Differential Calculus – Test 1 ♦ SOLUTION KEY ♦

Part I (Qs 1-5) – **no** calculator. total marks on test: **67**

1. Given that $f(x)=\frac{1}{x}$, find $f'(x)$ from first principles (i.e. limit definition of the derivative).

[7 marks]

$f(x)=\frac{1}{x}$ $\quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\left(\frac{-h}{x^2+hx}\cdot\frac{1}{h}\right)=\lim_{h\to 0}\left(\frac{-1}{x^2+hx}\right)$$

$$=\frac{-1}{x^2+0\cdot x}=-\frac{1}{x^2}$$

therefore, $f'(x)=-\frac{1}{x^2}$

2. The graph of the derivative of function g is below. The graph below has x-intercepts at the points $(-1,0),(1,0)$ & $(3,0)$; has absolute maxima at $x=-3$ and $x=2$; and has a relative minimum at $x=0$.

(a) State the interval(s) where function g is increasing: $-4<x<-1$ and $1<x<3$

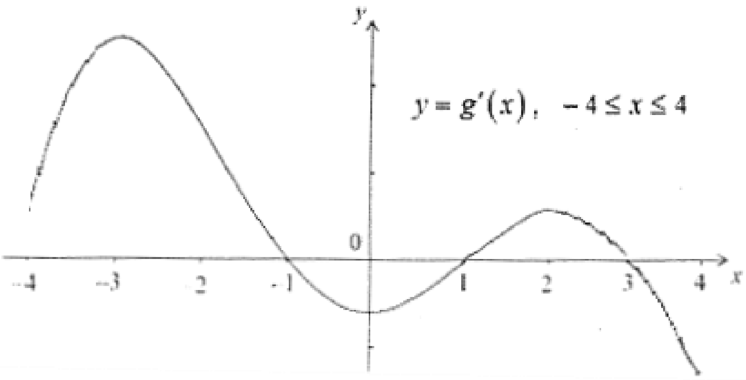
(b) State the interval(s) where function g is decreasing: $-1<x<1$ and $3<x<4$

(c) State the x value(s) where function g has a maximum: $x=-1$ and $x=3$

(d) State the x value(s) where function g has a minimum: $x=1$

(e) State the interval(s) where the graph of the function g is concave up: $-4<x<-3$ and $0<x<2$

(f) State the interval(s) where the graph of the function g is concave down: $-3<x<0$ and $2<x<4$



This is **NOT** the graph of function g

Differential Calculus – Test 1 ♦ SOLUTION KEY ♦

3. The line with equation $y = 17x - 27$ is tangent to the curve $y = x^3 + ax^2 + bx + 1$ at the point $(2, 7)$. Find the value of a and the value of b. **[7 marks]**

$y = 17x - 27 \Rightarrow$ slope $= 17$

$y = x^3 + ax^2 + bx + 1 \Rightarrow y' = 3x^2 + 2ax + b$

at $(2,7)$: $y' = 3(2)^2 + 2a(2) + b = 17 \Rightarrow 4a + b = 5 \quad (1)$

and $7 = (2)^3 + a(2)^2 + b(2) + 1 \Rightarrow 4a + 2b = -2 \quad (2)$

subtracting (1) from (2): $\underline{b = -7}$

using eq. (1): $4a - 7 = 5 \Rightarrow \underline{a = 3}$

$\therefore$ $\underline{a = 3 \text{ and } b = -7}$

4. Consider the function h defined as $h(x) = \dfrac{x^2}{2} - \dfrac{8}{x}$.

(a) Find the x-coordinate of the point A where $h'(x) = 0$. **[5 marks]**

(b) Determine, with justification, whether point A is a maximum, minimum or neither. **[4 marks]**

(a) $h(x) = \frac{1}{2}x^2 - 8x^{-1}$ $\quad h'(x) = x + 8x^{-2} = x + \dfrac{8}{x^2} = 0$ multiply through by x^2

$x^3 + 8 = 0 \rightarrow x^3 = -8 \rightarrow \underline{x = -2}$

(b) $h''(x) = 1 - 16x^{-3} = 1 - \dfrac{16}{x^3}$

$h''(-2) = 1 - \dfrac{16}{(-2)^3} = 1 - \dfrac{16}{-8} = 1 + 2 = 3 > 0$ therefore, graph of h is concave up at $x = -2$

thus, point A is a $\underline{\text{minimum}}$

Differential Calculus – Test 1 ♦ SOLUTION KEY ♦

5. Find the equation of the line tangent to the graph of $y = e^{x/3}$ at the point where $x = 6$. **[7 marks]**

$\frac{dy}{dx} = e^{x/3} \cdot \frac{1}{3} = \frac{1}{3}e^{x/3}$ at $x=6$: $\frac{dy}{dx} = \frac{1}{3}e^{6/3} = \frac{1}{3}e^{2}$

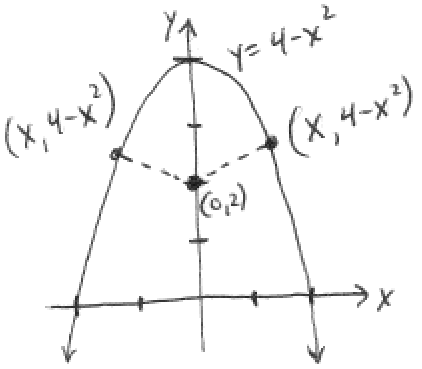
slope of tangent $= \frac{1}{3}e^2$ $y = e^{6/3} = e^2$ pt. of tangency is $(6, e^2)$

equation of tangent: $y - e^2 = \frac{1}{3}e^2(x-6) \Rightarrow y = \frac{1}{3}e^2 x - 2e^2 + e^2$

$y = \frac{1}{3}e^2 x - e^2$

Part II (Qs 6-8) – calculator allowed

6. Find the coordinates of the two points on the graph of $y = 4 - x^2$ that are closest to the point $(0,2)$. **[8 marks]**



distance **d** between $(0,2)$ and a point $(x, 4-x^2)$ on graph of $y = 4-x^2$ is given by

$d = \sqrt{(x-0)^2 + (4-x^2-2)^2} = \sqrt{x^4 - 3x^2 + 4}$

$d(x) = \sqrt{x^4 - 3x^2 + 4}$

function $d(x)$ will have a minimum when $f(x) = x^4 - 3x^2 + 4$ has a minimum

$f'(x) = 4x^3 - 6x = 2x(2x^2 - 3) = 0 \Rightarrow x = 0,\ x = \sqrt{\frac{3}{2}},\ x = -\sqrt{\frac{3}{2}}$

2nd derivative test

$f''(x) = 12x^2 - 6$

$f''(0) = 12(0)^2 - 6 = -6 < 0 \rightarrow$ max. at $x = 0$

$f''\left(\sqrt{\frac{3}{2}}\right) = 12\left(\sqrt{\frac{3}{2}}\right)^2 - 6 = 12 > 0 \rightarrow$ min. at $x = \sqrt{\frac{3}{2}}$

$f''\left(-\sqrt{\frac{3}{2}}\right) = 12\left(-\sqrt{\frac{3}{2}}\right)^2 - 6 = 12 > 0 \rightarrow$ min. at $x = -\sqrt{\frac{3}{2}}$

$f\left(\sqrt{\frac{3}{2}}\right) = 4 - \left(\sqrt{\frac{3}{2}}\right)^2 = 4 - \frac{3}{2} = \frac{5}{2}$ $f\left(-\sqrt{\frac{3}{2}}\right) = \frac{5}{2}$ also

thus, closest points on $y = 4 - x^2$ to the point $(0,2)$ are $\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ and $\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

Differential Calculus – Test 1 ♦ SOLUTION KEY ♦

7. Consider the function $y = \dfrac{8-4x^3}{x^4}$.

(a) Find $\dfrac{dy}{dx}$. **[4 marks]**

(b) Find the <u>exact</u> coordinates of any maximum or minimum points for the function. Indicate whether a point is a maximum or minimum and give a brief justification. **[7 marks]**

(c) State the domain and range of the function. **[4 marks]**

(a) $y = \dfrac{8}{x^4} - \dfrac{4x^3}{x^4} = 8x^{-4} - 4x^{-1}$ $\quad \dfrac{dy}{dx} = -32x^{-5} + 4x^{-2}$

$$\dfrac{dy}{dx} = -\dfrac{32}{x^5} + \dfrac{4}{x^2} = \dfrac{-32+4x^3}{x^5}$$

(b) $\dfrac{dy}{dx} = \dfrac{-32+4x^3}{x^5} = 0 \rightarrow -32+4x^3=0 \rightarrow 4x^3=32 \rightarrow x^3=8$

$\underline{x=2}$

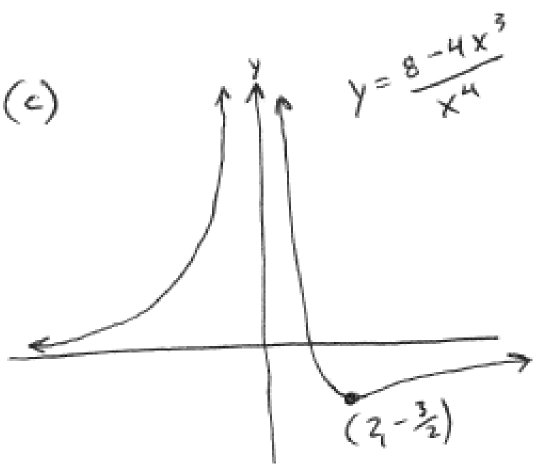
at $x=2$: $y = \dfrac{8-4(2)^3}{2^4} = \dfrac{8-32}{16} = \dfrac{-24}{16} = -\dfrac{3}{2}$ $\quad$ coordinates of extreme pt. $\left(2,-\frac{3}{2}\right)$

$\dfrac{d^2y}{dx^2} = 160x^{-6} - 8x^{-3} = \dfrac{160}{x^6} - \dfrac{8}{x^3}$ $\quad$ at $x=2$: $\dfrac{d^2y}{dx^2} = \dfrac{160}{2^6} - \dfrac{8}{2^3} = \dfrac{3}{2} > 0$

so, graph is <u>concave</u> up

thus, by 2nd derivative test the point $\left(2,-\frac{3}{2}\right)$ is a minimum

(c)



domain: $x \in \mathbb{R},\ x \neq 0$

range: $y \geq -\frac{3}{2}$

Differential Calculus – Test 1 ♦ SOLUTION KEY ♦

8. Consider the function $f(x)=\dfrac{2}{\sqrt{5-x^2}}$.

(a) Find the derivative of the function and express it with no negative exponents and no fractional exponents. **[4 marks]**

(b) Find the equation of the line that is <u>normal</u> to the graph of f at the point where $x=2$. Express the equation exactly. **[4 marks]**

(c) Find the equation of the line that is parallel to the line you found in (b) and intersects the graph of f at only one point. Express the equation exactly. **[4 marks]**

(a) $f(x)=2(5-x^2)^{-\frac{1}{2}}$ $\quad f'(x)=2\left[-\frac{1}{2}(5-x^2)^{-\frac{3}{2}}(-2x)\right]=\dfrac{2x}{\sqrt{(5-x^2)^3}}$

(b) $f'(2)=\dfrac{2(2)}{\sqrt{(5-(2)^2)^3}}=\dfrac{4}{\sqrt{1^3}}=4$ → slope of normal $=-\frac{1}{4}$

$f(2)=\dfrac{2}{\sqrt{5-(2)^2}}=\dfrac{2}{\sqrt{1}}=2$ point where normal intersects graph is $(2,2)$

equation of normal:

$y-2=-\frac{1}{4}(x-2)$ → $y=-\frac{1}{4}x+\frac{1}{2}+2$ → $y=-\frac{1}{4}x+\frac{5}{2}$

(c) where does line tangent to $f(x)$ have slope (derivative) equal to $-\frac{1}{4}$?

$f'(x)=\dfrac{2x}{\sqrt{(5-x^2)^3}}=-\frac{1}{4}$ solved on TI-Nspire

$\text{nSolve}\left(\dfrac{2\cdot x}{\sqrt{(5-x^2)^3}}=\dfrac{-1}{4},x\right)$ $\quad -1.$

$x=-1$

$f(-1)=\dfrac{2}{\sqrt{5-(-1)^2}}=\dfrac{2}{\sqrt{4}}=1$ pt. of tangency is $(-1,1)$

equation of tangent: $y-1=-\frac{1}{4}(x-(-1))$ → $y=-\frac{1}{4}x-\frac{1}{4}+1$ → $y=-\frac{1}{4}x+\frac{3}{4}$

<u>Bonus Question</u>:

At what point does the line that is normal to the graph of $y=x-x^2$ at the point $(2,-2)$ intersect the graph of the curve a second time? Express the coordinates of the point exactly. **[+4 marks]**

$\frac{dy}{dx}=1-2x$; for $x=2$: $\frac{dy}{dx}=1-2(2)=-3$

slope of normal $=\frac{1}{3}$

equation of normal: $y-(-2)=\frac{1}{3}(x-2)$

$y+2=\frac{1}{3}x-\frac{2}{3}$ → $y=\frac{1}{3}x-\frac{2}{3}-\frac{6}{3}$

eqn. of normal: $y=\frac{1}{3}x-\frac{8}{3}$

find intersection of $y=\frac{1}{3}x-\frac{8}{3}$ and $y=x-x^2$

$\frac{1}{3}x-\frac{8}{3}=x-x^2$ → $x^2+\frac{1}{3}x-\frac{3}{3}x-\frac{8}{3}=0$

$x^2-\frac{2}{3}x-\frac{8}{3}=0$ → $3x^2-2x-8=0$

$(3x+4)(x-2)=0$

$x=-\frac{4}{3}$ or $x=2$ ← point of tangency

for $x=-\frac{4}{3}$: $y=-\frac{4}{3}-\left(-\frac{4}{3}\right)^2$

$=-\frac{4}{3}-\frac{16}{9}=-\frac{12}{9}-\frac{16}{9}=-\frac{28}{9}$

intersection point is $\left(-\frac{4}{3},-\frac{28}{9}\right)$