■ sample question #1 ■

Worked Solution

(a) $k=0$: curve $y = xe^x$ and line $y=0$ (x-axis)

intersection: $xe^x = 0 \Rightarrow x=0$ or $e^x = 0$

$e^x > 0,\ x \in \mathbb{R}$, therefore $y = xe^x$ and $y=0$ intersect only one when $x=0$ and $y=0$ (at the origin)

(b) $k=1$: line is $y=x$

find equation of line tangent to $y = xe^x$ at $(0,\ 0)$

$\dfrac{dy}{dx} = xe^x + e^x$; at $(0,\ 0)$: $\dfrac{dy}{dx} = 0 + e^0 = 1$

equation of tangent line is $y - 0 = 1\cdot(x-0) \Rightarrow y = x$ **Q.E.D.**

(c) (i) $xe^x = kx \Rightarrow x(e^x - k) = 0 \Rightarrow x = 0$ or $x = \ln k$

when $k=1$, $x = \ln 1 = 0$ and there are not two distinct points of intersection

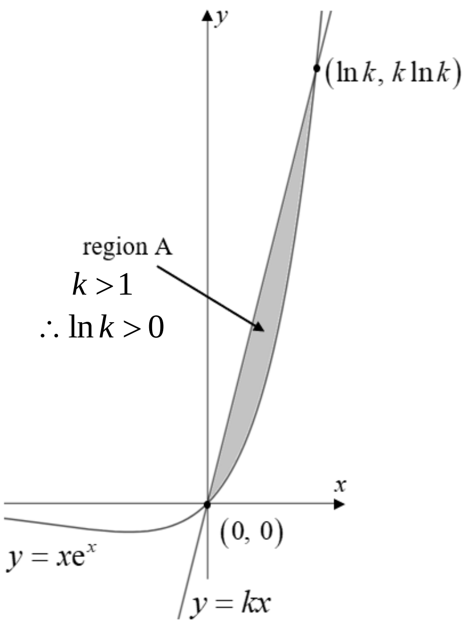
Therefore, there are two distinct points of intersection when $k > 0,\ k \neq 1$

(ii) $xe^x = kx \Rightarrow x(e^x - k) = 0 \Rightarrow x = 0$ or $x = \ln k$

when $x=0,\ y=0$; when $x = \ln k,\ y = k\ln k$

coordinates of points of intersection are $(0,\ 0)$ and $(\ln k,\ k\ln k)$

(d) (i) area of $\mathrm{A} = \int_0^{\ln k} \left(kx - xe^x\right) dx$



(ii) $k = e^2$: area of $\mathrm{A} = \int_0^{\ln(e^2)} \left(e^2 x - xe^x\right) dx$

area of $\mathrm{A} = \int_0^2 e^2 x\, dx - \int_0^2 xe^x\, dx$

$$= \left[\frac{e^2 x^2}{2}\right]_0^2 - \int_0^2 xe^x\, dx$$

Find $\int xe^x\, dx$ by integration by parts:

$u = x \Rightarrow du = dx$; $dv = e^x dx \Rightarrow v = e^x$

$\int xe^x\, dx = xe^x - \int e^x\, dx$

$= xe^x - e^x$

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(d) (ii) continued …

area of $\mathrm{A} = \left[\dfrac{\mathrm{e}^2 x^2}{2}\right]_0^2 - \left[x\mathrm{e}^x - \mathrm{e}^x\right]_0^2$

$= \left[2\mathrm{e}^2 - 0\right] - \left[\left(2\mathrm{e}^2 - \mathrm{e}^2\right) - (0-1)\right]$

$= 2\mathrm{e}^2 - 2\mathrm{e}^2 + \mathrm{e}^2 - 1$

Thus, when $k = \mathrm{e}^2$, area of $\mathrm{A} = \mathrm{e}^2 - 1$

(iii) $k = \mathrm{e}^n,\ n \in \mathbb{R}^+$

area of $\mathrm{A} = \int_0^{\ln(\mathrm{e}^n)} \left(\mathrm{e}^n x - x\mathrm{e}^x\right)\mathrm{d}x$

$= \left[\dfrac{\mathrm{e}^n x^2}{2} - \left(x\mathrm{e}^x - \mathrm{e}^x\right)\right]_0^2$

$= \left(\dfrac{n^2}{2}\mathrm{e}^n - n\mathrm{e}^n + \mathrm{e}^n\right) - (0 - 0 + 1)$

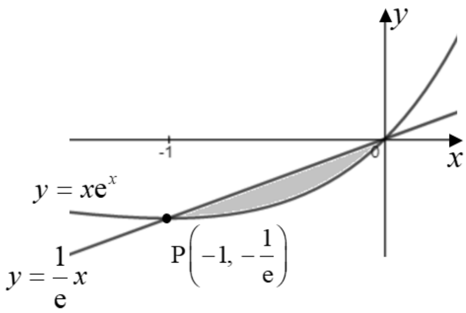
Thus, when $k = \mathrm{e}^n,\ n \in \mathbb{R}^+$, area of $\mathrm{A} = \mathrm{e}^n\left(\dfrac{n^2}{2} - n + 1\right) - 1$

(e) (i) $y = x\mathrm{e}^x \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^x + \mathrm{e}^x = \mathrm{e}^x(x+1) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = -1$

$y(-1) = -\mathrm{e}^{-1} = -\dfrac{1}{\mathrm{e}}$; therefore, coordinates of P are $\left(-1, -\dfrac{1}{\mathrm{e}}\right)$

gradient of line $= k = \dfrac{0 - \left(-\dfrac{1}{\mathrm{e}}\right)}{0 - (-1)} = \dfrac{1}{\mathrm{e}} \Rightarrow k = \dfrac{1}{\mathrm{e}}$

(ii) $k = \dfrac{1}{\mathrm{e}}$: area of enclosed region $= \int_{-1}^{0}\left(\dfrac{1}{\mathrm{e}}x - x\mathrm{e}^x\right)\mathrm{d}x = \left[\dfrac{\mathrm{e}^{-1}x^2}{2} - \left(x\mathrm{e}^x - \mathrm{e}^x\right)\right]_{-1}^{0}$



$= (0 - 0 + 1) - \left(\dfrac{1}{2\mathrm{e}} + \dfrac{1}{\mathrm{e}} + \dfrac{1}{\mathrm{e}}\right) = 1 - \left(\dfrac{1}{2\mathrm{e}} + \dfrac{2}{2\mathrm{e}} + \dfrac{2}{2\mathrm{e}}\right)$

area $= 1 - \dfrac{5}{2\mathrm{e}}$

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(f) since $0<k<1$, then $\ln k<0$ and $x=\ln k$ is lower limit of integration

area of $\mathrm{B}=\int_{\ln k}^{0}\left(kx-x\mathrm{e}^{x}\right)\mathrm{d}x$

$$\text{area of B}=\left[\frac{k}{2}x^{2}-\left(x\mathrm{e}^{x}-\mathrm{e}^{x}\right)\right]_{\ln k}^{0}=0-0+\mathrm{e}^{0}-\left(\frac{k}{2}(\ln k)^{2}-\ln k\left(\mathrm{e}^{\ln k}\right)+\mathrm{e}^{\ln k}\right)$$

$$=1-\frac{k}{2}(\ln k)^{2}-k\ln k+k$$

$$=1-\frac{k}{2}\left[(\ln k)^{2}-2\ln k+2\right]$$

$$=1-\frac{k}{2}\left[(\ln k)^{2}-2\ln k+1+1\right]$$

note: $\left[(\ln k-1)^{2}=(\ln k)^{2}-2\ln k+1\right]$

Hence, area of $\mathrm{B}=1-\frac{k}{2}\left[(\ln k-1)^{2}+1\right]$

$k>0$ and $(\ln k-1)^{2}+1>0$, therefore $\frac{k}{2}\left[(\ln k-1)^{2}+1\right]>0$

Thus, area of $\mathrm{B}<1$ **Q.E.D.**