HL / Maclaurin series and differential equations [56 marks]

1. [Maximum mark: 21]

The function f is defined by $f\left(x\right)=\mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f\left(x\right)$ and hence find the Maclaurin series for $f\left(x\right)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f\left(x\right)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x}-1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x}-1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x\to 0}\frac{f(x)-1}{\arctan(\mathrm{e}^{3x}-1)}$. [3]

2. [Maximum mark: 19]

This question investigates some applications of differential equations to modeling population growth.

One model for population growth is to assume that the rate of change of the population is proportional to the population, i.e. $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$, where $k\in\mathbb{R}$, t is the time (in years) and P is the population

(a) Show that the general solution of this differential equation is $P=A\mathrm{e}^{kt}$, where $A\in\mathbb{R}$. [5]

The initial population is 1000.

Given that $k=0.003$, use your answer from part (a) to find

(b) the population after 10

(b) the population after 10 years [2]

(c) the number of years it will take for the population to triple. [2]

(d) $\lim_{t \to \infty} P$ [1]

Consider now the situation when k is not a constant, but a function of time.

Given that $k = 0.003 + 0.002t$, find

(e) the solution of the differential equation, giving your answer in the form $P = f(t)$. [5]

(f) the number of years it will take for the population to triple. [4]

3. [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$, given that $y = 2$ when $x = 0$.

Give your answer in the form $y = f(x)$. [7]

4. [Maximum mark: 9]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a).

[1]