Paper 3 / Probability [50 marks]

1. [Maximum mark: 23] 25M.3.AHL.TZ1.1

This question asks you to use polynomial functions to model some situations in probability.

Two unbiased tetrahedral (four-sided) dice with faces labelled $1,\ 2,\ 3$ and 4 are thrown and the scores recorded.

The random variable M denotes the maximum of these two scores.

The probability distribution of M is given in the following table.

m	1	2	3	4
$\mathrm{P}(M=m)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

(a) Find $\mathrm{E}(M)$. [2]

An alternative way to represent the probability distribution of M is to use a polynomial function, G, where $G(t) = \sum_{m=1}^{4} \mathrm{P}(M=m)\,t^m$.

Hence, for the distribution of M, $G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$.

(b) Find $G(1)$. [1]

(c.i) Find $G'(t)$. [2]

(c.ii) Hence, show that $G'(1) = \mathrm{E}(M)$. [3]

A bag contains two red balls and three yellow balls.

Two balls are selected at random without replacement from the bag.

The random variable X denotes the total number of red balls selected.

The probability distribution of X can be represented by the polynomial function, G_X, where

$$G_x(t) = \sum_{x=0}^{2} \mathrm{P}(X = x)t^x.$$

(d) Show that $G_X(t) = \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$, making it clear how the coefficients of $G_X(t)$ have been determined. [5]

An unbiased coin and a biased coin are tossed.

The probability of obtaining a tail on the biased coin is p.

The random variable Y denotes the total number of tails obtained from tossing both coins.

The probability distribution of Y can be represented by the polynomial function, G_Y, where

$$G_Y(t) = \sum_{y=0}^{2} \mathrm{P}(Y = y)t^y.$$

(e) Given that the coefficient of t^2 in $G_Y(t)$ is $\frac{1}{3}$, find

(e.i) the value of p; [2]

(e.ii) an expression for $G_Y(t)$. [4]

The random variable Z denotes the sum of the total number of red balls selected, X, and the total number of tails obtained from tossing both coins, Y.

The probability distribution of Z can be represented by the function, G_Z, where

$$G_Z(t) = G_X(t)G_Y(t).$$

(f) For random variable Z, it can be shown that $G_Z\prime(1) = \mathrm{E}(Z)$.

Use this result to find $\mathrm{E}(Z)$. [4]

2. [Maximum mark: 27] 24M.3.AHL.TZ1.1

This question considers two possible models for the occurrence of random events in a computer game.

In a new computer game, each time a player performs an action, there is a random chance that the action will be *boosted*, meaning that it provides a benefit to the player.

The designer of this computer game is considering two possible models for when to boost an action.

In the first model, the probability that an action will be boosted is constant.

(a) Suppose the probability that an action will be boosted is 0.1.

(a.i) Find the probability that the first boost occurs on the third action. [2]

(a.ii) Find the probability that at least one boost occurs in the first six actions. [3]

(b) Suppose the probability that an action will be boosted is p, where $0 < p < 1$.

(b.i) Explain why the probability that the first boost occurs on the x^{th} action is $p(1-p)^{x-1}$. [1]

Let X be the number of actions until the first boost occurs.

(b.ii) Hence, write down an expression, using sigma notation, for $\mathrm{E}(X)$ in terms of x and p. [1]

Consider the sum of an infinite geometric sequence, with first term a and common ratio $r(|r| < 1)$,

$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$.

(c.i) By differentiating both sides of the above equation with respect to r, find an expression for $\sum_{n=1}^{\infty} nar^{n-1}$ in terms of a and r. [4]

(c.ii) Hence, show that $\mathrm{E}(X) = \frac{1}{p}$. [2]

It can be shown that $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ when $p = 0.1$. [2]

In the designer's second model, the initial probability that an action is boosted is 0.2, and each time an action occurs that is not boosted, the probability that the next action is boosted increases by 0.2. After an action has been boosted, the probability resets to 0.2 for the next action.

(e) Show that the probability that the first boost occurs on the third action is 0.288. [2]

Let Y be the number of actions until the first boost occurs.

(f) Explain why $Y \leq 5$. [1]

The following table shows the probability distribution of Y.

y	1	2	3	4	5
$\mathrm{P}(Y = y)$	0.2	m	0.288	n	0.0384

(g.i) Find the value of m and the value of n. [2]

(g.ii) Show that $\mathrm{E}(Y) = 2.5104$. [2]

(g.iii) Find $\mathrm{Var}(Y)$. [2]

(h.i) Use the expression given in (c)(ii) to find the value of p for which $\mathrm{E}(X) = \mathrm{E}(Y)$. [1]

(h.ii) Find $\mathrm{Var}(X)$ for this value of p. [1]

(h.iii) Hence determine, with a reason, which model provides a more consistent experience for the player with respect to boosted actions. [1]

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