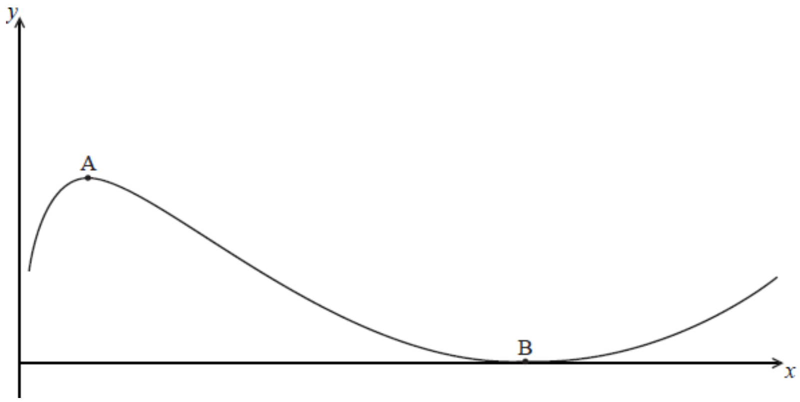
HL/Derivatives *[69 marks]*

The diagram shows the graph of the function defined by
$y = x(\ln x)^2$ for $x > 0$.



The function has a local maximum at the point A and a local minimum at the point B.

1a. Find the coordinates of the points A and B. *[5 marks]*

Markscheme

$f'(x) = (\ln x)^2 + \frac{2x\ln x}{x}\left(= (\ln x)^2 + 2\ln x = \ln x(\ln x + 2)\right)$ ***M1A1***

$f'(x) = 0$ $(\Rightarrow x = 1,\ x = e^{-2})$ ***M1***

Note: Award ***M1*** for an attempt to solve
$f'(x) = 0$.

$A(e^{-2}, 4e^{-2})$ **and** *B*(1, 0) ***A1A1***

Note: The final ***A1*** is independent of prior working.

[5 marks]

1b. Given that the graph of the function has exactly one point of inflexion, find its coordinates. *[3 marks]*

Markscheme

$f''(x) = \frac{2}{x}(\ln x + 1)$ ***A1***

$f''(x) = 0$ $\left(\Rightarrow x = e^{-1}\right)$ ***(M1)***

inflexion point
$(e^{-1},\ e^{-1})$ ***A1***

Note: ***M1*** for attempt to solve
$f''(x) = 0$.

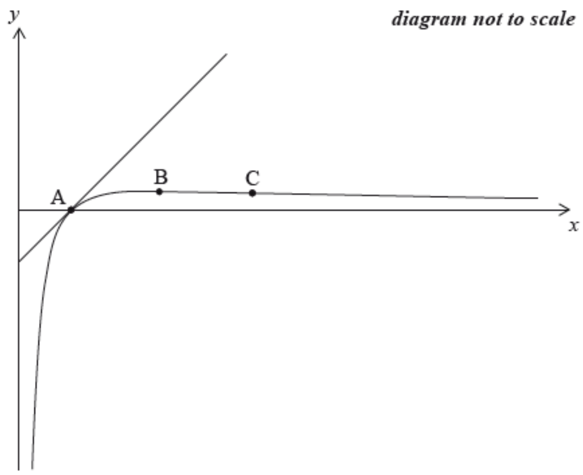
[3 marks]

Consider the function

$f(x) = \frac{\ln x}{x},\ x > 0.$

The sketch below shows the graph of

$y =\ f(x)$ and its tangent at a point A.



2a. Show that

$f'(x) = \frac{1-\ln x}{x^2}.$

[2 marks]

Markscheme

$f'(x) = \frac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \frac{1-\ln x}{x^2}$ ***AG***

[2 marks]

2b. Find the coordinates of B, at which the curve reaches its maximum value. *[3 marks]*

Markscheme

$\frac{1-\ln x}{x^2} = 0$ has solution

$x = \mathrm{e}$ ***M1A1***

$y = \frac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point

$\left(\mathrm{e},\ \frac{1}{\mathrm{e}}\right)$

[3 marks]

2c. Find the coordinates of C, the point of inflexion on the curve. *[5 marks]*

Markscheme

$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - 2x(1 - \ln x)}{x^4}$ ***M1A1***

$= \frac{2\ln x - 3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where
$f''(x) = 0$ ***M1***

so
$x = e^{\frac{3}{2}},\ y = \frac{3}{2}e^{\frac{-3}{2}}$ ***A1A1***

C has coordinates
$\left(e^{\frac{3}{2}},\ \frac{3}{2}e^{\frac{-3}{2}}\right)$

[5 marks]

2d. The graph of $y = f(x)$ crosses the x-axis at the point A. *[4 marks]*

Find the equation of the tangent to the graph of f at the point A.

Markscheme

$f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

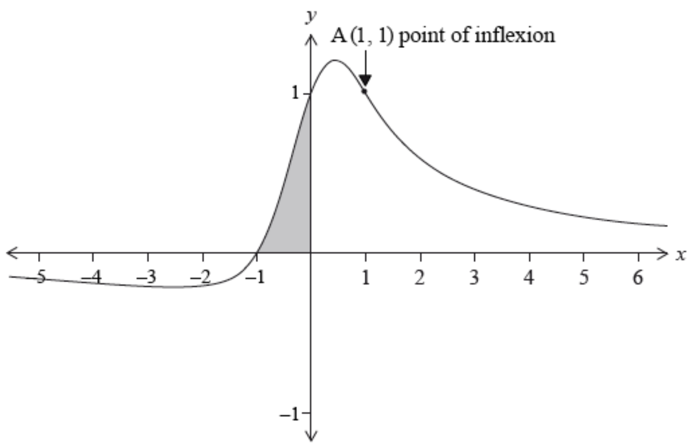
$y = x + c$ ***(M1)***

through (1, 0)

equation is
$y = x - 1$ ***A1***

[4 marks]

The graph of the function $f(x) = \frac{x+1}{x^2+1}$ is shown below.



3a. Find $f'(x)$. *[2 marks]*

Markscheme

(a)

$f'(x) = \frac{(x^2+1)-2x(x+1)}{(x^2+1)^2} \left(= \frac{-x^2-2x+1}{(x^2+1)^2}\right)$ ***M1A1***

[2 marks]

3b. Hence find the x-coordinates of the points where the gradient of the graph of f is zero. *[1 mark]*

Markscheme

$\frac{-x^2-2x+1}{(x^2+1)^2} = 0$

$x = -1 \pm \sqrt{2}$ ***A1***

[1 mark]

3c. Find $f''(x)$ expressing your answer in the form $\frac{p(x)}{(x^2+1)^3}$, where $p(x)$ is a polynomial of degree 3. *[3 marks]*

Markscheme

$f''(x) = \frac{(-2x-2)(x^2+1)^2-2(2x)(x^2+1)(-x^2-2x+1)}{(x^2+1)^4}$ ***A1A1***

Note: Award ***A1*** for $(-2x-2)(x^2+1)^2$ or equivalent.

Note: Award ***A1*** for $-2(2x)(x^2+1)(-x^2-2x+1)$ or equivalent.

$= \frac{(-2x-2)(x^2+1)-4x(-x^2-2x+1)}{(x^2+1)^3}$

$= \frac{2x^3+6x^2-6x-2}{(x^2+1)^3}$ ***A1***

$\left(= \frac{2(x^3+3x^2-3x-1)}{(x^2+1)^3}\right)$

[3 marks]

The point (1, 1) is a point of inflexion. There are two other points of inflexion.

3d. Find the x-coordinates of the other two points of inflexion. *[4 marks]*

Markscheme

recognition that
$(x-1)$ is a factor ***(R1)***

$(x-1)\left(x^2+bx+c\right)=\left(x^3+3x^2-3x-1\right)$ ***M1***

$\Rightarrow x^2+4x+1=0$ ***A1***

$x=-2\pm\sqrt{3}$ ***A1***

Note: Allow long division / synthetic division.

[4 marks]

Let
$f(x)=\sqrt{\frac{x}{1-x}},\ 0<x<1.$

4a. Show that
$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ and deduce that *f* is an increasing function. *[5 marks]*

Markscheme

EITHER

derivative of
$\frac{x}{1-x}$ is
$\frac{(1-x)-x(-1)}{(1-x)^2}$ ***M1A1***

$f'(x)=\frac{1}{2}\left(\frac{x}{1-x}\right)^{-\frac{1}{2}}\frac{1}{(1-x)^2}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all
$0<x<1$) so the function is increasing ***R1***

OR

$f(x)=\frac{x^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}$

$f'(x)=\frac{(1-x)^{\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}(-1)}{1-x}$ ***M1A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***A1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}[1-x+x]$ ***M1***

$=\frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$ ***AG***

$f'(x)>0$ (for all
$0<x<1$) so the function is increasing ***R1***

[5 marks]

4b. Show that the curve
$y=f(x)$ has one point of inflexion, and find its coordinates. *[6 marks]*

Markscheme

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^{-\frac{3}{2}}$

$\Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{1}{2}}(1-x)^{-\frac{5}{2}}$ ***M1A1***

$= -\frac{1}{4}x^{-\frac{3}{2}}(1-x)^{-\frac{5}{2}}[1-4x]$

$f''(x) = 0 \Rightarrow x = \frac{1}{4}$ ***M1A1***

$f''(x)$ changes sign at
$x = \frac{1}{4}$ hence there is a point of inflexion ***R1***

$x = \frac{1}{4} \Rightarrow y = \frac{1}{\sqrt{3}}$ ***A1***

the coordinates are
$\left(\frac{1}{4}, \frac{1}{\sqrt{3}}\right)$

[6 marks]

4c. Use the substitution
$x = \sin^2\theta$ to show that
$\int f(x)\mathrm{d}x = \arcsin\sqrt{x} - \sqrt{x - x^2} + c$. *[11 marks]*

Markscheme

$x = \sin^2\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sin\theta\cos\theta$ ***M1A1***

$\int\sqrt{\frac{x}{1-x}}\mathrm{d}x = \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}2\sin\theta\cos\theta\mathrm{d}\theta$ ***M1A1***

$= \int 2\sin^2\theta\mathrm{d}\theta$ ***A1***

$= \int 1 - \cos 2\theta\mathrm{d}\theta$ ***M1A1***

$= \theta - \frac{1}{2}\sin 2\theta + c$ ***A1***

$\theta = \arcsin\sqrt{x}$ ***A1***

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sqrt{x}\sqrt{1-x} = \sqrt{x - x^2}$ ***M1A1***

hence
$\int\sqrt{\frac{x}{1-x}}\mathrm{d}x = \arcsin\sqrt{x} - \sqrt{x - x^2} + c$ ***AG***

[11 marks]

The curve *C* has equation
$y = \frac{1}{8}(9 + 8x^2 - x^4)$.

5a. Find the coordinates of the points on *C* at which
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. *[4 marks]*

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \frac{1}{2}x^3$ ***A1***

$x\left(2 - \frac{1}{2}x^2\right) = 0$

$x = 0,\ \pm 2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at
$\left(0, \frac{9}{8}\right),\ \left(-2, \frac{25}{8}\right),\ \left(2, \frac{25}{8}\right)$ ***A1A1A1***

Note: Award ***A2*** for all three *x*-values correct with errors/omissions in *y*-values.

[4 marks]

5b. The tangent to *C* at the point P(1, 2) cuts the *x*-axis at the point T. Determine the coordinates of T. *[4 marks]*

Markscheme

at *x* =1, gradient of tangent
$= \frac{3}{2}$ ***(A1)***

Note: In the following, allow ***FT*** on incorrect gradient.

equation of tangent is
$y - 2 = \frac{3}{2}(x - 1)$ $\left(y = \frac{3}{2}x + \frac{1}{2}\right)$ ***(A1)***

meets *x*-axis when y = 0 ,
$-2 = \frac{3}{2}(x - 1)$ ***(M1)***

$x = -\frac{1}{3}$

coordinates of T are
$\left(-\frac{1}{3}, 0\right)$ ***A1***

[4 marks]

5c. The normal to *C* at the point P cuts the *y*-axis at the point N. Find the area of triangle PTN. *[7 marks]*

Markscheme

gradient of normal

$= -\frac{2}{3}$ ***(A1)***

equation of normal is

$y - 2 = -\frac{2}{3}(x - 1)$ $\left(y = -\frac{2}{3}x + \frac{8}{3}\right)$ ***(M1)***

at $x = 0$,

$y = \frac{8}{3}$ ***A1***

Note: In the following, allow FT on incorrect coordinates of T and N.

lengths of

$\text{PN} = \sqrt{\frac{13}{9}}$,

$\text{PT} = \sqrt{\frac{52}{9}}$ ***A1A1***

area of triangle

$\text{PTN} = \frac{1}{2} \times \sqrt{\frac{13}{9}} \times \sqrt{\frac{52}{9}}$ ***M1***

$= \frac{13}{9}$ (or equivalent *e.g.* $\frac{\sqrt{676}}{18}$) ***A1***

[7 marks]