20.3 $A = (43, 87)$, $\bar{a} = 45\bar{i} - 28\bar{j}$

$|\bar{a}| = \sqrt{45^2 + (-28)^2} = 53$

$\overline{AB} = 106\,\bar{a}^{\circ} = 106\,\dfrac{\bar{a}}{|\bar{a}|} = 106\,\dfrac{\bar{a}}{53} = 2\bar{a} = 2(45\bar{i} - 28\bar{j}) = 90\bar{i} - 56\bar{j}$

$\Rightarrow B = (43+90,\ 87-56) = \underline{(133, 31)}$

20.4 $\bar{u} = 7\bar{i} - 13\bar{j}$, $\bar{v} = -28\bar{i} - 54\bar{j}$

$\bar{u} \parallel \bar{v} \Leftrightarrow \bar{u} = t\bar{v}$

$\Leftrightarrow 7\bar{i} - 13\bar{j} = t(-28\bar{i} - 54\bar{j}) = -28t\bar{i} - 54t\bar{j}$

$\Rightarrow \begin{cases} 7 = -28t & \Leftrightarrow t = -\frac{1}{4} \\ -13 = -54t & \Leftrightarrow t = \frac{13}{54} \end{cases}$ ↯ $\Rightarrow \underline{\bar{u} \nparallel \bar{v}}$

20.5 $\bar{u} = s\bar{i} - 5\bar{j}$ $\quad \bar{v} = -7\bar{i} + \bar{j}$

$\bar{u} \parallel \bar{v} \Leftrightarrow \bar{u} = t\bar{v}$

$\Leftrightarrow s\bar{i} - 5\bar{j} = t(-7\bar{i} + \bar{j}) = -7t\bar{i} + t\bar{j}$

$\Rightarrow \begin{cases} s = -7t & \Rightarrow s = -7\cdot(-5) = \underline{35} \\ -5 = t \text{ sij.} \end{cases}$

Tällöin $\bar{u} = -5\bar{v}$, $-5 < 0 \Rightarrow \underline{\bar{u} \uparrow\downarrow \bar{v}}$

20.9 a) $\bar{a} = 2\bar{i} + 3\bar{j}$, $\bar{b} = 4\bar{i} + 6\bar{j}$ $\Rightarrow \bar{b} = 2\bar{a}$ $\Rightarrow$ $\underline{1 \text{ ja } 2}$

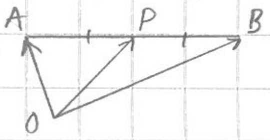
b) $\bar{a} = 10\bar{i} - 16\bar{j}$, $\bar{b} = -5\bar{i} + 8\bar{j}$ $\Rightarrow \bar{a} = -2\bar{b}$ $\Rightarrow$ $\underline{1 \text{ ja } 3}$

c) $\bar{a} = -7\bar{i} + 2\bar{j}$, $\bar{b} = 14\bar{i} - 6\bar{j}$

$\bar{a} \parallel \bar{b} \Leftrightarrow \bar{a} = t\bar{b} \Leftrightarrow -7\bar{i} + 2\bar{j} = 14t\bar{i} - 6t\bar{j}$

$\Rightarrow \begin{cases} -7 = 14t & \Leftrightarrow t = -\frac{1}{2} \\ 2 = -6t & \Leftrightarrow t = -\frac{1}{3} \end{cases}$ ↯ $\Rightarrow \bar{a} \nparallel \bar{b}$ $\Rightarrow \underline{4}$

20.10 $\overline{OA} = -7\bar{i} + 15\bar{j}$, $\overline{OB} = 11\bar{i} - 9\bar{j}$

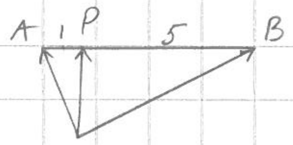
a) 

$\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + \frac{1}{2}\overline{AB}$

$= \overline{OA} + \frac{1}{2}(-\overline{OA} + \overline{OB}) = \frac{1}{2}(\overline{OA} + \overline{OB})$

$= \frac{1}{2}\left((-7\bar{i} + 15\bar{j}) + (11\bar{i} - 9\bar{j})\right)$

$= \frac{1}{2}(4\bar{i} + 6\bar{j}) = \underline{2\bar{i} + 3\bar{j}}$

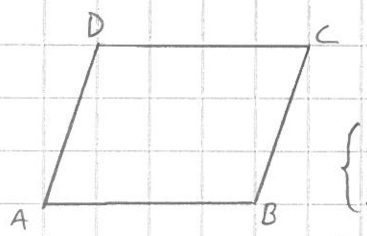
b) 

$\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + \frac{1}{6}\overline{AB}$

$= \overline{OA} + \frac{1}{6}(-\overline{OA} + \overline{OB}) = \frac{5}{6}\overline{OA} + \frac{1}{6}\overline{OB}$

$= \frac{5}{6}(-7\bar{i} + 15\bar{j}) + \frac{1}{6}(11\bar{i} - 9\bar{j})$

$= \underline{-4\bar{i} + 11\bar{j}}$

20.17 

$A = (4,5)$, $B = (12,9)$, $C = (-4,16)$

$D = (-12, 12)$

$\begin{cases} \overline{AB} = 8\bar{i} + 4\bar{j} = \overline{DC} & \Rightarrow \overline{AB} \parallel \overline{DC} \\ \overline{AD} = -16\bar{i} + 7\bar{j} = \overline{BC} & \Rightarrow \overline{AD} \parallel \overline{BC} \end{cases}$

$\Rightarrow$ ABCD on suunnikas

21.1 a) $\bar{a} = 3\bar{i} + 5\bar{j}$, $\bar{b} = 7\bar{i} - 3\bar{j}$

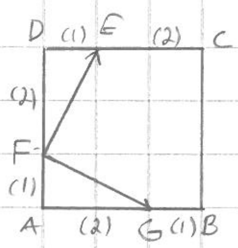
$\bar{a}\cdot\bar{b} = 3\cdot 7 + 5\cdot(-3) = 21 - 15 = \underline{6} \neq 0 \Rightarrow \underline{\bar{a} \not\perp \bar{b}}$

b) $\bar{a} = -2\bar{i} - \bar{j}$, $\bar{b} = 4\bar{i} - 8\bar{j}$

$\bar{a}\cdot\bar{b} = -2\cdot 4 + (-1)\cdot(-8) = -8 + 8 = \underline{0} \Rightarrow \underline{\bar{a} \perp \bar{b}}$

21.3 $\bar{a} = 2\bar{i} - 5\bar{j}$, $\bar{b} = t\bar{i} + 6\bar{j}$

$\bar{a} \perp \bar{b} \Leftrightarrow \bar{a}\cdot\bar{b} = 2\cdot t + (-5)\cdot 6 = 2t - 30 = 0 \Leftrightarrow \underline{t = 15}$

21.5 

$\overline{AB} = \bar{a}$, $\overline{AD} = \bar{b}$

$\begin{cases} \overline{FE} = \frac{2}{3}\bar{b} + \frac{1}{3}\bar{a} \\ \overline{FG} = -\frac{1}{3}\bar{b} + \frac{2}{3}\bar{a} \end{cases}$

$\Rightarrow \overline{FE}\cdot\overline{FG} = (\frac{2}{3}\bar{b} + \frac{1}{3}\bar{a})\cdot(-\frac{1}{3}\bar{b} + \frac{2}{3}\bar{a})$

$= -\frac{2}{9}\underbrace{\bar{b}\cdot\bar{b}}_{|\bar{b}|^2} + \frac{4}{9}\bar{b}\cdot\bar{a} - \frac{1}{9}\bar{a}\cdot\bar{b} + \frac{2}{9}\underbrace{\bar{a}\cdot\bar{a}}_{|\bar{a}|^2}$

$= -\frac{2}{9}\underbrace{|\bar{b}|^2}_{|\bar{a}|^2} + \frac{2}{9}|\bar{a}|^2 = -\frac{2}{9}|\bar{a}|^2 + \frac{2}{9}|\bar{a}|^2 = 0$

$\Rightarrow \underline{\overline{FE} \perp \overline{FG}}$

21.6 $A = (1,5)$, $B = (4,12)$, $C = (-1,10)$

b) $\overline{AB} = 3\bar{i} + 7\bar{j}$, $\overline{AC} = -2\bar{i} + 5\bar{j}$, $\overline{BC} = -5\bar{i} - 2\bar{j}$

$\overline{AC}\cdot\overline{BC} = -2\cdot(-5) + 5\cdot(-2) = 10 - 10 = 0 \Rightarrow \overline{AC} \perp \overline{BC}$

$\Rightarrow$ kolmio on suorakulmainen

21.9 $\bar{a} = 5\bar{i} + \bar{j}$, $\bar{b} = -2\bar{i} - 4\bar{j}$

a) $\bar{a}\cdot\bar{b} = 5\cdot(-2) + 1\cdot(-4) = -10 - 4 = \underline{-14}$

b) $\bar{a} + \bar{b} = 3\bar{i} - 3\bar{j} \Rightarrow \bar{a}\cdot(\bar{a} + \bar{b}) = 5\cdot 3 + 1\cdot(-3) = 15 - 3 = \underline{12}$

c) $\bar{a} - \bar{b} = 7\bar{i} + 5\bar{j} \Rightarrow (\bar{a} + \bar{b})\cdot(\bar{a} - \bar{b}) = 3\cdot 7 + (-3)\cdot 5 = 21 - 15 = \underline{6}$

21.10 $|\bar{a}| = 3$, $|\bar{b}| = 7$, $\bar{a}\cdot\bar{b} = 13$

$|\bar{a} + \bar{b}|^2 = (\bar{a} + \bar{b})\cdot(\bar{a} + \bar{b}) = \bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b}$

$= |\bar{a}|^2 + 2\bar{a}\cdot\bar{b} + |\bar{b}|^2$

$= 3^2 + 2\cdot 13 + 7^2 = 84$

$\Rightarrow |\bar{a} + \bar{b}| = \sqrt{84} = \sqrt{4\cdot 21} = \underline{2\sqrt{21}}$

21.13 $\bar{a} = -3t\bar{i} - 3\bar{j}$, $\bar{b} = 4\bar{i} + (t-1)\bar{j}$

$\bar{a} \perp \bar{b} \Leftrightarrow \bar{a}\cdot\bar{b} = -3t\cdot 4 + (-3)\cdot(t-1) = -12t - 3t + 3 = 0$

$\Leftrightarrow -15t = -3 \Leftrightarrow \underline{t = \frac{1}{5}}$

22.1 a) $\bar{a} = 2\bar{i} + \bar{j}$, $\bar{b} = \bar{i} + 3\bar{j}$

$|\bar{a}| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $|\bar{b}| = \sqrt{1^2 + 3^2} = \sqrt{10}$

$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos(\bar{a}, \bar{b})$

$\Leftrightarrow \cos(\bar{a}, \bar{b}) = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{2\cdot 1 + 1\cdot 3}{\sqrt{5}\sqrt{10}} = \dfrac{5}{\sqrt{50}} = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \sphericalangle(\bar{a}, \bar{b}) = \underline{45{,}0^{\circ}}$

b) $\bar{a} = -3\bar{i} + 5\bar{j}$, $\bar{b} = 2\bar{i} - 6\bar{j}$

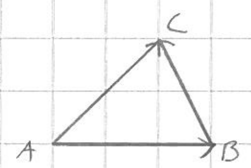
$|\bar{a}| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$, $|\bar{b}| = \sqrt{2^2 + (-6)^2} = \sqrt{40}$

$\cos(\bar{a}, \bar{b}) = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{-3\cdot 2 + 5\cdot(-6)}{\sqrt{34}\sqrt{40}} = \dfrac{-36}{\sqrt{34}\sqrt{40}}$

$\Rightarrow \sphericalangle(\bar{a}, \bar{b}) \approx \underline{167{,}5^{\circ}}$

22.2 $|\bar{a}| = \sqrt{14}$, $|\bar{b}| = \sqrt{14}$, $\bar{a}\cdot\bar{b} = -7$

$\cos(\bar{a}, \bar{b}) = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{-7}{\sqrt{14}\sqrt{14}} = \dfrac{-7}{14} = -\dfrac{1}{2} \Rightarrow \sphericalangle(\bar{a}, \bar{b}) = \underline{120^{\circ}}$

22.3 

$A = (-1, 2)$, $B = (3, -1)$, $C = (7, 3)$

$\overline{AB} = 4\bar{i} - 3\bar{j}$, $|\overline{AB}| = \sqrt{4^2 + (-3)^2} = 5$

$\overline{AC} = 8\bar{i} + \bar{j}$, $|\overline{AC}| = \sqrt{8^2 + 1^2} = \sqrt{65}$

$\overline{BC} = 4\bar{i} + 4\bar{j}$, $|\overline{BC}| = \sqrt{4^2 + 4^2} = \sqrt{32}$

$\cos(\overline{AB}, \overline{AC}) = \dfrac{\overline{AB}\cdot\overline{AC}}{|\overline{AB}||\overline{AC}|} = \dfrac{4\cdot 8 + (-3)\cdot 1}{5\sqrt{37}} = \dfrac{29}{5\sqrt{37}} \Rightarrow \sphericalangle A = \sphericalangle(\overline{AB}, \overline{AC})$

$\approx 43{,}99^{\circ} \approx \underline{44^{\circ}}$

$\cos(\overline{BA}, \overline{BC}) = \dfrac{-\overline{AB}\cdot\overline{BC}}{|-\overline{AB}||\overline{BC}|} = \dfrac{-4\cdot 4 + 3\cdot 4}{5\sqrt{32}} = \dfrac{-4}{5\sqrt{32}} \Rightarrow \sphericalangle B = \sphericalangle(\overline{BA}, \overline{BC})$

$\approx 98{,}13^{\circ} \approx \underline{98^{\circ}}$

$\cos(\overline{CA}, \overline{CB}) = \dfrac{-\overline{AC}\cdot(-\overline{BC})}{|-\overline{AC}||-\overline{BC}|} = \dfrac{-8\cdot(-4) + (-1)\cdot(-4)}{\sqrt{65}\sqrt{32}} = \dfrac{36}{\sqrt{65}\sqrt{32}}$

$\Rightarrow \sphericalangle C = \sphericalangle(\overline{CA}, \overline{CB}) \approx 37{,}87^{\circ} \approx \underline{38^{\circ}}$

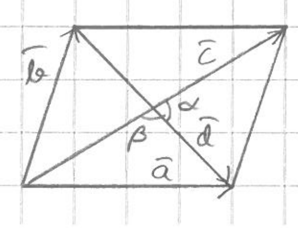
(Tark. $44^{\circ} + 98^{\circ} + 38^{\circ} = 180^{\circ}$ ✓.)

22.5 b) $\bar{a} = 4\bar{j}$, $\bar{b} = t\bar{i} + \bar{j}$

$\cos(\bar{a}, \bar{b}) = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \dfrac{0\cdot t + 4\cdot 1}{4\sqrt{t^2 + 1^2}} = \dfrac{4}{4\sqrt{t^2+1}} = \dfrac{1}{\sqrt{t^2+1}} = \cos 60^{\circ} = \dfrac{1}{2}$

$\Leftrightarrow \sqrt{t^2+1} = 2$ $\quad |(\ )^2$ mol. puol. ≥ 0

$\Leftrightarrow t^2 + 1 = 4 \Leftrightarrow t^2 = 3 \quad |\sqrt{\ } \quad \Leftrightarrow \underline{t = \pm\sqrt{3}}$

22.18 

$\bar{a} = -3\bar{i} + \bar{j}$, $\bar{b} = 2\bar{i} - 4\bar{j}$

$\bar{c} = \bar{a} + \bar{b} = -\bar{i} - 3\bar{j} \Rightarrow |\bar{c}| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$\bar{d} = -\bar{b} + \bar{a} = -5\bar{i} + 5\bar{j} \Rightarrow |\bar{d}| = \sqrt{(-5)^2 + 5^2} = \sqrt{50}$