SL/Derivatives *[87 marks]*

Let
$g(x) = \frac{\ln x}{x^2}$, for
$x > 0$.

1a. Use the quotient rule to show that
$g'(x) = \frac{1-2\ln x}{x^3}$. *[4 marks]*

Markscheme

$\frac{d}{dx}\ln x = \frac{1}{x}$,
$\frac{d}{dx}x^2 = 2x$ (seen anywhere) ***A1A1***

attempt to substitute into the quotient rule (do **not** accept product rule) ***M1***

e.g.
$\frac{x^2\left(\frac{1}{x}\right)-2x\ln x}{x^4}$

correct manipulation that clearly leads to result ***A1***

e.g.
$\frac{x-2x\ln x}{x^4}$,
$\frac{x(1-2\ln x)}{x^4}$,
$\frac{x}{x^4}$,
$\frac{2x\ln x}{x^4}$

$g'(x) = \frac{1-2\ln x}{x^3}$ ***AG*** ***N0***

[4 marks]

1b. The graph of *g* has a maximum point at A. Find the *x*-coordinate of A. *[3 marks]*

Markscheme

evidence of setting the derivative equal to zero ***(M1)***

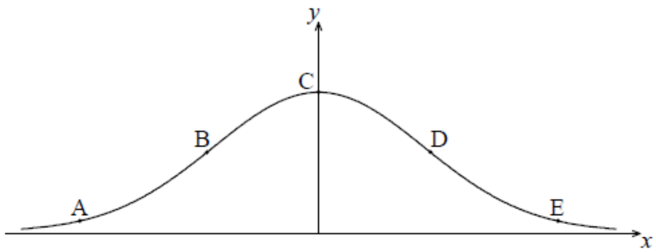
e.g.
$g'(x) = 0$,
$1 - 2\ln x = 0$

$\ln x = \frac{1}{2}$ ***A1***

$x = e^{\frac{1}{2}}$ ***A1*** ***N2***

[3 marks]

The following diagram shows the graph of
$f(x) = \mathrm{e}^{-x^2}$.



The points A, B, C, D and E lie on the graph of f . Two of these are points of inflexion.

2a. Identify the **two** points of inflexion. *[2 marks]*

Markscheme

B, D ***A1A1*** ***N2***

[2 marks]

2b. (i) Find
$f'(x)$. *[5 marks]*

(ii) Show that
$f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$.

Markscheme

(i)
$f'(x) = -2x\mathrm{e}^{-x^2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for
e^{-x^2} and ***A1*** for
$-2x$.

(ii) finding the derivative of
$-2x$, i.e.
-2 ***(A1)***

evidence of choosing the product rule ***(M1)***

e.g.
$-2\mathrm{e}^{-x^2}$
$-2x \times -2x\mathrm{e}^{-x^2}$

$-2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2}$ ***A1***

$f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2}$ ***AG*** ***N0***

[5 marks]

2c. Find the *x*-coordinate of each point of inflexion. *[4 marks]*

Markscheme

valid reasoning ***R1***

e.g.
$f''(x) = 0$

attempting to solve the equation ***(M1)***

e.g.
$(4x^2 - 2) = 0$, sketch of
$f''(x)$

$p = 0.707$
$\left(= \frac{1}{\sqrt{2}}\right)$,
$q = -0.707$
$\left(= -\frac{1}{\sqrt{2}}\right)$ ***A1A1 N3***

[4 marks]

2d. Use the second derivative to show that one of these points is a point of inflexion. *[4 marks]*

Markscheme

evidence of using second derivative to test values on either side of POI ***M1***

e.g. finding values, reference to graph of
f'' , sign table

correct working ***A1A1***

e.g. finding any two correct values either side of POI,

checking sign of
f'' on either side of POI

reference to sign change of
$f''(x)$ ***R1 N0***

[4 marks]

Consider
$f(x) = \ln(x^4 + 1)$.

3a. Find the value of
$f(0)$. *[2 marks]*

Markscheme

substitute
0 into
f ***(M1)***

eg
$\ln(0 + 1)$,
$\ln 1$

$f(0) = 0$ ***A1 N2***

[2 marks]

3b. Find the set of values of
x for which
f is increasing. *[5 marks]*

Markscheme

$f'(x) = \frac{1}{x^4+1} \times 4x^3$ (seen anywhere) ***A1A1***

Note: Award ***A1*** for
$\frac{1}{x^4+1}$ and ***A1*** for
$4x^3$.

recognizing
f increasing where
$f'(x) > 0$ (seen anywhere) ***R1***

eg
$f'(x) > 0$, diagram of signs

attempt to solve
$f'(x) > 0$ ***(M1)***

eg
$4x^3 = 0$,
$x^3 > 0$

f increasing for
$x > 0$ (accept
$x \geq 0$) ***A1*** ***N1***

[5 marks]

The second derivative is given by
$f''(x) = \frac{4x^2(3-x^4)}{(x^4+1)^2}$.

The equation
$f''(x) = 0$ has only three solutions, when
$x = 0$,
$\pm\sqrt[4]{3}$
$(\pm 1.316\ldots)$.

3c. (i) Find
$f''(1)$.

[5 marks]

(ii) **Hence**, show that there is no point of inflexion on the graph of
f at
$x = 0$.

Markscheme

(i) substituting
$x = 1$ into
f'' ***(A1)***

eg
$\frac{4(3-1)}{(1+1)^2}$,
$\frac{4\times 2}{4}$

$f''(1) = 2$ ***A1*** ***N2***

(ii) valid interpretation of point of inflexion (seen anywhere) ***R1***

eg no change of sign in
$f''(x)$, no change in concavity,

f' increasing both sides of zero

attempt to find
$f''(x)$ for
$x < 0$ ***(M1)***

eg
$f''(-1)$,
$\frac{4(-1)^2(3-(-1)^4)}{((-1)^4+1)^2}$, diagram of signs

correct working leading to positive value ***A1***

eg
$f''(-1) = 2$, discussing signs of numerator **and** denominator

there is no point of inflexion at
$x = 0$ ***AG*** ***N0***

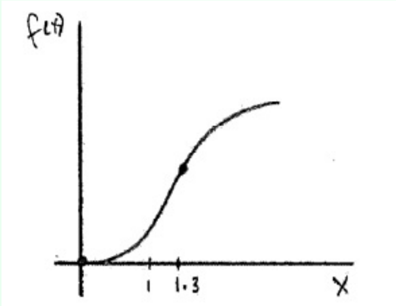
[5 marks]

3d. There is a point of inflexion on the graph of
f at
$x = \sqrt[4]{3}$
$(x = 1.316\ldots)$.

[3 marks]

Sketch the graph of
f , for
$x \geq 0$.

Markscheme



A1A1A1 ***N3***

Notes: Award ***A1*** for shape concave up left of POI and concave down right of POI.

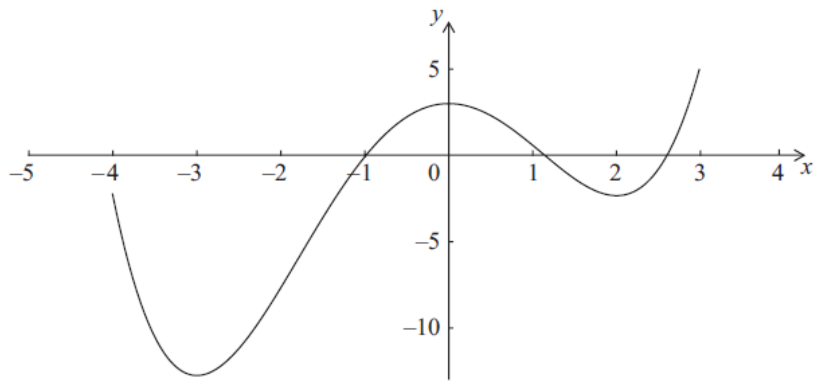
Only if this ***A1*** is awarded, then award the following:

A1 for curve through (
0,
0) , ***A1*** for increasing throughout.

Sketch need not be drawn to scale. Only essential features need to be clear.

[3 marks]

A function *f* is defined for
$-4 \le x \le 3$. The graph of *f* is given below.



The graph has a local maximum when
$x = 0$, and local minima when
$x = -3$,
$x = 2$.

4a. Write down the *x*-intercepts of the graph of the **derivative** function,
f' . *[2 marks]*

Markscheme

x-intercepts at
$-3, 0, 2$ ***A2*** ***N2***

[2 marks]

4b. Write down all values of *x* for which
$f'(x)$ is positive. *[2 marks]*

Markscheme

$-3 < x < 0$,
$2 < x < 3$ ***A1A1 N2***

[2 marks]

4c. At point D on the graph of f , the x-coordinate is -0.5. Explain why $f''(x) < 0$ at D. *[2 marks]*

Markscheme

correct reasoning ***R2***

e.g. the graph of *f* is **concave-down** (accept convex), the first derivative is decreasing

therefore the second derivative is negative ***AG***

[2 marks]

Consider the function *f* with second derivative $f''(x) = 3x - 1$. The graph of *f* has a minimum point at A(2, 4) and a maximum point at $\mathrm{B}\left(-\frac{4}{3}, \frac{358}{27}\right)$.

5a. Use the second derivative to justify that B is a maximum. *[3 marks]*

Markscheme

substituting into the second derivative ***M1***

e.g.
$3 \times \left(-\frac{4}{3}\right) - 1$

$f''\left(-\frac{4}{3}\right) = -5$ ***A1***

since the second derivative is negative, B is a maximum ***R1 N0***

[3 marks]

5b. Given that $f'(x) = \frac{3}{2}x^2 - x + p$, show that $p = -4$. *[4 marks]*

Markscheme

setting
$f'(x)$ equal to zero ***(M1)***

evidence of substituting
$x = 2$ (or
$x = -\frac{4}{3}$) ***(M1)***

e.g.
$f'(2)$

correct substitution ***A1***

e.g.
$\frac{3}{2}(2)^2 - 2 + p$,
$\frac{3}{2}\left(-\frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right) + p$

correct simplification

e.g.
$6 - 2 + p = 0$,
$\frac{8}{3} + \frac{4}{3} + p = 0$,
$4 + p = 0$ ***A1***

$p = -4$ ***AG*** ***N0***

[4 marks]

5c. Find
$f(x)$. *[7 marks]*

Markscheme

evidence of integration ***(M1)***

$f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - 4x + c$ ***A1A1A1***

substituting (2, 4) or
$\left(-\frac{4}{3}, \frac{358}{27}\right)$ into **their** expression ***(M1)***

correct equation ***A1***

e.g.
$\frac{1}{2} \times 2^3 - \frac{1}{2} \times 2^2 - 4 \times 2 + c = 4$,
$\frac{1}{2} \times 8 - \frac{1}{2} \times 4 - 4 \times 2 + c = 4$,
$4 - 2 - 8 + c = 4$

$f(x) = \frac{1}{2}x^3 - \frac{1}{2}x^2 - 4x + 10$ ***A1*** ***N4***

[7 marks]

Consider
$f(x) = x^2 + \frac{p}{x}$,
$x \neq 0$, where *p* is a constant.

6a. Find
$f'(x)$. *[2 marks]*

Markscheme

$f'(x) = 2x - \frac{p}{x^2}$ ***A1A1*** ***N2***

Note: Award ***A1*** for $2x$, ***A1*** for $-\frac{p}{x^2}$.

[2 marks]

6b. There is a minimum value of $f(x)$ when $x = -2$. Find the value of p. *[4 marks]*

Markscheme

evidence of equating derivative to 0 (seen anywhere) ***(M1)***

evidence of finding $f'(-2)$ (seen anywhere) ***(M1)***

correct equation ***A1***

e.g.
$-4 - \frac{p}{4} = 0$,
$-16 - p = 0$

$p = -16$ ***A1*** ***N3***

[4 marks]

Let $f'(x) = -24x^3 + 9x^2 + 3x + 1$.

7a. There are two points of inflexion on the graph of f. Write down the x-coordinates of these points. *[3 marks]*

Markscheme

valid approach ***R1***

e.g.
$f''(x) = 0$, the max and min of f' gives the points of inflexion on f

$-0.114, 0.364$ (accept ($-0.114, 0.811$) and ($0.364, 2.13$)) ***A1A1*** ***N1N1***

[3 marks]

7b. Let $g(x) = f''(x)$. Explain why the graph of g has no points of inflexion. *[2 marks]*

Markscheme

METHOD 1

graph of g is a quadratic function ***R1*** ***N1***

a quadratic function does not have any points of inflexion ***R1*** ***N1***

METHOD 2

graph of g is concave down over entire domain ***R1*** ***N1***

therefore no change in concavity ***R1*** ***N1***

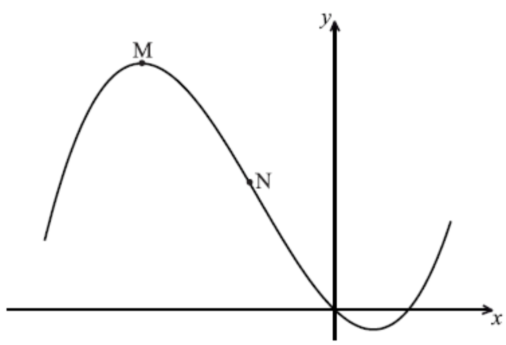
METHOD 3

$g''(x) = -144$ ***R1 N1***

therefore no points of inflexion as
$g''(x) \neq 0$ ***R1*** ***N1***

[2 marks]

Consider
$f(x) = \frac{1}{3}x^3 + 2x^2 - 5x$. Part of the graph of f is shown below. There is a maximum point at M, and a point of inflexion at N.



8a. Find
$f'(x)$. *[3 marks]*

Markscheme

$f'(x) = x^2 + 4x - 5$ ***A1A1A1*** ***N3***

[3 marks]

8b. Find the *x*-coordinate of M. *[4 marks]*

Markscheme

evidence of attempting to solve
$f'(x) = 0$ ***(M1)***

evidence of correct working ***A1***

e.g.
$(x+5)(x-1)$,
$\frac{-4\pm\sqrt{16+20}}{2}$, sketch

$x = -5,$
$x = 1$ ***(A1)***

so
$x = -5$ ***A1*** ***N2***

[4 marks]

8c. Find the *x*-coordinate of N. *[3 marks]*

Markscheme

METHOD 1

$f''(x) = 2x + 4$ (may be seen later) ***A1***

evidence of setting second derivative = 0 ***(M1)***

e.g.
$2x + 4 = 0$

$x = -2$ ***A1*** ***N2***

METHOD 2

evidence of use of symmetry ***(M1)***

e.g. midpoint of max/min, reference to shape of cubic

correct calculation ***A1***

e.g.
$\frac{-5+1}{2}$

$x = -2$ ***A1*** ***N2***

[3 marks]

8d. The line *L* is the tangent to the curve of *f* at $(3, 12)$. Find the equation of *L* in the form $y = ax + b$. *[4 marks]*

Markscheme

attempting to find the value of the derivative when
$x = 3$ ***(M1)***

$f'(3) = 16$ ***A1***

valid approach to finding the equation of a line ***M1***

e.g.
$y - 12 = 16(x - 3)$,
$12 = 16 \times 3 + b$

$y = 16x - 36$ ***A1*** ***N2***

[4 marks]

A function *f* has its first derivative given by
$f'(x) = (x-3)^3$.

9a. Find the second derivative. *[2 marks]*

Markscheme

METHOD 1

$f''(x) = 3(x-3)^2$ ***A2 N2***

METHOD 2

attempt to expand
$(x-3)^3$ ***(M1)***

e.g.
$f'(x) = x^3 - 9x^2 + 27x - 27$

$f''(x) = 3x^2 - 18x + 27$ ***A1 N2***

[2 marks]

9b. Find
$f'(3)$ and
$f''(3)$. *[1 mark]*

Markscheme

$f'(3) = 0$,
$f''(3) = 0$ ***A1 N1***

[1 mark]

9c. The point P on the graph of *f* has *x*-coordinate
3. Explain why P is not a point of inflexion. *[2 marks]*

Markscheme

METHOD 1

f'' does not change sign at P ***R1***

evidence for this ***R1 N0***

METHOD 2

f' changes sign at P so P is a maximum/minimum (i.e. not inflexion) ***R1***

evidence for this ***R1 N0***

METHOD 3

finding
$f(x) = \frac{1}{4}(x-3)^4 + c$ and sketching this function ***R1***

indicating minimum at
$x = 3$ ***R1 N0***

[2 marks]