Differential Calculus – Test 3

SOLUTION KEY

total marks on test: 80

Section 1 (questions 1-4) - **NO** calculator allowed

1. Find $\frac{dy}{dx}$ for the curve with the equation $2x^2y+3y^2=16$. **[8 marks]**

$$2\frac{d}{dx}\left(x^2y\right)+3\frac{d}{dx}\left(y^2\right)=\frac{d}{dx}(16)$$

$$2\left[2xy+x^2\cdot\frac{dy}{dx}\right]+3\left[2y\cdot\frac{dy}{dx}\right]=0$$

$$4xy+2x^2\frac{dy}{dx}+6y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(2x^2+6y\right)=-4xy$$

$$\frac{dy}{dx}=\frac{-4xy}{2x^2+6y}$$

$$\frac{dy}{dx}=\frac{-2xy}{x^2+3y}$$

2. Find the equation of the line tangent to the curve $y=xe^{2x}$ at the point where $x=1$. Express the answer exactly in slope-intercept form; that is, $y=mx+c$. **[8 marks]**

$y=xe^{2x}$ when $x=1$: $y=1\cdot e^{2\cdot 1}=e^2$

point of tangency is $(1,e^2)$

$$\frac{dy}{dx}=e^{2x}+x\left(e^{2x}\cdot 2\right)=e^{2x}+2xe^{2x}$$

when $x=1$: $\frac{dy}{dx}=e^2+2e^2=3e^2$

eqn. of tangent line: $y-e^2=3e^2(x-1)$

$$y=3e^2x-3e^2+e^2$$

$$y=3e^2x-2e^2$$

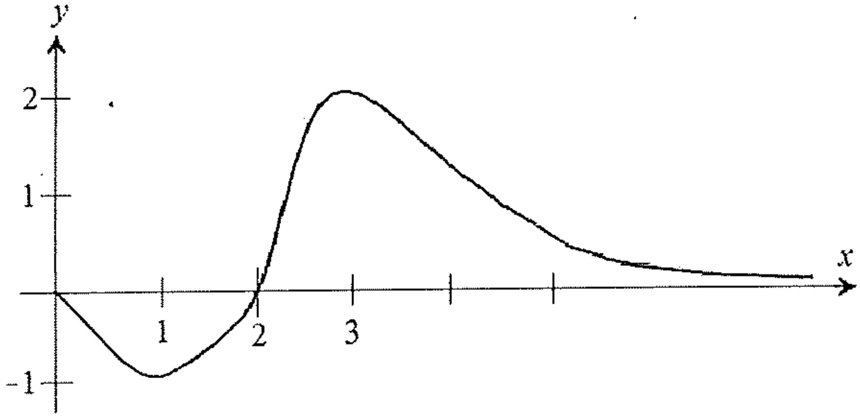
$\rightarrow y = x(1-x^3)^{\frac{1}{2}}$

3. Given that $y = x\sqrt{1-x^3}$, find $\dfrac{dy}{dx}$ and write it in the form of a single rational expression. **[8 marks]**

$$\frac{dy}{dx} = (1-x^3)^{\frac{1}{2}} + x\left[\tfrac{1}{2}(1-x^3)^{-\frac{1}{2}}(-3x^2)\right] = (1-x^3)^{-\frac{1}{2}}\left[(1-x^3) - \tfrac{3}{2}x^3\right]$$

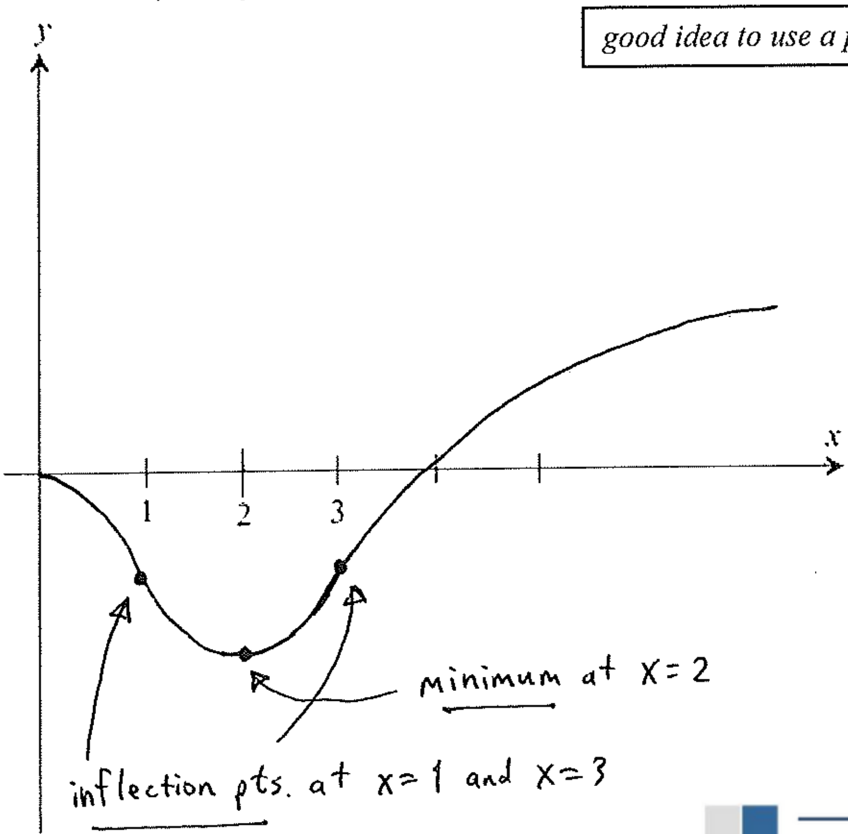
$$= \frac{1-\frac{5}{2}x^3}{\sqrt{1-x^3}} \qquad \frac{dy}{dx} = \frac{2-5x^3}{2\sqrt{1-x^3}}$$

4. The diagram shows the graph of $y = f'(x)$, the **derivative** of $f(x)$. The graph of $f'(x)$ passes through $(2,0)$, has a minimum at $(1,-1)$, a maximum at $(3,2)$ and a horizontal asymptote at $y = 0$. **[8 marks]**



On the axes below, given that $f(0) = 0$, sketch the graph of $y = f(x)$ clearly indicating the location (i.e. the x-coordinate) of any maxima, minima or inflection points.

good idea to use a pencil for sketch



- concave down from 0 to 1
- concave up from 1 to 3
- concave down for $x > 3$

minimum at $x=2$

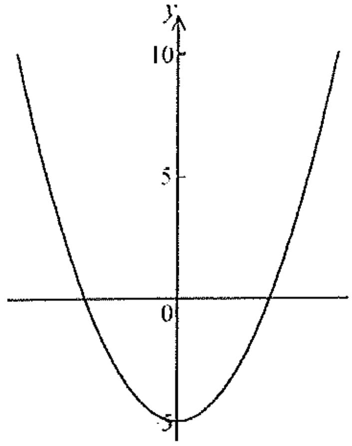
inflection pts. at $x=1$ and $x=3$

Section 2 (questions 5-8) - calculator *is* allowed

5. The curve $y = x^2 - 5$ is graphed at right. A point P on the curve has an x-coordinate of a.

(a) Show that the distance from the origin to P is given by $\mathrm{OP} = \sqrt{a^4 - 9a^2 + 25}$. **[3 marks]**

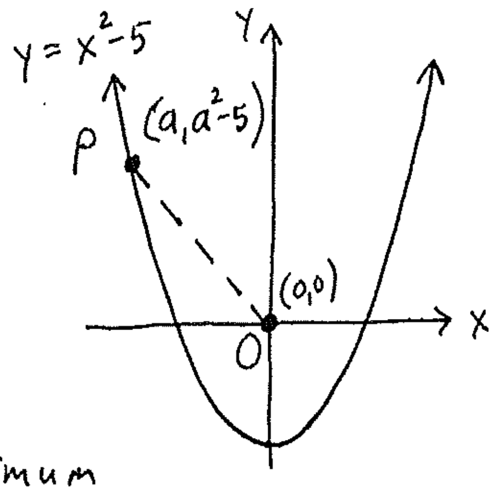
(b) Find the values of a for which the curve is closest to the origin. **[8 marks]**



(a) $OP = \sqrt{(a-0)^2 + (a^2-5-0)^2}$

$= \sqrt{a^2 + a^4 - 10a^2 + 25}$

$OP = \sqrt{a^4 - 9a^2 + 25}$



(b) the value of a that gives a minimum for $\sqrt{a^4 - 9a^2 + 25}$ will also give a minimum for $a^4 - 9a^2 + 25$

$f(a) = a^4 - 9a^2 + 25$ $\quad f'(a) = 4a^3 - 18a = 0$

$2a(2a^2 - 9) = 0$

$a = 0$ OR $a^2 = \frac{9}{2}$

$a = \pm \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm \frac{3\sqrt{2}}{2}$

$f''(a) = 12a^2 - 18$

$f''(0) = -18 < 0 \rightarrow$ local max. at $a = 0$

$f''\left(\pm\frac{3\sqrt{2}}{2}\right) = 12\left(\frac{9}{2}\right) - 18 = 36 > 0 \rightarrow$ Min. at $a = \frac{3\sqrt{2}}{2}$ and $a = -\frac{3\sqrt{2}}{2}$

$a \approx 2.12$ $\quad a \approx -2.12$

6. Air is being pumped into a spherical balloon at a rate of $10\ \text{cm}^3$ per second. When the radius is 3 cm, find: $\left[\text{Volume of a sphere } V = \frac{4}{3}\pi r^3\right]$

(a) the rate of increase of the radius of the balloon (include units). **[6 marks]**

(b) the rate of increase of the surface area of the balloon (include units). $[\text{surface area} = 4\pi r^2]$

[6 marks]

$$\frac{dV}{dt} = 10\ \frac{cm^3}{sec}$$

(a) $V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = \frac{4}{3}\pi\left[3r^2 \cdot \frac{dr}{dt}\right] = 4\pi r^2 \cdot \frac{dr}{dt}$

when $r=3$: $\quad 10 = 4\pi(3)^2 \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{10}{36\pi} = \frac{5}{18\pi} \approx 0.0884\ \frac{cm}{sec}$

(b) $A = 4\pi r^2 \qquad \frac{dA}{dt} = 4\pi\left[2r \cdot \frac{dr}{dt}\right] = 8\pi r \frac{dr}{dt}$

when $r=3$: $\quad \frac{dA}{dt} = 8\pi(3)\left(\frac{5}{18\pi}\right) = \frac{120}{18} = \frac{20}{3}\ \frac{cm^2}{sec} \quad \left[6.\bar{6}\ \frac{cm^2}{sec}\right]$

7. Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$. **[8 marks]**

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

[2 marks]

(a) $2x + y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$

$$\frac{dy}{dx}(x + 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$$

at $(-1, k)$: $\quad \frac{dy}{dx} = \frac{-2(-1) - k}{-1 + 2k} = \frac{2-k}{2k-1}$

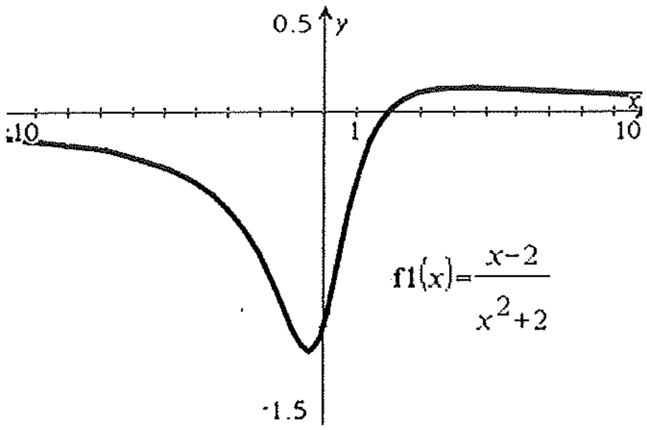
(b) $\frac{dy}{dx} = \frac{2-k}{2k-1} = 0 \rightarrow 2 - k = 0 \rightarrow k = 2$

8. The graph of the function $y = \dfrac{x-2}{x^2+2}$ is shown below.

(a) Find $f'(x)$. **[3 marks]**

(b) Hence, find the **exact** coordinates of the points where the gradient of the graph of f is zero. **[6 marks]**

(c) Find $f''(x)$ expressing your answer in the form $\dfrac{p(x)}{\left(x^2+2\right)^3}$, where $p(x)$ is a cubic polynomial. **[6 marks]**



(a) $f'(x) = \dfrac{(x^2+2)-(x-2)(2x)}{(x^2+2)^2} = \dfrac{x^2+2-2x^2+4x}{(x^2+2)^2} = \dfrac{-x^2+4x+2}{(x^2+2)^2}$

(b) $f'(x) = \dfrac{-x^2+4x+2}{(x^2+2)^2} = 0 \Rightarrow -x^2+4x+2 = 0 \Rightarrow x^2-4x-2=0$

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2-4(1)(-2)}}{2} = \dfrac{4 \pm \sqrt{16+8}}{2} = \dfrac{4 \pm \sqrt{24}}{2} = \dfrac{4 \pm 2\sqrt{6}}{2} = 2 \pm \sqrt{6}$

$f(2+\sqrt{6}) = \dfrac{2+\sqrt{6}-2}{(2+\sqrt{6})^2+2} = \dfrac{\sqrt{6}}{4+4\sqrt{6}+6+2} = \dfrac{\sqrt{6}}{12+4\sqrt{6}} \cdot \dfrac{12-4\sqrt{6}}{12-4\sqrt{6}} = \dfrac{12\sqrt{6}-24}{144-96} = \dfrac{12\sqrt{6}-24}{48} = \dfrac{\sqrt{6}-2}{4}$

$f(2-\sqrt{6}) = \dfrac{2-\sqrt{6}-2}{(2-\sqrt{6})^2+2} = \dfrac{-\sqrt{6}}{4-4\sqrt{6}+6+2} = \dfrac{-\sqrt{6}}{12-4\sqrt{6}} \cdot \dfrac{12+4\sqrt{6}}{12+4\sqrt{6}} = \dfrac{-12\sqrt{6}-24}{48} = \dfrac{-\sqrt{6}-2}{4}$

coordinates of points where $f'(x)=0$

$\left(2+\sqrt{6},\ \dfrac{-2+\sqrt{6}}{4}\right)$ and $\left(2-\sqrt{6},\ \dfrac{-2-\sqrt{6}}{4}\right)$

OVER

8. (continued)

(c) $$f'(x) = \frac{-x^2+4x+2}{(x^2+2)^2}$$

$$f''(x) = \frac{(x^2+2)^2(-2x+4) - (-x^2+4x+2)\left[2(x^2+2)2x\right]}{(x^2+2)^4}$$

$$= \frac{(x^2+2)(-2x+4) - (-x^2+4x+2)(4x)}{(x^2+2)^3}$$

$$= \frac{-2x^3+4x^2-4x+8+4x^3-16x^2-8x}{(x^2+2)^3}$$

$$f''(x) = \frac{2x^3-12x^2-12x+8}{(x^2+2)^3}$$