Paper 3 / 10.10.2024 [59 marks]

1. [Maximum mark: 29] EXM.3.AHL.TZ0.3

This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \; n \in \mathbb{N}.$

(a) Find the exact values of I_0, I_1 and I_2. [6]

(b.i) Use integration by parts to show that
$I_n = \frac{n-1}{n} I_{n-2}, \; n \geqslant 2.$ [5]

(b.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

(c) Hence, find the exact values of I_3 and I_4. [2]

Let $J_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx, \; n \in \mathbb{N}.$

(d) Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. [4]

(e) Hence, find the exact values of J_5 and J_6 [2]

Let $T_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \; n \in \mathbb{N}.$

(f) Find the exact values of T_0 and T_1. [3]

(g.i) Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that
$T_n = \frac{1}{n-1} - T_{n-2}, \; n \geqslant 2.$ [3]

(g.ii) Explain where the condition $n \geqslant 2$ was used in your proof. [1]

(h) Hence, find the exact values of T_2 and T_3. [2]

2. [Maximum mark: 30] EXN.3.AHL.TZ0.2

A **Gaussian integer** is a complex number, z, such that $z = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\beta = 1 - 2\mathrm{i}$, such that $\gamma = \alpha\beta$ for some Gaussian integer γ.

(a) Find γ. [2]

Now consider two Gaussian integers, $\alpha = 3 + 4\mathrm{i}$ and $\gamma = 11 + 2\mathrm{i}$.

(b) Determine whether $\frac{\gamma}{\alpha}$ is a Gaussian integer. [3]

The norm of a complex number z, denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3\mathrm{i}$ then $N(2 + 3\mathrm{i}) = 2^2 + 3^2 = 13$.

(c) On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3. [2]

(d) Given that $\alpha = a + b\mathrm{i}$ where $a,\ b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$. [1]

A **Gaussian prime** is a Gaussian integer, z, that **cannot** be expressed in the form $z = \alpha\beta$ where $\alpha,\ \beta$ are Gaussian integers with $N(\alpha),\ N(\beta) > 1$.

(e) By expressing the positive integer $n = c^2 + d^2$ as a product of two Gaussian integers each of norm $c^2 + d^2$, show that n is not a Gaussian prime. [3]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

(f) Verify that 2 is not a Gaussian prime. [2]

(g) Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. [2]

Let α, β be Gaussian integers.

(h) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. [6]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

(i) Hence show that $1 + 4\mathrm{i}$ is a Gaussian prime. [3]

(j) Use proof by contradiction to prove that a prime number, p, that is not of the form $a^2 + b^2$ is a Gaussian prime. [6]