Revision / Complex numbers [70 marks]

1. [Maximum mark: 18] SPM.1.AHL.TZ0.11

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that

$$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0.$$ [4]

2. [Maximum mark: 6] 19N.2.AHL.TZ0.H_6

Let $P(z)=az^3-37z^2+66z-10$, where $z\in\mathbb{C}$ and $a\in\mathbb{Z}$.

One of the roots of $P(z)=0$ is $3+\mathrm{i}$. Find the value of a. [6]

3. [Maximum mark: 5] 21M.2.AHL.TZ2.8

Consider $z=\cos\theta+\mathrm{i}\sin\theta$ where $z\in\mathbb{C},\ z\neq1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right)=0$. [5]

4. [Maximum mark: 8] 21M.1.AHL.TZ1.7

Consider the quartic equation

$z^4 + 4z^3 + 8z^2 + 80z + 400 = 0,\ z \in \mathbb{C}.$

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

Find the possible values of a. [8]

5. [Maximum mark: 6] 22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\mathrm{i}$ and $z_2 = \left(1 - b^2\right) - 2b\mathrm{i}$, where $b \in \mathbb{R},\ b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

6. [Maximum mark: 10] 24M.2.AHL.TZ2.8

Let $z = 1 + \cos 2\theta + \mathrm{i} \sin 2\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that

(a.i) $\arg z = \theta$; [3]

(a.ii) $|z| = 2 \cos \theta$. [4]

(b) Hence or otherwise, find the value of θ such that $\arg\left(z^2\right) = \left|z^3\right|$. [3]

7. [Maximum mark: 17] 23N.1.AHL.TZ2.12

(a) Find the binomial expansion of $(\cos\,\theta + \mathrm{i}\,\sin\,\theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\,\theta$ and $\cos\,\theta$. [4]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin\ 5\theta \equiv 16\ \sin^5\theta - 20\ \sin^3\theta + 5\ \sin\,\theta$. [6]

(c.i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\ \sin^4\theta - 20\ \sin^2\theta + 5 = 0$. [3]

(c.ii) Hence. show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5}{4}}$ [4]

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