Differential Calculus – Test 2 SOLUTION KEY

Part I - Questions 1-4: NO calculator

7 questions >> 65 total marks

1. Given that $f(x)=x^2-3x$, find $f'(x)$ from first principles; that is, use the limit definition of the derivative $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. **[8 marks]**

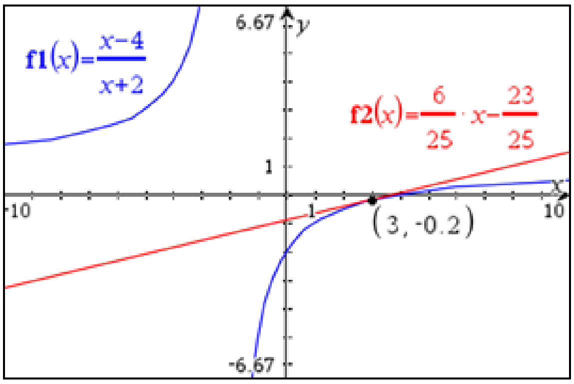
$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-3(x+h)-(x^2-3x)}{h}=\lim_{h\to 0}\frac{x^2+2hx+h^2-3x-3h-x^2+3x}{h}=$$

$$=\lim_{h\to 0}\frac{2hx+h^2-3h}{h}=\lim_{h\to 0}\frac{h(2x+h-3)}{h}=\lim_{h\to 0}(2x+h-3)=2x-3$$

2. Find the equation of the line tangent to the curve $y=\dfrac{x-4}{x+2}$ at the point where $x=3$. **[8 marks]**

$$\frac{dy}{dx}=\frac{(x+2)(1)-(1)(x-4)}{(x+2)^2}=\frac{6}{(x+2)^2}\qquad \text{at } x=3:\ \frac{dy}{dx}=\frac{6}{(3+2)^2}=\frac{6}{25},\ \text{ and } y(3)=\frac{3-4}{3+2}=-\frac{1}{5}$$

eq of tangent line: $y+\dfrac{1}{5}=\dfrac{6}{25}(x-3)\ \Rightarrow\ y=\dfrac{6}{25}x-\dfrac{18}{25}-\dfrac{5}{25}\ \Rightarrow\ y=\dfrac{6}{25}x-\dfrac{23}{25}$

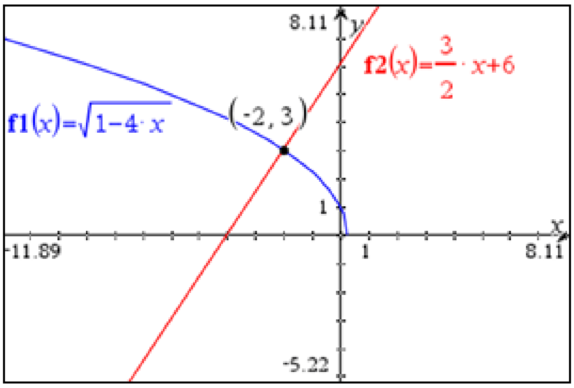


3. Find the equation of the line normal to $y=\sqrt{1-4x}$ at the point where $x=-2$. **[8 marks]**

$$y=\sqrt{1-4x}=(1-4x)^{1/2}\ \Rightarrow\ \frac{dy}{dx}=\frac{1}{2}(1-4x)^{-1/2}(-4)=-\frac{2}{\sqrt{1-4x}}$$

at $x=-2$: $y(-2)=\sqrt{1-4(-2)}=3$ and $\dfrac{dy}{dx}=-\dfrac{2}{\sqrt{1+8}}=-\dfrac{2}{3}\ \Rightarrow$ slope of normal $=\dfrac{3}{2}$

eq. of normal: $y-3=\dfrac{3}{2}(x+2)\ \Rightarrow\ y=\dfrac{3}{2}x+3+3\ \Rightarrow\ y=\dfrac{3}{2}x+6$



4. The line L with equation $y = x - 5$ intersects the graph of $g(x) = 2x^3 + ax^2 + bx + 3$ at $(-1, -6)$. Line L is also tangent to the graph of g at $x = 2$. Find the value of a and the value of b. **[8 marks]**

$$g(-1) = -6 \;\Rightarrow\; 2(-1)^3 + a(-1)^2 + b(-1) + 3 = -6 \;\Rightarrow\; -2 + a - b + 3 = -6 \;\Rightarrow\; a - b = -7$$

slope of line L is 1, thus

$$g'(-1) = 1 \;\Rightarrow\; g'(x) = 6x^2 + 2ax + b \;\Rightarrow\; g'(2) = 6(2)^2 + 2a(2) + b = 1 \;\Rightarrow\; 4a + b = -23$$

solving system of equations: $a = -6,\; b = 1$

Part II - Questions 5-8: Calculator allowed

5. Consider the function $y = \mathrm{e}^x \cos x$ with **domain** $0 < x < 5$. **[15 marks]**

(a) Determine $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the approximate coordinates of any maximum or minimum point for the function in the indicated domain. Provide a brief justification for each.

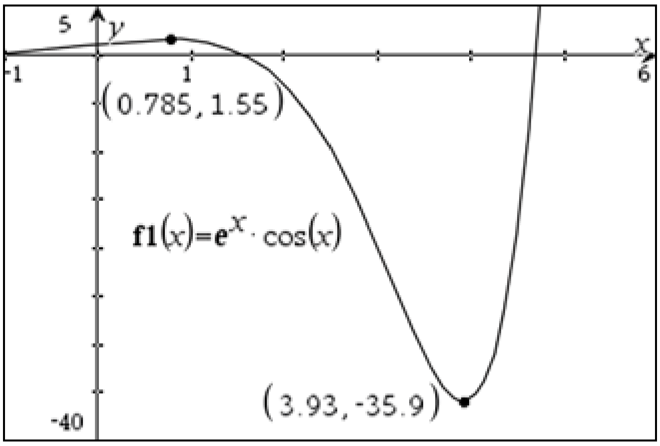
(c) Find the exact coordinates of any inflexion point for the function in the indicated domain. Provide a brief justification for each.

(a) $\dfrac{dy}{dx} = \mathrm{e}^x \cos x + \mathrm{e}^x(-\sin x) \;=\; \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\mathrm{e}^x \cos x - \mathrm{e}^x \sin x\right) = \frac{d}{dx}\left(\mathrm{e}^x \cos x\right) - \frac{d}{dx}\left(\mathrm{e}^x \sin x\right) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \left(\mathrm{e}^x \sin x - \mathrm{e}^x \cos x\right)$$

$$\frac{d^2y}{dx} = -2\mathrm{e}^x \sin x$$

(b)



at $\underline{x \approx 0.785}$: $\dfrac{d^2y}{dx} = -2\mathrm{e}^{0.785}\sin(0.785) \approx -3.10 < 0 \;\Rightarrow\;$ curve concave down at $x \approx 0.785$

$y(0.785) = e^{0.785}\cos(0.785) \approx 1.55$; therefore, there is a maximum at $(0.785, 1.55)$

at $\underline{x \approx 0.393}$: $\dfrac{d^2y}{dx} = -2\mathrm{e}^{03.93}\sin(3.93) \approx 71.8 > 0 \;\Rightarrow\;$ curve concave up at $x \approx 3.93$

$y(3.93) = e^{3.93}\cos(3.93) \approx -35.9$; therefore, there is a minimum at $(3.93, -35.9)$

[question 5 continued]

(c) $\dfrac{d^2y}{dx} = -2e^x \sin x = 0 \;\Rightarrow\; \sin x = 0$ in the interval $0 < x < 5 \;\Rightarrow\; x = \pi$

$y(\pi) = e^{\pi}\cos\pi = -e^{\pi}$

at $x = \dfrac{\pi}{2}$: $\dfrac{d^2y}{dx} = -2e^{\pi/2}\sin\dfrac{\pi}{2} = -2e^{\pi/2} < 0$ at $x = \dfrac{3\pi}{2}$: $\dfrac{d^2y}{dx} = -2e^{\pi/2}\sin\dfrac{3\pi}{2} = 2e^{\pi/2} > 0$

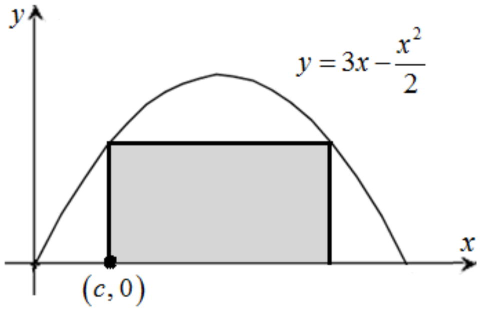
$\dfrac{d^2y}{dx}$ changes sign at $x = \pi$; therefore, there is an inflexion point at $(\pi, -e^{\pi})$

6. A rectangle is circumscribed by the graph of the parabola $y = 3x - \dfrac{x^2}{2}$ such that the two upper vertices of the rectangle are on the graph of the parabola and the two lower vertices are on the x-axis (as shown in the diagram). The lower left vertex of the rectangle has coordinates $(c, 0)$.

(a) Show that the function, A, for the area of the rectangle expressed in terms of c is $A(c) = c^3 - 9c^2 + 18c$.

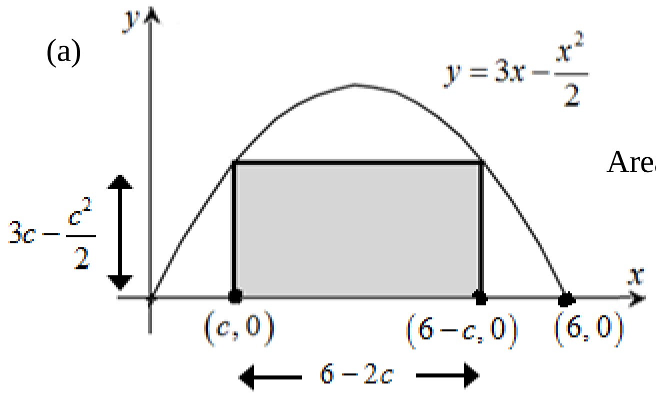
(b) Find the width and height of the rectangle with the maximum area.

(c) Find the maximum area.



[10 marks]

(a)



Area $= A(c) = (6-2c)\left(3c - \dfrac{c^2}{2}\right) = 18c - 3c^2 - 6c^2 + c^3$

$= c^3 - 9c^2 + 18c, \quad 0 < c < 3$ ***QED***

(b) $A'(c) = 3c^2 - 18c + 18 = 0 \;\Rightarrow\; A'(c) = 3(c^2 - 6c + 6) = 0$

$\text{polyRoots}(c^2 - 6\cdot c + 6, c)$
$\{1.26795, 4.73205\}$

solution of $c \approx 4.73$ is outside domain

thus, $A'(c) = 0$ at $c \approx 1.27$

$A''(c) = 6c - 18 \;\Rightarrow\; A''(1.27) \approx 6(1.27) - 18 < 0$, thus $A(c)$ has a maximum at $c \approx 1.27$

width $= 6 - 2c \approx 6 - 2(1.26795...) \approx 3.4641$; height $= 3c - \dfrac{c^2}{2} \approx 3(1.26795...) - \dfrac{(1.26795...)^2}{2} = 3$

dimensions of rectangle with max. area are 3.46 units by 3 units

(c) max. area $= 3(3.4641...) \approx 10.4$ units^2

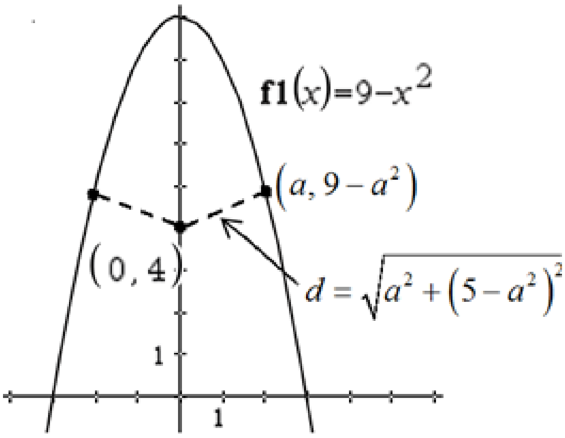
7. Find the coordinates of the two points on the graph of $y = 9 - x^2$ that are closest to $(0, 4)$.

[8 marks]

Let a be the x-coordinate of one of the points on the graph of $y = 9 - x^2$ closest to $(0, 4)$. d is the distance from $(0, 4)$ to $(a, 9-a^2)$, giving:

$$d(a) = \sqrt{a^2 + (5-a^2)^2} = \sqrt{a^4 - 9a^2 + 25}, \quad a > 0$$

The distance d will have a minimum at the value of a that makes $a^4 - 9a^2 + 25$ have a minimum.



$f(a) = a^4 - 9a^2 + 25 \Rightarrow f'(a) = 4a^3 - 18a$ need to solve: $4a^3 - 18a = 2a(2a^2 - 9) = 0$

since $a > 0$, then $a = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}$

$f''(a) = 12a^2 - 18 \Rightarrow f''\left(\pm\frac{3\sqrt{2}}{2}\right) = 12\left(\pm\frac{3\sqrt{2}}{2}\right)^2 - 18 = 39 > 0 \Rightarrow$ verifies d is min. at $a = \pm\frac{3\sqrt{2}}{2}$

for $a = \pm\frac{3\sqrt{2}}{2}$: $9 - a^2 = 9 - \left(\pm\frac{3\sqrt{2}}{2}\right)^2 = \frac{9}{2}$

Therefore, coordinates of the points on $y = 9 - x^2$ closest to $(0, 4)$ are: $\left(\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$ & $\left(-\frac{3\sqrt{2}}{2}, \frac{9}{2}\right)$