ ***A1***

$V = \frac{\pi h^2}{2} \Rightarrow h\sqrt{\frac{2V}{\pi}}$ ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\pi\sqrt{\frac{2V}{\pi}}$ ***A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$ ***AG***

**Note:** Assuming that

$\frac{\mathrm{d}h}{\mathrm{d}t} = -3$ without justification gains no marks.

***[6 marks]***

5c. If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*

# Markscheme

$V_0 = 5000\pi$ (
$= 15700$ cm$^3$) ***A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$

attempting to separate variables ***M1***

**EITHER**

$\int \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int \mathrm{d}t$ ***A1***

$2\sqrt{V} = -3\sqrt{2\pi}t + c$ ***A1***

$c = 2\sqrt{5000\pi}$ ***A1***

$V = 0$ ***M1***

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

**OR**

$\int_{5000\pi}^{0} \frac{\mathrm{d}V}{\sqrt{V}} = -3\sqrt{2\pi}\int_0^T \mathrm{d}t$ ***M1A1A1***

**Note:** Award ***M1*** for attempt to use definite integrals, ***A1*** for correct limits and ***A1*** for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = 3\sqrt{2\pi}T$ ***A1***

$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\frac{1}{3}$ hours ***A1***

***[7 marks]***

6. The function *f* is defined by
$f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\frac{x}{3}\right)$. *[21 marks]*

(a) Write down the largest possible domain, for each of the two terms of the function, *f* , and hence state the largest possible domain, *D* , for *f* .

(b) Find the volume generated when the region bounded by the curve *y* = *f*(*x*) , the *x*-axis, the *y*-axis and the line *x* = 2.8 is rotated through $2\pi$ radians about the *x*-axis.

(c) Find
$f'(x)$ in simplified form.

(d) **Hence** show that
$\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where
$p \in D$ .

(e) Find the value of *p* which maximises the value of the integral in (d).

(f) (i) Show that
$f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that *f*(*x*) has a point of inflexion at *x* = 0 , but not at
$x = \pm\sqrt{\frac{25}{2}}$ .

# Markscheme

(a) For
$x\sqrt{9-x^2}$,
$-3 \leqslant x \leqslant 3$ and for
$2\arcsin\left(\frac{x}{3}\right)$,
$-3 \leqslant x \leqslant 3$ ***A1***

$\Rightarrow D$ is $-3 \leqslant x \leqslant 3$ ***A1***

***[2 marks]***

(b)

$V = \pi \int_0^{2.8} \left(x\sqrt{9-x^2} = 2\arcsin\frac{x}{3}\right)^2 \mathrm{d}x$ ***M1A1***

= 181 ***A1***

***[3 marks]***

(c)

$\frac{\mathrm{d}y}{\mathrm{d}x} = (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$ ***M1A1***

$= (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}}$ ***A1***

$= \frac{11-2x^2}{\sqrt{9-x^2}}$ ***A1***

***[5 marks]***

(d)

$\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}} \mathrm{d}x = \left[x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right]_{-p}^{p}$ ***M1***

$= p\sqrt{9-p^2} + 2\arcsin\frac{p}{3} + p\sqrt{9-p^2} + 2\arcsin\frac{p}{3}$ ***A1***

$= 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$ ***AG***

***[2 marks]***

(e)

$11 - 2p^2 = 0$ ***M1***

$p = 2.35 \quad \left(\sqrt{\frac{11}{2}}\right)$ ***A1***

**Note:** Award ***A0*** for
$p = \pm 2.35$ .

***[2 marks]***

(f) (i)

$f''(x) = \frac{(9-x^2)^{\frac{1}{2}}(-4x) + x(11-2x^2)(9-x^2)^{-\frac{1}{2}}}{9-x^2}$ ***M1A1***

$= \frac{-4x(9-x^2) + x(11-2x^2)}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{-36x + 4x^3 + 11x - 2x^3}{(9-x^2)^{\frac{3}{2}}}$ ***A1***

$= \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$ ***AG***

(ii) **EITHER**

When
$0 < x < 3$,
$f''(x) < 0$. When
$-3 < x < 0$,
$f''(x) > 0$. ***A1***

**OR**

$f''(0) = 0$ ***A1***

**THEN**

Hence
$f''(x)$ changes sign through *x* = 0 , giving a point of inflexion. ***R1***

**EITHER**

$x = \pm\sqrt{\frac{25}{2}}$ is outside the domain of *f*. ***R1***

**OR**

$x = \pm\sqrt{\frac{25}{2}}$ is not a root of
$f''(x) = 0$ . ***R1***

***[7 marks]***

***Total [21 marks]***