HL / Revision / 21.11.2022 *[36 marks]*

1. Consider the function $f\left(x\right)=x\,\mathrm{e}^{2x}$, where $x\in\mathbb{R}$. The n^{th} derivative of $f\left(x\right)$ is denoted by $f^{\left(n\right)}\left(x\right)$. *[7 marks]*

   Prove, by mathematical induction, that $f^{\left(n\right)}\left(x\right)=\left(2^{n}x+n2^{n-1}\right)\mathrm{e}^{2x}$, $n\in\mathbb{Z}^{+}$.

Markscheme

$f'(x) = \mathrm{e}^{2x} + 2x\mathrm{e}^{2x}$ ***A1***

Note: This must be obtained from the candidate differentiating $f(x)$.

$= \left(2^1 x + 1 \times 2^{1-1}\right)\mathrm{e}^{2x}$ ***A1***

(hence true for $n = 1$)

assume true for $n = k$: ***M1***

$f^{(k)}(x) = \left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}$

Note: Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}\right)$

attempt to differentiate $f^{(k)}(x)$ ***M1***

$f^{(k+1)}(x) = 2^k\mathrm{e}^{2x} + 2\left(2^k x + k2^{k-1}\right)\mathrm{e}^{2x}$ ***A1***

$f^{(k+1)}(x) = \left(2^k + 2^{k+1}x + k2^k\right)\mathrm{e}^{2x}$

$f^{(k+1)}(x) = \left(2^{k+1}x + (k+1)2^k\right)\mathrm{e}^{2x}$ ***A1***

$= \left(2^{k+1}x + (k+1)2^{(k+1)-1}\right)\mathrm{e}^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n\ (\in \mathbb{Z}^+)$ ***R1***

Note: Do not award final ***R1*** if the two previous ***M1s*** are not awarded. Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

The times taken for male runners to complete a marathon can be modelled by a normal distribution with a mean 196 minutes and a standard deviation 24 minutes.

2a. Find the probability that a male runner selected at random will complete the marathon in less than 3 hours. *[2 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$T \sim N(196,\ 24^2)$

$\mathrm{P}(T < 180) = 0.252$ ***(M1)A1***

[2 marks]

It is found that 5% of the male runners complete the marathon in less than T_1 minutes.

2b. Calculate T_1. *[2 marks]*

Markscheme

$\mathrm{P}(T < T_1) = 0.05$ ***(M1)***

$T_1 = 157$ ***A1***

[2 marks]

The times taken for female runners to complete the marathon can be modelled by a normal distribution with a mean 210 minutes. It is found that 58% of female runners complete the marathon between 185 and 235 minutes.

2c. Find the standard deviation of the times taken by female runners. *[4 marks]*

Markscheme

$F \sim N(210,\ \sigma^2)$

$\mathrm{P}(F < 235) = 0.79$ ***(M1)***

$\frac{235-210}{\sigma} = 0.806421$ or equivalent ***(M1)(A1)***

$\sigma = 31.0$ ***A1***

[4 marks]

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} k\sin\left(\frac{\pi x}{6}\right), & 0 \leqslant x \leqslant 6 \\ 0, & \text{otherwise} \end{cases}.$$

3a. Find the value of k. *[4 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

attempt to equate integral to 1 (may appear later) ***M1***

$$k\int_0^6 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x = 1$$

correct integral ***A1***

$$k\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^6 = 1$$

substituting limits ***M1***

$$-\frac{6}{\pi}(-1-1) = \frac{1}{k}$$

$k = \frac{\pi}{12}$ ***A1***

[4 marks]

3b. By considering the graph of *f* write down the mean of X; *[1 mark]*

Markscheme

mean $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

[1 mark]

3c. By considering the graph of *f* write down the median of X; *[1 mark]*

Markscheme

median $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

[1 mark]

3d. By considering the graph of *f* write down the mode of X. *[1 mark]*

Markscheme

mode $= 3$ ***A1***

Note: Award ***A1A0A0*** for three equal answers in $(0,\ 6)$.

[1 mark]

3e. Show that $P(0 \leqslant X \leqslant 2) = \frac{1}{4}$. *[4 marks]*

Markscheme

$\frac{\pi}{12}\int_0^2 \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x$ ***M1***

$= \frac{\pi}{12}\left[-\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^2$ ***A1***

Note: Accept without the $\frac{\pi}{12}$ at this stage if it is added later.

$\frac{\pi}{12}\left[-\frac{6}{\pi}\left(\cos\frac{\pi}{3} - 1\right)\right]$ ***M1***

$= \frac{1}{4}$ ***AG***

[4 marks]

3f. Hence state the interquartile range of X. *[2 marks]*

Markscheme

from (c)(i) $Q_1 = 2$ ***(A1)***

as the graph is symmetrical about the middle value $x = 3 \Rightarrow Q_3 = 4$ ***(A1)***

so interquartile range is

$4 - 2$

$= 2$ ***A1***

[3 marks]

3g. Calculate $P(X \leqslant 4 | X \geqslant 3)$. *[2 marks]*

Markscheme

$P(X \leqslant 4 | X \geqslant 3) = \frac{P(3 \leqslant X \leqslant 4)}{P(X \geqslant 3)}$

$= \frac{\frac{1}{4}}{\frac{1}{2}}$ ***(M1)***

$= \frac{1}{2}$ ***A1***

[2 marks]

The points A and B are given by $\mathrm{A}(0, 3, -6)$ and $\mathrm{B}(6, -5, 11)$.

The plane *Π* is defined by the equation $4x - 3y + 2z = 20$.

4a. Find a vector equation of the line *L* passing through the points A and B. *[3 marks]*

Markscheme

* This question is from an exam for a previous syllabus, and may contain minor differences in marking or structure.

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***(A1)***

r = $\begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ or ***r*** = $\begin{pmatrix} 6 \\ -5 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix}$ ***M1A1***

Note: Award ***M1A0*** if ***r*** = is not seen (or equivalent).

[3 marks]

4b. Find the coordinates of the point of intersection of the line *L* with the plane *Π*. *[3 marks]*

Markscheme

substitute line L in $\Pi: 4(6\lambda) - 3(3 - 8\lambda) + 2(-6 + 17\lambda) = 20$ ***M1***

$82\lambda = 41$

$\lambda = \frac{1}{2}$ ***(A1)***

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 6 \\ -8 \\ 17 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$$

so coordinate is $\left(3,\ -1,\ \frac{5}{2}\right)$ ***A1***

Note: Accept coordinate expressed as position vector $\begin{pmatrix} 3 \\ -1 \\ \frac{5}{2} \end{pmatrix}$.

[3 marks]