IB Mathematics SL

Differential Calculus - Test 1

SOLUTION KEY

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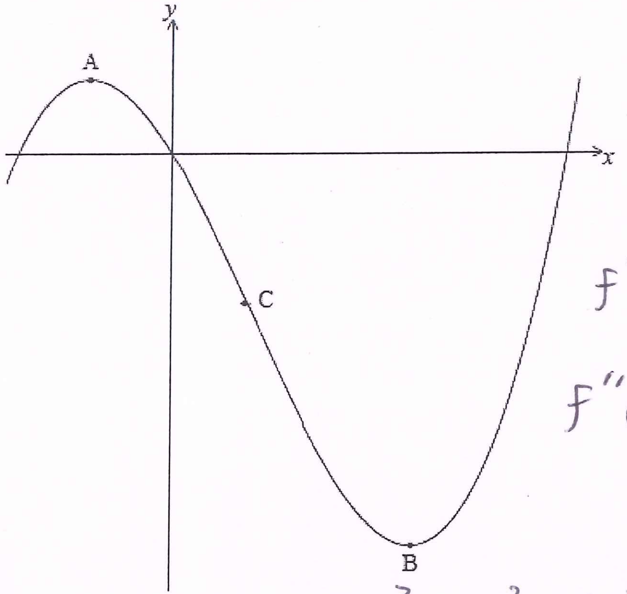
Part I - <u>No calculator</u> **allowed for questions 1 – 4.** **total marks on test: 55**

1. Let $f(x)=\frac{1}{3}x^3-x^2-3x$. The graph of f has a maximum at point A, a minimum at point B, and an inflexion point at point C. Find the coordinates of the three points A, B and C. **[6 marks]**



$f'(x) = x^2-2x-3 = 0$

$(x+1)(x-3)=0$

$f'(x)=0$ at $x=-1$, $x=3$

$f''(x)= 2x-2 = 2(x-1) = 0$

$f''(x)=0$ at $x=1$

$f(-1) = \frac{1}{3}(-1)^3-(-1)^2-3(-1) = \frac{1}{3}(-1)-1+3 = -\frac{1}{3}+2 = -\frac{1}{3}+\frac{6}{3} = \frac{5}{3}$

$f(3) = \frac{1}{3}(3)^3 - 3^2-3(3) = \frac{1}{3}(27)-9-9 = 9-18 = -9$

$f(1) = \frac{1}{3}(1)^3-1^2-3(1) = \frac{1}{3}-1-3 = \frac{1}{3}-\frac{12}{3} = -\frac{11}{3}$

coordinates of A, B + C are $A(-1,\frac{5}{3})$, $B(3,-9)$ and $C(1,-\frac{11}{3})$

2. Consider the function $g(x)=8x^2+\frac{c}{x}$, $x\neq 0$, where c is a constant. **[7 marks]**

(a) Find $g'(x)$

(b) There is a minimum value of $g(x)$ when $x=\frac{1}{2}$. Find the value of c.

(a) $g(x) = 8x^2+cx^{-1}$ $\quad g'(x) = 16x - cx^{-2}$

$g'(x) = 16x - \frac{c}{x^2}$

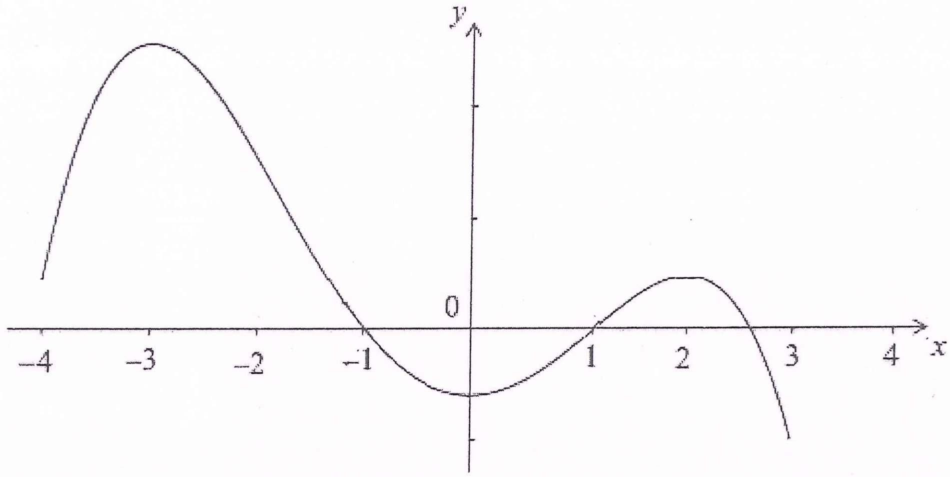
(b) $g'(x)=0$ when $x=\frac{1}{2}$

$g'(\frac{1}{2}) = 16(\frac{1}{2}) - \frac{c}{(\frac{1}{2})^2} = 0$

$8 - \frac{c}{\frac{1}{4}} = 0 \Rightarrow 8-4c=0$

$c=2$

3. A function $h(x)$ is defined for $-4 \le x \le 3$. The graph of $h(x)$ is given below. **[8 marks]**



The graph of $h(x)$ has a local minimum at $x = 0$, and local maxima when $x = -3$ and $x = 2$

(a) Write down the x-intercepts of the graph of the **derivative** function, $h'(x)$.

x-intercepts of derivative function are $(-3, 0)$, $(0, 0)$ and $(2, 0)$

(b) Write down all values of x for which $h'(x)$ is positive.

$-4 \le x < -3$, $0 < x < 2$

(c) Explain why $h''(2.5) < 0$.

at $x = 2.5$, the graph of $h(x)$ is concave down
therefore $h''(2.5)$ must be negative

4. Consider the function $f(x) = x^2 - 5x + 2$. Use the formula $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ to show that the derivative of $f(x)$ is $2x - 5$.

[6 marks]

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 5(x+h) + 2 - (x^2 - 5x + 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 \cancel{- 5x} - 5h \cancel{+ 2} \cancel{- x^2} \cancel{+ 5x} \cancel{- 2}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2 - 5h}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x + h - 5)}{\cancel{h}} = \lim_{h\to 0} (2x + h - 5)$$

$$= 2x + 0 - 5 = 2x - 5$$

therefore, $f'(x) = 2x - 5$

Q.E.D.

Part II – Calculator allowed – Qs 5-8

5. Consider the cubic function $f(x) = 2x^3 + 3x^2 - 12x + 4$. **[8 marks]**

(a) Find the coordinates of any stationary points and confirm whether they are a maximum, minimum or neither.

(b) Find the coordinates of any inflexion point. Justify that the concavity of f changes at any inflexion point you find.

(a) $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 0$

$= (x+2)(x-1) = 0$

$f'(x) = 0$ at $x = -2,\ x = 1$

$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 4 = 24$

$f(1) = 2(1)^3 + 3(1)^2 - 12(1) + 4 = -3$

$f''(-2) = -18 < 0$ concave down

$f''(x) = 12x + 6 = 6(2x+1)$

$f''(1) = 18 > 0$ concave up

therefore, f has a max. at $(-2, 24)$ and a min. at $(1, -3)$

(b) $f'(x) = 0$ when $x = -\frac{1}{2}$ $\quad f(-\frac{1}{2}) = \frac{21}{2}$

let $x = -1$ be left test pt., and $x = 0$ be right test point

$f'(-1) = -6$ and $f'(0) = 6$, therefore concavity changes at $x = -\frac{1}{2}$

and f has an inflexion pt. at $(-\frac{1}{2}, \frac{21}{2})$

6. Find the equation of the line tangent to $y = \sqrt{2-4x}$ at the point $\left(\frac{1}{4}, 1\right)$. **[8 marks]**

$y = (2-4x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(2-4x)^{-\frac{1}{2}}(-4) = \frac{-2}{\sqrt{2-4x}}$

at $x = \frac{1}{4}$, $\frac{dy}{dx} = \frac{-2}{\sqrt{2-4(\frac{1}{4})}} = \frac{-2}{\sqrt{2-1}} = -2$

tangent line has slope $(m) = -2$ and passes through $(\frac{1}{4}, 1)$

$y - 1 = -2(x - \frac{1}{4}) \Rightarrow y - 1 = -2x + \frac{1}{2} \Rightarrow y = -2x + \frac{1}{2} + \frac{2}{2}$

equation of tangent line is $y = -2x + \frac{3}{2}$

7. Explain why the graph of the function $g(x)=\dfrac{1}{3x+6}$ has no stationary points. **[5 marks]**

$g(x)=(3x+6)^{-1}$ $\quad g'(x)=-(3x+6)^{-2}(3)=\dfrac{-3}{(3x+6)^2}$

$g'(x)=\dfrac{-3}{(3x+6)^2}=0$ has no solutions

[the fraction $\dfrac{-3}{(3x+6)^2}$ can never be zero]

thus, since there are no x-values that solve $g'(x)=0$ then $g(x)$ has no stationary points (i.e. no points on its graph where slope of tangent is zero)

8. The curve $y=x^3+ax^2+2x-1$ has an inflexion point at $x=-3$. Find the value of a. **[7 marks]**

$y'=3x^2+2ax+2$

$y''=6x+2a$ $\quad y''$ must be zero when $x=-3$

thus, $6(-3)+2a=0$

$-18+2a=0 \Rightarrow a=9$