Revision / Complex numbers [70 marks]

1. [Maximum mark: 18] SPM.1.AHL.TZ0.11

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Markscheme

attempt to find modulus ***(M1)***

$r=2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta=\pi+\arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$=\frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i}=\sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(=2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leqslant\pi$. [5]

Markscheme

attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi\mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi\mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi\mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

Markscheme

METHOD 1

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

Area $= 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

METHOD 2

$\text{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\text{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using Area = $\frac{1}{2}$ × UV × VW × sin α for example ***M1***

Area $= \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) By considering the sum of the roots u, v and w, show that

$$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0.$$

[4]

Markscheme

$u + v + w = 0$ ***R1***

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \mathrm{i}\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + \mathrm{i}\sin\frac{17\pi}{18}\right) = 0$$ ***A1***

consideration of real parts ***M1***

$$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{17\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$ ***AG***

[4 marks]

2. [Maximum mark: 6] 19N.2.AHL.TZ0.H_6

Let $P(z) = az^3 - 37z^2 + 66z - 10$, where $z \in \mathbb{C}$ and $a \in \mathbb{Z}$.

One of the roots of $P(z) = 0$ is $3 + \mathrm{i}$. Find the value of a. [6]

Markscheme

METHOD 1

one other root is $3 - \mathrm{i}$ ***A1***

let third root be α ***(M1)***

considering sum or product of roots ***(M1)***

sum of roots $= 6 + \alpha = \frac{37}{a}$ ***A1***

product of roots $= 10\alpha = \frac{10}{a}$ ***A1***

hence $a = 6$ ***A1***

METHOD 2

one other root is $3 - \mathrm{i}$ ***A1***

quadratic factor will be $z^2 - 6z + 10$ ***(M1)A1***

$P(z) = az^3 - 37z^2 + 66z - 10 = (z^2 - 6z + 10)(az - 1)$ ***M1***

comparing coefficients ***(M1)***

hence $a = 6$ ***A1***

METHOD 3

substitute $3 + \mathrm{i}$ into $P(z)$ ***(M1)***

$a(18 + 26\mathrm{i}) - 37(8 + 6\mathrm{i}) + 66(3 + \mathrm{i}) - 10 = 0$ ***(M1)A1***

equating real or imaginary parts or dividing ***M1***

$18a - 296 + 198 - 10 = 0$ or $26a - 222 + 66 = 0$ or $\frac{10 - 66(3+\mathrm{i}) + 37(8+6\mathrm{i})}{18+26\mathrm{i}}$ ***A1***

hence $a = 6$ ***A1***

[6 marks]

3. [Maximum mark: 5] 21M.2.AHL.TZ2.8

Consider $z = \cos\theta + \mathrm{i}\sin\theta$ where $z \in \mathbb{C},\ z \neq 1$.

Show that $\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$. [5]

Markscheme

$\frac{1+z}{1-z} = \frac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta}$

attempt to use the complex conjugate of their denominator ***M1***

$= \frac{(1+\cos\theta+\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}{(1-\cos\theta-\mathrm{i}\sin\theta)(1-\cos\theta+\mathrm{i}\sin\theta)}$ ***A1***

$\mathrm{Re}\left(\frac{1+z}{1-z}\right) = \frac{1-\cos^2\theta-\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta} \quad \left(= \frac{1-\cos^2\theta-\sin^2\theta}{2-2\cos\theta}\right)$ ***M1A1***

Note: Award ***M1*** for expanding the numerator and ***A1*** for a correct numerator. Condone either an incorrect denominator or the absence of a denominator.

using $\cos^2\theta + \sin^2\theta = 1$ to simplify the numerator ***(M1)***

$\mathrm{Re}\left(\frac{1+z}{1-z}\right) = 0$ ***AG***

[5 marks]

4. [Maximum mark: 8] 21M.1.AHL.TZ1.7

Consider the quartic equation $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0,\ z \in \mathbb{C}$.

Two of the roots of this equation are $a + b\mathrm{i}$ and $b + a\mathrm{i}$, where $a,\ b \in \mathbb{Z}$.

Find the possible values of a. [8]

Markscheme

METHOD 1

other two roots are $a - b\mathrm{i}$ and $b - a\mathrm{i}$ ***A1***

sum of roots $= -4$ and product of roots $= 400$ ***A1***

attempt to set sum of four roots equal to -4 or 4 OR
attempt to set product of four roots equal to 400 ***M1***

$a + b\mathrm{i} + a - b\mathrm{i} + b + a\mathrm{i} + b - a i = -4$

$2a + 2b = -4 (\Rightarrow a + b = -2)$ ***A1***

$(a + b\mathrm{i})(a - b\mathrm{i})\ (b + a\mathrm{i})(b - a\mathrm{i}) = 400$

$\left(a^2 + b^2\right)^2 = 400$ ***A1***

$a^2 + b^2 = 20$

attempt to solve simultaneous equations *(M1)*

$a = 2$ or $a = -4$ ***A1A1***

METHOD 2

other two roots are $a - b\text{i}$ and $b - a\text{i}$ ***A1***

$(z - (a + b\text{i}))(z - (a - b\text{i}))(z - (b + a\text{i}))(z - (b - a\text{i}))(= 0)$ ***A1***

$\left((z - a)^2 + b^2\right)\left((z - b)^2 + a^2\right)(= 0)$

$\left(z^2 - 2az + a^2 + b^2\right)\left(z^2 - 2bz + b^2 + a^2\right)(= 0)$ ***A1***

Attempt to equate coefficient of z^3 and constant with the given quartic equation ***M1***

$-2a - 2b = 4$ and $\left(a^2 + b^2\right)^2 = 400$ ***A1***

attempt to solve simultaneous equations *(M1)*

$a = 2$ or $a = -4$ ***A1A1***

[8 marks]

5. [Maximum mark: 6] 22M.1.AHL.TZ1.9

Consider the complex numbers $z_1 = 1 + b\text{i}$ and $z_2 = \left(1 - b^2\right) - 2b\text{i}$, where $b \in \mathbb{R},\ b \neq 0$.

(a) Find an expression for $z_1 z_2$ in terms of b. [3]

Markscheme

$z_1 z_2 = (1 + b\text{i})\left(\left(1 - b^2\right) - (2b)\text{i}\right)$

$= \left(1 - b^2 - 2\text{i}^2 b^2\right) + \text{i}\left(-2b + b - b^3\right)$ ***M1***

$= \left(1 + b^2\right) + \text{i}\left(-b - b^3\right)$ ***A1A1***

Note: Award ***A1*** for $1 + b^2$ and A1 for $-b\text{i} - b^3\text{i}$.

[3 marks]

(b) Hence, given that $\arg(z_1 z_2) = \frac{\pi}{4}$, find the value of b. [3]

Markscheme

$\arg(z_1 z_2) = \arctan\left(\frac{-b-b^3}{1+b^2}\right) = \frac{\pi}{4}$ ***(M1)***

EITHER

$\arctan(-b) = \frac{\pi}{4}$ (since $1 + b^2 \neq 0$, for $b \in \mathbb{R}$) ***A1***

OR

$-b - b^3 = 1 + b^2$ (or equivalent) ***A1***

THEN

$b = -1$ ***A1***

[3 marks]

6. [Maximum mark: 10] 24M.2.AHL.TZ2.8

Let $z = 1 + \cos 2\theta + \mathrm{i} \sin 2\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(a) Show that

(a.i) $\arg z = \theta$; [3]

Markscheme

METHOD 1

$\arg z = \arctan\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right) \left(\tan(\arg z) = \frac{\sin 2\theta}{1+\cos 2\theta}\right)$ ***A1***

uses $2 \sin \theta \cos \theta$ in the numerator and any double angle identity for $\cos 2\theta$ in the denominator ***M1***

$\arg z = \arctan\left(\frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta}\right) \left(\tan(\arg z) = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta}\right)$

$\Rightarrow \arg z = \arctan(\tan \theta) \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ ***A1***

$= \theta$ ***AG***

[3 marks]

METHOD 2 - covers answers to both parts a(i) and a(ii)

$z = 1 + 2\cos^2\theta - 1 + 2\sin\theta\cos\theta\mathrm{i}$ ***M1A1A1***

$z = 2\cos^2\theta + 2\sin\theta\cos\theta\mathrm{i}$ ***A1*** attempt to form $z = r\,cis\theta$ *M1*

$z = 2\cos\theta(\cos\theta + \mathrm{i}\sin\theta)$ *A1A1*

$\therefore |z| = 2\cos\theta$ and $\arg z = \theta$. *AG*

[3 marks]

(a.ii) $|z| = 2\cos\theta$. [4]

Markscheme

attempts to express $|z|$ in the form $\sqrt{(\mathrm{Re}\ z)^2 + (\mathrm{Im}\ z)^2}$ *(M1)*

$|z| = \sqrt{(1+\cos 2\theta)^2 + \sin^2 2\theta}$

attempts to expand $(1+\cos 2\theta)^2$ and then uses

$\cos^2 2\theta + \sin^2 2\theta = 1$ in an attempt to simplify *(M1)*

$|z| = \sqrt{2 + 2\cos 2\theta}$ *A1*

$|z| = \sqrt{4\cos^2\theta}\ (= 2|\cos\theta|)$ *A1*

$= 2\cos\theta\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ *AG*

[4 marks]

METHOD 2 - covers answers to both parts a(i) and a(ii)

$z = 1 + 2\cos^2\theta - 1 + 2\sin\theta\cos\theta\mathrm{i}$ *M1A1A1*

$z = 2\cos^2\theta + 2\sin\theta\cos\theta\mathrm{i}$ *A1*

attempt to form $z = r\,cis\theta$ *M1*

$z = 2\cos\theta(\cos\theta + \mathrm{i}\sin\theta)$ *A1A1*

$\therefore |z| = 2\cos\theta$ and $\arg z = \theta$. *AG*

[4 marks]

(b) Hence or otherwise, find the value of θ such that $\arg\left(z^2\right) = |z^3|$. [3]

Markscheme

$2\theta = (2\cos\theta)^3$ *(A1)*

attempts to solve for θ *(M1)*

$\theta = 0.913236\ldots$

$\theta = 0.913$ ***A1***

Note: Award all marks for $\theta = 0.913$ found directly without using part (a).

Note: Award ***(A1)(M1)A0*** for $\theta = 3.97^\circ$.

[3 marks]

7. [Maximum mark: 17] 23N.1.AHL.TZ2.12

(a) Find the binomial expansion of $(\cos\ \theta + \mathrm{i}\ \sin\ \theta)^5$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\ \theta$ and $\cos\ \theta$. [4]

Markscheme

attempt to expand using binomial theorem: ***(M1)***

Note: Award ***(M1)*** for seeing at least one term with a product of a binomial coefficient, power of $\mathrm{i}\ \sin\ \theta$ and a power of $\cos\ \theta$.

$(\cos\ \theta + \mathrm{i}\ \sin\ \theta)^5 = \cos^5\theta + {}^5C_1\mathrm{i}\ \cos^4\theta\ \sin\ \theta + {}^5C_2\mathrm{i}^2\cos^3\theta\ \sin^2\theta$

$+{}^5C_3\mathrm{i}^3\cos^2\theta\ \sin^3\theta + {}^5C_4\mathrm{i}^4\cos\ \theta\ \sin^4\theta + \mathrm{i}^5\sin^5\theta$ ***A1***

$= \left(\cos^5\theta - 10\ \cos^3\theta\ \sin^2\theta + 5\ \cos\ \theta\ \sin^4\theta\right) + \mathrm{i}\left(5\ \cos^4\theta\ \sin\ \theta - 10\ \cos^2\theta\ \sin^3\theta + \sin^5\theta\right)$
A1A1

Note: Award ***A1*** for correct real part and ***A1*** for correct imaginary part.

[4 marks]

(b) By using De Moivre's theorem and your answer to part (a), show that $\sin\ 5\theta \equiv 16\ \sin^5\theta - 20\ \sin^3\theta + 5\ \sin\ \theta$. [6]

Markscheme

$(\cos\ \theta + \mathrm{i}\ \sin\ \theta)^5 = \cos\ 5\theta + \mathrm{i}\ \sin\ 5\theta$ ***(A1)***

equate imaginary parts: ***(M1)***

$\sin\ 5\theta = 5\ \cos^4\theta\ \sin\ \theta - 10\ \cos^2\theta\ \sin^3\theta + \sin^5\theta$ ***A1***

substitute $\cos^2\theta = 1 - \sin^2\theta$ ***(M1)***

$\sin\ 5\theta = 5\left(1 - \sin^2\theta\right)^2\sin\ \theta - 10\ \sin^3\theta\left(1 - \sin^2\theta\right) + \sin^5\theta$ ***A1***

$\sin 5\theta = 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta(1 - \sin^2\theta) + \sin^5\theta$ ***A1***

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ ***AG***

Note: Some of this working may be seen in part (a). Allow for awarding marks in part (b).

[6 marks]

(c.i) Hence, show that $\theta = \frac{\pi}{5}$ and $\theta = \frac{3\pi}{5}$ are solutions of the equation $16\ \sin^4\theta - 20\ \sin^2\theta + 5 = 0$. [3]

Markscheme

factorising $16\ \sin^5\ \theta - 20\ \sin^3\theta + 5\ \sin\ \theta$ ***M1***

$(\sin\ 5\ \theta =)\sin\ \theta(16\ \sin^4\theta - 20\ \sin^2\theta + 5)$

EITHER

$\sin\ 5\left(\frac{\pi}{5}\right) = 0$ and $\sin\ 5\left(\frac{3\pi}{5}\right) = 0$ ***R1***

Note: The ***R1*** is independent of the ***M1*****.**

OR

solving $\sin\ 5\theta = 0$

$\theta = \frac{k\pi}{5}$ where $k \in \mathbb{Z}$ ***R1***

Note: The ***R1*** is independent of the ***M1***.

THEN

therefore either $\sin\ \theta = 0$ OR $16\ \sin^4\ \theta - 20\ \sin^2\ \theta + 5 = 0$

$\sin\frac{\pi}{5} \neq 0$ and $\sin\frac{3\pi}{5} \neq 0$ (or only solution to $\sin\ \theta = 0$ is $\theta = 0$) ***R1***

therefore $\frac{\pi}{5}, \frac{3\pi}{5}$ are solutions of $16\ \sin^4\theta - 20\ \sin^2\theta + 5 = 0$ ***AG***

Note: The final ***R1*** is dependent on both previous marks.

[3 marks]

(c.ii) Hence. show that $\sin\frac{\pi}{5}\sin\frac{3\pi}{5} = \sqrt{\frac{5}{4}}$ [4]

Markscheme

METHOD 1

attempt to use quadratic formula: ***(M1)***

$\sin^2\theta = \frac{20 \pm \sqrt{80}}{32}$ ***A1***

$= \frac{5 \pm \sqrt{5}}{8}$

$\sin\,\theta = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$

$\Rightarrow \sin\,\frac{\pi}{5}\sin\,\frac{3\pi}{5} = \sqrt{\frac{5+\sqrt{5}}{8}}\sqrt{\frac{5-\sqrt{5}}{8}}$ ***M1***

$= \sqrt{\frac{20}{64}}$ ***A1***

$= \sqrt{\frac{5}{4}}$ ***AG***

METHOD 2

roots of quartic are $\sin\,\frac{\pi}{5}, \sin\,\frac{2\pi}{5}, \sin\,\frac{3\pi}{5}, \sin\,\frac{4\pi}{5}$ ***A1***

attempt to set product of roots equal to $\pm\frac{5}{16}$ ***M1***

$\sin\,\frac{\pi}{5}\sin\,\frac{2\pi}{5}\sin\,\frac{3\pi}{5}\sin\,\frac{4}{5} = \frac{5}{16}$ ***A1***

recognition that $\sin\,\frac{\pi}{5} = \sin\,\frac{4\pi}{5}$ and $\sin\,\frac{2\pi}{5} = \sin\,\frac{3\pi}{5}$

$\sin^2\,\frac{\pi}{5}\sin^2\,\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\,\frac{\pi}{5}\sin\,\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

METHOD 3

Consider $16\,\sin^4\,\theta - 20\,\sin^2\theta + 5 = 0$ as a quadratic in $\sin^2\,\theta$ ***M1***

$\left(\theta = \frac{\pi}{5}, \frac{3\pi}{5}\text{ are roots}\right)$, so $\sin^2\,\frac{\pi}{5}$ and $\sin^2\,\frac{3\pi}{5}$ are roots of the quadratic. ***A1***

Consider product of roots: ***M1***

$\Rightarrow \sin^2\,\frac{\pi}{5}\sin^2\,\frac{3\pi}{5} = \frac{5}{16}$ ***A1***

$\sin\,\frac{\pi}{5}\sin\,\frac{3\pi}{5} = \frac{\sqrt{5}}{4}$ ***AG***

[4 marks]