HL/ Differential equations [56 marks]

1. [Maximum mark: 7] 24M.2.AHL.TZ1.7

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$, given that $y = 2$ when $x = 0$.

Give your answer in the form $y = f(x)$. [7]

2. [Maximum mark: 9] 23N.2.AHL.TZ1.9

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

(b) By solving the differential equation, show that $y = 4 - 2\mathrm{e}^{-\frac{x}{10}}$. [5]

(c) Find the absolute value of the error in your approximation in part (a). [1]

3. [Maximum mark: 10] 22N.1.AHL.TZ0.9

Consider the homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2-2x^2}{xy}$, where $x,\ y \neq 0$.

It is given that $y = 2$ when $x = 1$.

(a) By using the substitution $y = vx$, solve the differential equation. Give your answer in the form $y^2 = f(x)$. [8]

The points of zero gradient on the curve $y^2 = f(x)$ lie on two straight lines of the form $y = mx$ where $m \in \mathbb{R}$.

(b) Find the values of m. [2]

4. [Maximum mark: 30] 21N.3.AHL.TZ0.2

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - y$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = ax + y$,

where $x,\ y,\ t \in \mathbb{R}^+$ and a is a parameter.

First consider the case where $a = 0$.

(a.i) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y$, show that $y = A\mathrm{e}^t$ where A is a constant. [3]

(a.ii) Show that $\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$. [1]

(a.iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a = -1$.

(b.i) By differentiating $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ with respect to t, show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2\frac{\mathrm{d}y}{\mathrm{d}t}$. [3]

(b.ii) By substituting $Y = \frac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y = B\mathrm{e}^{2t}$ where B is a constant.

[3]

(b.iii) Hence find y as a function of t. [2]

(b.iv) Hence show that $x = -\frac{B}{2}\mathrm{e}^{2t} + C$, where C is a constant. [3]

Now consider the case where $a = -4$.

(c.i) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [3]

From previous cases, we might conjecture that a solution to this differential equation is $y = F\mathrm{e}^{\lambda t}$, $\lambda \in \mathbb{R}$ and F is a constant.

(c.ii) Find the two values for λ that satisfy $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$. [4]

(c.iii) Let the two values found in part (c)(ii) be λ_1 and λ_2.

Verify that $y = F\mathrm{e}^{\lambda_1 t} + G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i),where G is a constant. [4]