The estimation of the path between two points on the Earth using a small circle, great circle, and spherical law of cosines.

Contents

Aim

The aim of this investigation is to determine the shortest distance between two points on the Earth using small circle, great circle, and spherical law of cosines.

Introduction

During my summer vacation, I like to read different books. One of the books that I read in the summer is “Astronomy for entertainment” written by Yakov Perelman. Yakov Perelman included great circles and great circles routes in his book. According to this book, the great circle route is shorter than the small circle route, although the great circle route looked curved on the 2D map. I found this interesting. After this, I found out more about Spherical geometry and the spherical law of cosines and I quickly understood that Euclidian geometry cannot be applied to spheres. I decided to investigate how the shortest distance can be determined by using the spherical law of cosines, great and small circles. I started to make a plan about what kind of different situations can be introduced. I came up with three situations: points that are located at the same longitude, points that are located at the same latitude, and points that are located on random parts of the Earth. In addition, Spherical geometry is relevant to me, because I am intended to study astronomy in the future.

The Earth is approximately spherical. Trigonometry and Spherical Geometry can be used to determine distances between points on the Earth if it is imagined that the Earth is a perfect sphere. Spherical geometry is a geometry of a sphere. The difference between Spherical and Euclidian Geometry is that the segment between two points can be stretched to infinity, however, in Spherical geometry, the segment eventually will reach the starting point. Another important difference between Spherical and Euclidian Geometry is that the sum of the angles of the triangle in Euclidian geometry is always 180°, but in Spherical geometry, they will exceed this number.[1]

The ends of the axis around which Earth spins are called the South and North pole. The earth is divided into longitude lines which run north-south and latitude lines which run east-west and are parallel to the equator. The equator is a great circle that has 0 degrees latitude. It divides Earth into northern and southern hemispheres.[2] The great circle is a circle that has the same center and radius as the sphere. Another example of the great circle on the Earth is the prime meridian which has a longitude of 0 degrees. It divides Earth into Eastern and Western hemispheres.[3] In Euclidian Geometry the shortest path between two points is a line segment of gradient 0. However, the shortest distance between two points on the sphere is the great circle route. This distance is commonly used in navigation. In addition, the sphere has small circles which always have a smaller radius than the great circles. Small circles do not have a

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[1] *The Three Geometries - EscherMath*. (2015)

[2] *What is latitude?* (n.d.)

[3] *Great Circle*. (n.d.)

center of the sphere. The small circle route can be used to prove that the great circle route is the shortest distance between two points on the sphere.[4]

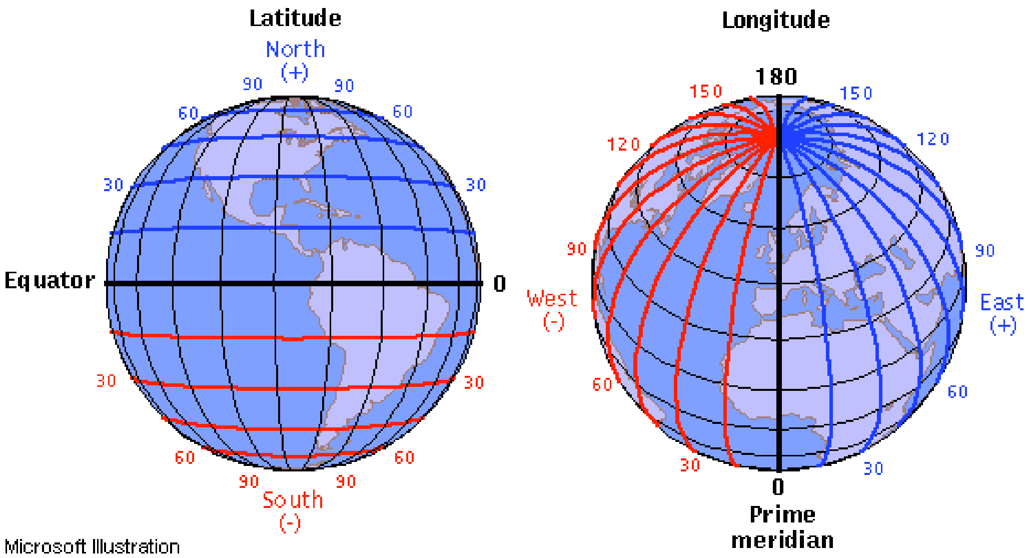

Figure 1 Longitude and latitude lines[5]

Method

For the sake of the investigation, the approximation of Earth's radius of 6400 km will be used, and the Earth will be assumed to be a perfect sphere. In addition, four significant figures will be used for the approximation of final distances between two points. Coordinates used in calculations were all taken from the website "Latitude.to.".[6] Three situations will be introduced: points that are located on the same longitude, points that are located on the same latitude, and points that are located on random parts of the Earth.

1. *Distance between two points on the same longitude on the same side of the equator.*

[4] Roy, M. (2022).
[5] https://journeynorth.org/tm/LongitudeIntro.html
[6] https://latitude.to./

As was said before, each longitude, also called meridian is a great circle, and its radius is equal to the radius of the Earth. The approximated radius of the Earth is 6400 km.

Helsinki (Finland) and Tallin (Estonia) both have the same longitude of approximately 25 ° E. Helsinki lies on approximately 60.17° N and Tallin lies on approximately 59.44° N.

Both points are on the same side of equator, so separation is used.

The angle between the latitude of Helsinki and Tallin is: 60.17° - 59.44° = 0.73°

To find the distance the arc formula is needed:

$$s = \frac{\theta}{360^\circ} \times (2\pi r)$$ [7]

Where s is the arc length, θ is the central angle and r is the radius of a circle. The formula can be simplified.

Units are canceled:

$$s = \frac{\theta}{360°} \times (2\pi r)$$

$$s = \frac{\theta}{180°} \times \pi r$$

After this, the values are plugged into the formula:

$$s = \frac{0.73^\circ}{180^\circ} \times \pi \times 6400 = 82.5418 \ldots \approx 82.54 \text{ (km)}$$

[7] La Rondie et al. (2019)

Another example will introduce the situation, where two points on the same longitude located on the different sides of the equator.

Wladyslawowo (Poland) and Cape Town (South Africa) have the same longitude of approximately 18° E. Wladyslawowo lies on approximately 54.78° N and Cape town lies on approximately 33.93° S.

Angles are added together because points are on the different sides of equator:

$$33.93^\circ + 54.78^\circ = 88.71^\circ$$

The values are plugged into the formula:

$$s = \frac{88.71^\circ}{180^\circ} \times \pi \times 6400 = 9909.0 \ldots \approx 9909\ (\text{km})$$

2. *Distance between points which share the same latitude.*

Equator is a great circle and has a 0 degrees latitude.

Macapa (Brazil) and Quito (Ecuador) lie on the equator on the same side of the Prime Meridian. Macapa lies on approximately 51.07° W and Quito lies on approximately 78.51° W.

So,

$$78.51^\circ - 51.07^\circ = 27.44^\circ$$

The same arc formula is used to calculate the distance between two points:

$$s = \frac{\theta}{180^\circ} \times \pi r$$

Numbers are plugged into the formula:

$$s = \frac{27.44^{\circ}}{180^{\circ}} \times \pi \times 6400 = 3065.08 \ldots \approx 3065 \text{ (km)}$$

If the points are located on the different sides of the Prime Meridian, the angles are always added together.

For example, Kahala (Indonesia) and Macapa (Brazil) lie on the equator on the different sides of the Prime Meridian. Kahala lies on approximately 116.4° W and Macapa lies on approximately 51.07° W.

So,

$$51.07^{\circ} + 116.4^{\circ} = 167.47^{\circ} \approx 167.5^{\circ}$$

Numbers are plugged into the formula:

$$s = \frac{167.5^{\circ}}{180^{\circ}} \times \pi \times 6400 = 18709.9 \ldots \approx 18710 \text{ (km)}$$

Another example will investigate two points which do not lie on the equator.

Sydney (Australia) and Margaret River (Australia) both share the same latitude of approximately 34° S. Sydney lies on approximately 151.2° E and Margaret River lies on approximately 115.1° E.

At first the small circle distance should be calculated.

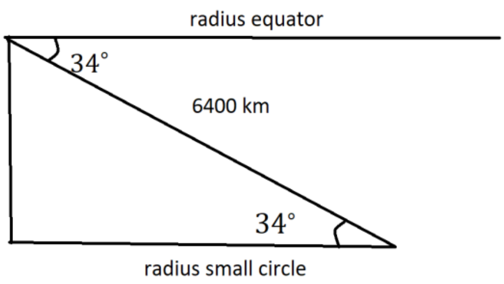

Figure 2 The radius of equator and the radius of small circle

The radius (r) of the small circle is:

$$\cos 34^{\circ} = \frac{r}{6400}$$

$$r = 6400 \times \cos(34°)$$

$$r = 5305.84 \approx 5306 \text{ (km)}$$

After this the longitude difference between Sydney and Margaret River should be calculated:

$$151.2° - 115.1° = 36.1°$$

The arc length formula should be used to determine the distance between points by the small circle.

$$s = \frac{\theta}{180°} \times \pi r$$

$$s = \frac{36.1^{\circ}}{180°} \times \pi \times 5306$$

$$s = 3343.12\ldots \approx 3343 \text{ (km)}$$

The distance between points by the small circle is 3343 km.

Great circle distance can be calculated by cosine rule, where c, a and b are the sides of the triangle. C is the angle opposite side c.

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$ [8]

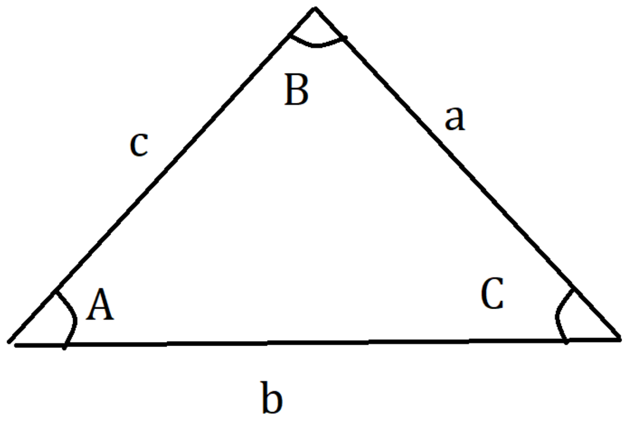

Figure 3 Triangle with sides a, b, c and angles A, B, C

At first, the line segment d between Sydney and Margaret River is needed.

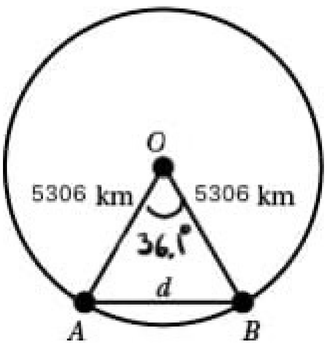

Figure 4 The line segment d between Sydney and Margaret River. The 5306 km is the radius of the small circle distance.

[8] Rondie et al. (2019)

The line segment (d) between Sydney and Margaret River is:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$d^2 = 5306^2 + 5306^2 - 2 \times 5306 \times 5306 \times \cos(36.1°)$$

$$d = \sqrt{(5306^2 + 5306^2 - 2 \times 5306 \times 5306 \times \cos(36.1°))}$$

$$d = 3288.09 \ldots \approx 3288 \text{ (km)}$$

After this, the angle between Sydney and Margaret River can be calculated.

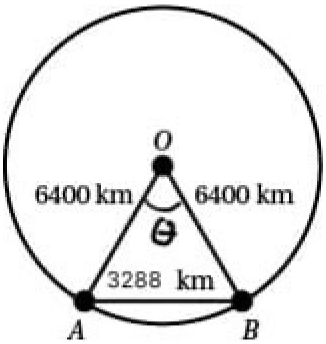

Figure 5 The angle θ between Sydney and Margaret River. The 6400 km is the radius of the Earth and the radius of the great circle distance.

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\cos(C) = \frac{(a^2 + b^2 - c^2)}{2ab}$$

$$\cos(\theta) = \frac{(6400^2 + 6400^2 - 3288^2)}{2 \times 6400 \times 6400}$$

$$\theta = \cos^{-1}\left(\frac{(6400^2 + 6400^2 - 3288^2)}{2 \times 6400 \times 6400}\right)$$

The angle is:

$$\theta = 29.77°$$

The great circle distance can be calculated by using arc formula:

$$s = \frac{\theta}{180°} \times \pi r$$

The great circle distance is:

$$s = \frac{29.77^{\circ}}{180^{\circ}} \times \pi \times 6400$$

$$s = 3325.34 \ldots \approx 3325 \text{ (km)}$$

In addition, the difference between great circle distance and small circle distance can be calculated.

$$3343 - 3325 = 18 \text{ (km)}$$

The great circle distance is 3325 km and small circle distance is 3343 km. This can prove that the great circle distance is shorter than the small circle distance.

3. *The spherical law of cosines and the distance between points that are located on random parts of the Earth.*

The spherical law of cosines is a Spherical geometry. It is a theorem similar to the ordinary law of cosines from trigonometry. This theorem relates to the angles and sides of the triangle on the sphere. On the unit sphere, where the radius is 1, three sides a, b and c of triangle are created by three different great circles across the sphere. The C is the angle that lays opposite to the c side of spherical triangle.

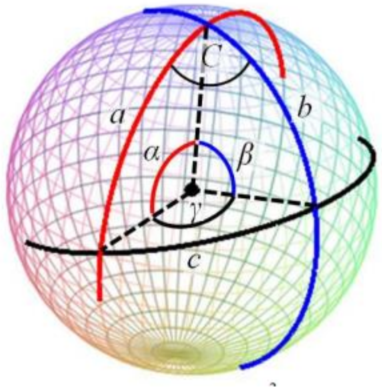

Figure 6 The triangle on the sphere created by great circles.[9]

The unit sphere, where the radius (r) is 1, is used to investigate the spherical law of cosines. The sides of triangle a, b, and c are simultaneously the central angles (in radians) shown on the picture as α, β, and γ subtended by those sides a, b, and c from the center of the sphere.

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

If the C is equal $\frac{\pi}{2}$, then cos C = 0. This leads to analogous of Pythagorean theorem:

$$\cos(c) = \cos(a)\cos(b)$$

The cos(a) in the formulas cos(c) = cos(a) cos(b) and cos(c) = cos(a) cos(b) + sin(a) sin(b) cos(C) could be replaced with cos $(\frac{a}{r})$, cos(b) with cos $(\frac{b}{r})$, and cos(c) with cos $(\frac{c}{r})$ where r is radius, because $(\frac{a}{r})$, $(\frac{b}{r})$, and $(\frac{c}{r})$ will be the angles based on the arcs of the great circle of length a, b, and c.

Formulas become:

[9] https://www.theoremoftheday.org/GeometryAndTrigonometry/SphericalCos/TotDSphericalCos.pdf

$$\cos(\frac{c}{r}) = \cos(\frac{a}{r})\cos(\frac{b}{r})$$

and

$$\cos(\frac{c}{r}) = \cos(\frac{a}{r})\cos(\frac{b}{r}) + \sin(\frac{a}{r})\sin(\frac{b}{r})\cos(C)$$

Any spherical triangle satisfies:

$$\cos(\frac{c}{r}) = \cos(\frac{a}{r})\cos(\frac{b}{r}) + \sin(\frac{a}{r})\sin(\frac{b}{r})\cos(C)$$ [10]

Therefore, it also satisfies:

$$\cos(\frac{b}{r}) = \cos(\frac{a}{r})\cos(\frac{c}{r}) + \sin(\frac{a}{r})\sin(\frac{c}{r})\cos(B)$$

and

$$\cos(\frac{a}{r}) = \cos(\frac{b}{r})\cos(\frac{c}{r}) + \sin(\frac{b}{r})\sin(\frac{c}{r})\cos(A)$$

Let A, B, and C be points that connect the spherical triangle (△ABC). The △A′B′C′ is a polar triangle of (△ABC) and its sides are a′, b′, and c′. A polar triangle is a triangle formed by the arcs of three great circles. Each pole of the arcs of three great circles is the vertex of a given spherical triangle.[11] On a sphere, two spherical triangles are mutually polar if each vertex of one is the pole of an edge of the other. The polar triangle lies in the same hemisphere as the original spherical triangle. [12]

[10] Hetyei (2018)
[11] Merriam-Webster. (n.d.).
[12] Russel et al (n.d)

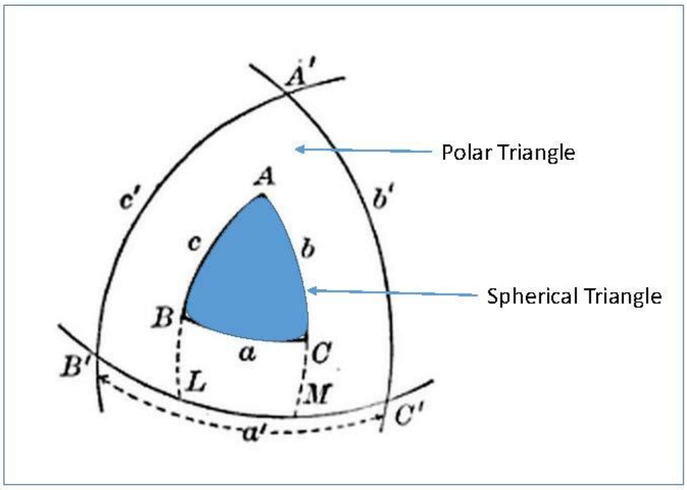

Figure 7 Polar triangle and spherical triangle[13]

To get the proof that any spherical triangle satisfies $\cos(\frac{c}{r}) = \cos(\frac{a}{r}) \cos(\frac{b}{r}) + \sin(\frac{a}{r})$ $\sin(\frac{b}{r}) \cos(C)$ and obtain the spherical law of cosines for angles the polar triangle △A′B′C′ of spherical triangle △ABC can be used. Where a' = (π − A)r, b' = (π − B)r and c' = (π − C)r and angles $A' = \pi - \frac{a}{r}$, $B' = \pi - \frac{b}{r}$ and $C' = \pi - \frac{c}{r}$

According to spherical law of cosines:

$$\cos(\frac{a}{r}) = \cos(\frac{b}{r}) \cos(\frac{c}{r}) + \sin(\frac{b}{r}) \sin(\frac{c}{r}) \cos(A)$$

This formula is applied to the polar triangle:

$$\cos(\frac{a'}{r}) = \cos(\frac{b'}{r}) \cos(\frac{c'}{r}) + \sin(\frac{b'}{r}) \sin(\frac{c'}{r}) \cos(A')$$

This is the same as:

$$\cos(\pi - A) = \cos(\pi - B) \cos(\pi - C) + \sin(\pi - B) \sin(\pi - C) \cos(\pi - \frac{a}{r})$$

[13]https://math.stackexchange.com/questions/1365018/why-is-the-polar-triangle-useful-in-spherical-geometry

Using the $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x) = \sin(x)$ and multiplying both sides by -1 the formula for angles is obtained:

$\cos(A) = -\cos(B)\cos(C) + \sin(B)\sin(C)\cos(\frac{a}{r})$ [14]

The spherical law of cosines can be used to determine the great circle distance between two points towards North pole on the globe. Let A = $(x_1°N, y_1°E)$ and B = $(x_2°N, y_2°E)$, where $x_1°N$, $y_1°E$ are the coordinates of place A and $x_2°N$, $y_2°E$ are the coordinates of place B. The C is the north pole. The spherical triangle CAB is formed.

The cosine rule of Spherical geometry:

$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C)$$

$$= \cos(90° - x_2°)\cos(90° - x_1°) + \sin(90° - x_2°)\sin(90° - x_1°)\cos(y_2° - y_1°)$$

Using trigonometric identities $\cos(x) = \sin(90° - x)$ and $\sin(x) = \cos(90° - x)$ the equation become:

$$\cos(c) = \sin x_2° \sin x_1° + \cos x_2° \cos x_1° \cos(y_2° - y_1°)$$

$$c = \cos^{-1}(\sin x_2° \sin x_1° + \cos x_2° \cos x_1° \cos(y_2° - y_1°))$$ [15]

The value of c can be used to determine the shortest distance between two points by using the arc formula:

$$s = \frac{\theta}{180°} \times \pi r$$

[14] Hetyei (2018)
[15] Zohora, Akter (2018)

In this case, s is the arc and the distance between two points, r is the radius of the Earth and θ is the c.

To give an example, Madrid (Spain) has approximately 40.42° N and 3.704° W and New York (USA) has approximately 40.71° N, 74.00° W. Let Madrid be a point A and New York a point B.

$$c = \cos^{-1}(\sin x_2° \sin x_1° + \cos x_2° \cos x_1° \cos(y_2° - y_1°))$$

$$c = \cos^{-1}(\sin 40.71° \sin 40.42° + \cos 40.71° \cos 40.42° \cos(74.00° - 3.704°))$$

The length of the arc:

$$s = \pi \times \frac{6400}{180°} \times \cos^{-1}(\sin 40.71° \sin 40.42° + \cos 40.71° \cos 40.42° \cos(74.00° - 3.704°))$$

$$s = 5793.76\ldots \approx 5794 \text{ (km)}$$

This is the distance between Madrid and New York.

Conclusion

In conclusion, the aim of this investigation was to determine the distance between two points on the Earth using small circle, great circle, and spherical law of cosines. The three situations were explored: Distance between two points on the same longitude on the same side of the equator, distance between points which share the same latitude, and the distance between points that are located on random parts of the Earth.

Since the Earth is not a perfect sphere, distances obtained are rough approximations. Four significant figures were used for distances. In addition, the investigation showed that the great circle distance is shorter than the small circle distance.

Distances obtained through calculations were checked and compared to the values obtained from the internet from different sources. The values obtained were close to the values from different sources, despite rough approximations.

For example, according to the website "Distance.to"[16], the distance between New York and Madrid is approximately 5768 km. This is close to the obtained value of 5794 km.

Throughout the course of this exploration, I learned a lot about the Spherical geometry and spherical law of cosines and their applications and importance in navigation.

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