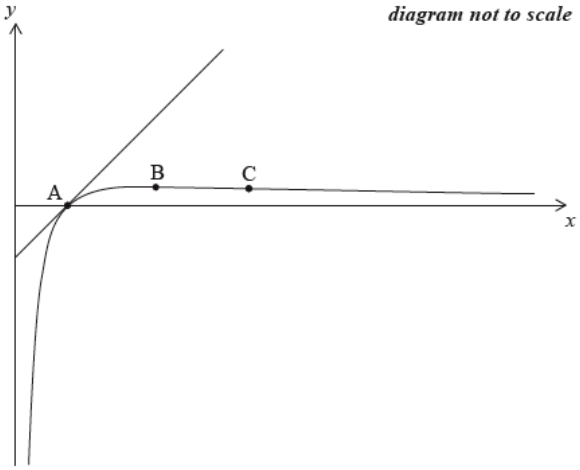
HL / Integration *[94 marks]*

Consider the function
$f(x) = \frac{\ln x}{x}$, $x > 0$.
The sketch below shows the graph of
$y = f(x)$ and its tangent at a point A.



1a. Show that
$f'(x) = \frac{1-\ln x}{x^2}$. *[2 marks]*

1b. Find the coordinates of B, at which the curve reaches its maximum value. *[3 marks]*

1c. Find the coordinates of C, the point of inflexion on the curve. *[5 marks]*

1d. The graph of
$y = f(x)$ crosses the
x-axis at the point A. *[4 marks]*

Find the equation of the tangent to the graph of
f at the point A.

1e. The graph of
$y = f(x)$ crosses the
x-axis at the point A. *[7 marks]*

Find the area enclosed by the curve
$y = f(x)$, the tangent at A, and the line
$x = \text{e}$.

The graph below shows the two curves $y = \frac{1}{x}$ and $y = \frac{k}{x}$, where $k > 1$.

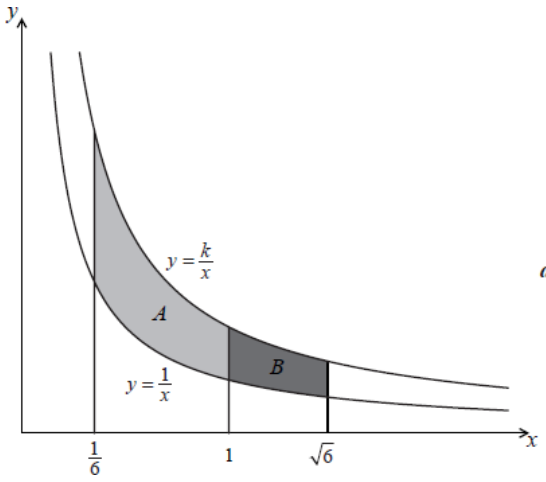


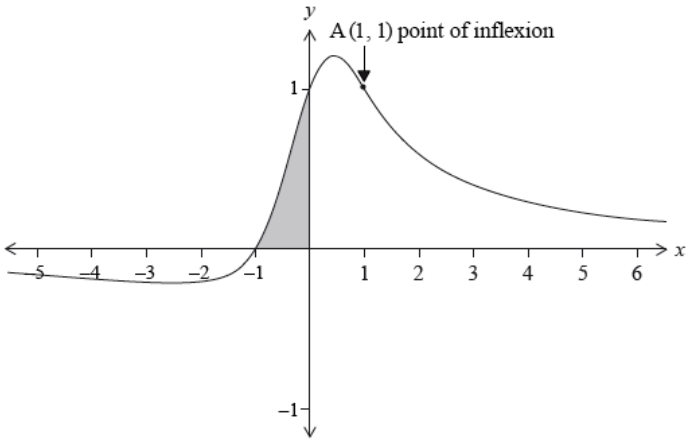
diagram not to scale

2a. Find the area of region *A* in terms of *k*. *[3 marks]*

2b. Find the area of region *B* in terms of *k*. *[2 marks]*

2c. Find the ratio of the area of region *A* to the area of region *B*. *[3 marks]*

The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



3a. Find $f'(x)$. *[2 marks]*

3b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero. *[1 mark]*

3c. Find $f''(x)$ expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. *[3 marks]*

The point (1, 1) is a point of inflexion. There are two other points of inflexion.

3d. Find the x-coordinates of the other two points of inflexion. *[4 marks]*

3e. Find the area of the shaded region. Express your answer in the form $\frac{\pi}{a} - \ln\sqrt{b}$, where a and b are integers. *[6 marks]*

Particle *A* moves such that its velocity $v \text{ ms}^{-1}$, at time *t* seconds, is given by $v(t) = \frac{t}{12+t^4},\ t \geqslant 0$.

4a. Sketch the graph of $y = v(t)$. Indicate clearly the local maximum and write down its coordinates. *[2 marks]*

4b. Use the substitution $u = t^2$ to find $\int \frac{t}{12+t^4}\mathrm{d}t$. *[4 marks]*

4c. Find the exact distance travelled by particle A between $t = 0$ and $t = 6$ seconds. *[3 marks]*

Give your answer in the form $k\arctan(b),\ k,\ b \in \mathbb{R}$.

Particle *B* moves such that its velocity $v \text{ ms}^{-1}$ is related to its displacement s m, by the equation $v(s) = \arcsin\left(\sqrt{s}\right)$.

4d. Find the acceleration of particle B when $s = 0.1$ m. *[3 marks]*

An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the *y*-axis. Units on the coordinate axes are defined to be in centimetres.

5a. When the glass contains water to a height h cm, find the volume V of water in terms of h . *[3 marks]*

5b. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, $\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where t is measured in hours. *[6 marks]*

5c. If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*

6. The function *f* is defined by
$f(x) = x\sqrt{9-x^2} + 2\arcsin\left(\frac{x}{3}\right)$. *[21 marks]*

(a) Write down the largest possible domain, for each of the two terms of the function, *f* , and hence state the largest possible domain, *D* , for *f* .

(b) Find the volume generated when the region bounded by the curve *y* = *f*(*x*) , the *x*-axis, the *y*-axis and the line *x* = 2.8 is rotated through 2π radians about the *x*-axis.

(c) Find
$f'(x)$ in simplified form.

(d) **Hence** show that
$\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}}\mathrm{d}x = 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$, where
$p \in D$.

(e) Find the value of *p* which maximises the value of the integral in (d).

(f) (i) Show that
$f''(x) = \frac{x(2x^2-25)}{(9-x^2)^{\frac{3}{2}}}$.

(ii) Hence justify that *f*(*x*) has a point of inflexion at *x* = 0 , but not at
$x = \pm\sqrt{\frac{25}{2}}$.