5 Measuring change: differentiation

Skills check

1 **a** $\dfrac{-3-0}{-4-0}=\dfrac{3}{4}$

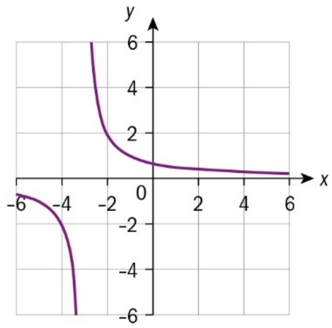
b $\dfrac{-1-2}{4-\left(-\frac{3}{4}\right)}=-\dfrac{12}{19}$

2 **a** $7\sqrt{x}=7x^{\frac{1}{2}}$

b $\dfrac{1}{x^2}=x^{-2}$

c $\dfrac{8}{5\sqrt{x^3}}=\dfrac{8}{5}x^{-\frac{3}{2}}$

3



4 Since $\left|\frac{1}{2}\right|<1$,

$$\sum_{n=0}^{\infty}5\left(\frac{1}{2}\right)^n=5\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{5(1)}{1-\frac{1}{2}}=10$$

Exercise 5A

1 $\lim\limits_{x\to3^-}\left(x^2+1\right)=\lim\limits_{x\to3^+}\left(x^2+1\right)=10$

2 $\lim\limits_{x\to1^-}(5-2x)=\lim\limits_{x\to1^+}(5-2x)=3$

3 $\lim\limits_{x\to0^-}\left(\dfrac{2x^2-x}{x}\right)=\lim\limits_{x\to0^+}\left(\dfrac{2x^2-x}{x}\right)=-1$

4 $\lim\limits_{x\to1^-}\left(\dfrac{x^2-x}{x-1}\right)=\lim\limits_{x\to1^+}\left(\dfrac{x^2-x}{x-1}\right)=1$

Exercise 5B

1 Vertical asymptote at $x=\dfrac{1}{6}$

since $\lim\limits_{x\to\frac{1}{6}^-}f(x)=-\infty$ and $\lim\limits_{x\to\frac{1}{6}^+}f(x)=\infty$

Horizontal asymptote at $y=\dfrac{1}{2}$

since $\lim\limits_{x\to\pm\infty}f(x)=\dfrac{1}{2}$

2 Vertical asymptotes at $x=\pm\sqrt{3}$

since $\lim\limits_{x\to-\sqrt{3}^-}=-\infty$ and $\lim\limits_{x\to-\sqrt{3}^+}=\infty$

and $\lim\limits_{x\to\sqrt{3}^-}=\infty$ and $\lim\limits_{x\to\sqrt{3}^+}=-\infty$

Horizontal asymptote at $y=-1$
since $\lim\limits_{x\to\pm\infty}g(x)=-1$

3 Vertical asymptote at $x=1$

since $\lim\limits_{x\to1^-}f(x)=\infty$ and $\lim\limits_{x\to1^+}f(x)=-\infty$

Horizontal asymptote at $y=-1$
since $\lim\limits_{x\to\pm\infty}h(x)=-1$

4 Vertical asymptotes at $x=\pm\sqrt{2}$

since $\lim\limits_{x\to-\sqrt{2}^-}=\infty$ and $\lim\limits_{x\to-\sqrt{2}^+}=-\infty$

and $\lim\limits_{x\to\sqrt{2}^-}=\infty$ and $\lim\limits_{x\to\sqrt{2}^+}=-\infty$

Horizontal asymptote at $y=0$

since $\lim\limits_{x\to\pm\infty}\left(-\dfrac{5x}{x^2-2}\right)=0$

Exercise 5C

1 $f'(x)=7x^{7-1}=7x^6$

2 $f'(x)=18x^{18-1}=18x^{17}$

3 $f'(x)=-\dfrac{1}{2}x^{-\frac{1}{2}-1}=-\dfrac{1}{2}x^{-\frac{3}{2}}$

4 $f(x)=\sqrt[5]{x}=x^{\frac{1}{5}}\Rightarrow f'(x)=\dfrac{1}{5}x^{\frac{1}{5}-1}=\dfrac{1}{5}x^{-\frac{4}{5}}$

5 $f(x)=\dfrac{1}{\sqrt{x}}=x^{-\frac{1}{2}}\Rightarrow f'(x)=-\dfrac{1}{2}x^{-\frac{1}{2}-1}=-\dfrac{1}{2}x^{-\frac{3}{2}}$

6 $f(x)=\sqrt[4]{x^3}=x^{\frac{3}{4}}\Rightarrow f'(x)=\dfrac{3}{4}x^{\frac{3}{4}-1}=\dfrac{3}{4}x^{-\frac{1}{4}}$

Exercise 5D

1 **a** $\dfrac{dy}{dx}=4x^3-x$

b $f(x)=5x(x^2-1)=5x^3-5x$

$\therefore f'(x)=15x^2-5$

c $f'(x)=24x^3-6x$ **d** $\dfrac{ds}{dt}=4t+3$

e $\dfrac{dv}{dt}=-9.8$ **f** $\dfrac{dc}{dx}=24$

2 a $f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}} \quad \therefore f'(x) = 3x^{-\frac{1}{2}}$

b $f(x) = 5\sqrt[5]{x^3} = 5x^{\frac{3}{5}} \quad \therefore f'(x) = 3x^{-\frac{2}{5}}$

c $f(x) = \frac{2}{x} - 3\sqrt{x} = 2x^{-1} - 3x^{\frac{1}{2}}$

$\therefore f'(x) = -2x^{-2} - \frac{3}{2}x^{-\frac{1}{2}}$

3 a $f(x) = \frac{3}{2x^2} = \frac{3}{2}x^{-2} \quad \therefore f'(x) = -3x^{-3}$

b $f(x) = \frac{3}{(2x)^2} = \frac{3}{4x^2} = \frac{3}{4}x^{-2}$

$\therefore f'(x) = -\frac{3}{2}x^{-3}$

c $f'(x) = 12\pi x^2$

d $f(x) = (x+1)^2 = x^2 + 2x + 1$

$\therefore f'(x) = 2x + 2$

e $f(x) = \frac{x^3 + x - 3}{x} = x^2 + 1 - \frac{3}{x}$

$= x^2 + 1 - 3x^{-1} \therefore f'(x) = 2x + 3x^{-2}$

f $f(x) = (2x-1)(x^2+3)$

$= 2x^3 - x^2 + 6x - 3$

$\therefore f'(x) = 6x^2 - 2x + 6$

4 a $y = 1 + x\sqrt{x} = 1 + x^{\frac{3}{2}} \quad \therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

b $y = \frac{7}{x^2} - \frac{1}{\sqrt{x}} = 7x^{-2} - x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -14x^{-3} + \frac{1}{2}x^{-\frac{3}{2}}$

c $y = \sqrt[3]{x} + \sqrt[4]{x} = x^{\frac{1}{3}} + x^{\frac{1}{4}}$

$\therefore \frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-\frac{3}{4}}$

Exercise 5E

1 a $\frac{dy}{dx} = 2x - 4$ so the gradient at

$x = -1$ is $2(-1) - 4 = -6$

b $y = \frac{2x^5 - 5}{x} = 2x^4 - \frac{5}{x} = 2x^4 - 5x^{-1}$

$\therefore \frac{dy}{dx} = 8x^3 + 5x^{-2}$

so the gradient at $(1, -3)$ is

$8(1)^3 + 5(1)^{-2} = 13$

c $f(x) = \sqrt[4]{x} + \frac{8}{\sqrt{x}} = x^{\frac{1}{4}} + 8x^{-\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{4}x^{-\frac{3}{4}} - 4x^{-\frac{3}{2}}$

so $f'(1) = \frac{1}{4}(1)^{-\frac{3}{4}} - 4(1)^{-\frac{3}{2}} = -\frac{15}{4}$

2 $f'(x) = 2x^2 - 9x - 3 \quad \therefore 2 = 2x^2 - 9x - 3$

$\Rightarrow 2x^2 - 9x - 5 = (2x+1)(x-5) = 0$

so $x = -\frac{1}{2}$ or $x = 5$

when $x = -\frac{1}{2}$, $y = f\left(-\frac{1}{2}\right) = \frac{199}{24}$

when $x = 5$, $y = f(5) = -\frac{217}{6}$

so $\left(-\frac{1}{2}, \frac{199}{24}\right)$ and $\left(5, -\frac{217}{6}\right)$

Exercise 5F

1 $y = \frac{1-2x}{x^2} = \frac{1}{x^2} - \frac{2}{x} = x^{-2} - 2x^{-1}$

$\therefore \frac{dy}{dx} = -2x^{-3} + 2x^{-2}$

Therefore, the gradient at $\left(2, -\frac{3}{4}\right)$

is $-2(2)^{-3} + 2(2)^{-2} = \frac{1}{4}$

So the gradient of the normal at this point is -4

$\therefore y - \left(-\frac{3}{4}\right) = -4(x-2) \Rightarrow y = -4x + \frac{29}{4}$

2 $\frac{dy}{dx} = -3x^2 + 4x$

So the gradient at $x = -1$ is -7

$\frac{dy}{dx} = -7 \Rightarrow 3x^2 - 4x - 7$

$= (3x-7)(x+1) = 0$

$\therefore x = \frac{7}{3}$ or $x = -1$ (i.e. the tangent itself)

$y\left(\frac{7}{3}\right) = -\frac{22}{27}$

$\therefore y + \frac{22}{27} = -7\left(x - \frac{7}{3}\right) \Rightarrow y = -7x + \frac{419}{27}$

3 $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$

$\dfrac{dy}{dx} = -3 \Rightarrow 1 - \dfrac{1}{x^2} = -3 \Rightarrow x = \pm\dfrac{1}{2}$

$y\left(\pm\dfrac{1}{2}\right) = \pm\dfrac{5}{2}$

Gradient of normal is $\dfrac{1}{3}$

$\therefore y - \left(\pm\dfrac{5}{2}\right) = \dfrac{1}{3}\left(x - \left(\pm\dfrac{1}{2}\right)\right)$

$\Rightarrow y = \dfrac{1}{3}x \pm \dfrac{5}{2} \mp \dfrac{1}{6}$

$\therefore y = \dfrac{1}{3}x + \dfrac{7}{3}$ and $y = \dfrac{1}{3}x - \dfrac{7}{3}$

4 $f'(x) = 6x - 2k$

$f'(1) = 6 - 2k = 10 \Rightarrow k = -2$

5 $f'(x) = 3x^2 - 2x - 2 = 0$

$\Rightarrow x^2 - \dfrac{2}{3}x - \dfrac{2}{3} = \left(x - \dfrac{1}{3}\right)^2 = \dfrac{7}{9}$

$\therefore x = \dfrac{1 \pm \sqrt{7}}{3}$

Coordinates are

$\left(\dfrac{1+\sqrt{7}}{3}, \dfrac{7 - 14\sqrt{7}}{27}\right), \left(\dfrac{1-\sqrt{7}}{3}, \dfrac{7 + 14\sqrt{7}}{27}\right)$

6 $g(x) = \dfrac{1}{x^n} = x^{-n} \quad \therefore g'(x) = -nx^{-n-1}$

$\Rightarrow xg'(x) + ng(x) = x(-nx^{-n-1}) + nx^{-n}$

$= -nx^{-n} + nx^{-n} = 0$

7 a $f'(x) = 15ax^2 - 4bx + 4c$

b $f'(x) \geq 0 \Rightarrow 15ax^2 - 4bx + 4c \geq 0$

$\Rightarrow x^2 - \dfrac{4b}{15a}x + \dfrac{4c}{15a} \geq 0$

$\Rightarrow \left(x - \dfrac{2b}{15a}\right)^2 - \dfrac{4b^2}{225a^2} + \dfrac{4c}{15a} \geq 0$

The LHS is valid for all real x and attains its minimum at $x = \dfrac{2b}{15a}$ so

$-\dfrac{4b^2}{225a^2} + \dfrac{4c}{15a} \geq 0 \Rightarrow b^2 \leq 15ac$

8 $f(x) = -20x^{-1} + 1 \qquad$ for $x > 0$

$\therefore f'(x) = 20x^{-2}, g'(x) = 5$ for $x \in \mathbb{R}$

$f'(x) = g'(x)$ when $20x^{-2} = 5$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

But $x > 0$ so $x = 2$ only

Exercise 5G

1 a $y = u^5$ where $u = 2x + 3$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (5u^4)(2) = 10u^4$

$= 10(2x+3)^4$

b $y = \sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$

$y = u^{\frac{1}{2}}$ where $u = 1 - 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(\dfrac{1}{2}u^{-\frac{1}{2}}\right)(-2)$

$= -u^{-\frac{1}{2}} = -(1-2x)^{-\frac{1}{2}}$

c $y = -\dfrac{3}{\sqrt{2x^2 - 1}} = -3u^{-\frac{1}{2}}$

where $u = 2x^2 - 1$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(\dfrac{3}{2}u^{-\frac{3}{2}}\right)(4x)$

$= 6x(2x^2 - 1)^{-\frac{3}{2}}$

d $y = 2\left(x^2 - \dfrac{2}{x}\right)^3 = 2u^3$

where $u = x^2 - \dfrac{2}{x}$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (6u^2)\left(2x + \dfrac{2}{x^2}\right)$

$= 12\left(x^2 - \dfrac{2}{x}\right)^2\left(x + \dfrac{1}{x^2}\right)$

2 At $x = 0$, $y = 6$ so tangent passes through $(0,6)$

$y = 6(1-2x)^{\frac{1}{3}} = 6u^{\frac{1}{3}}$ where $u = 1 - 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \left(2u^{-\frac{2}{3}}\right)(-2) = -4u^{-\frac{2}{3}}$

$= -4(1-2x)^{-\frac{2}{3}}$

so the gradient at $x = 0$ is -4

$\therefore y - 6 = -4(x - 0) \Rightarrow y = -4x + 6$

3 When $x = 1$, $y = 1$ so $a = \sqrt{1+b}$

$y = a(1+bx)^{-\frac{1}{2}} = au^{-\frac{1}{2}}$ where $u = 1 + bx$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{a}{2}u^{-\frac{3}{2}}\right)(b) = -\frac{ab}{2}u^{-\frac{3}{2}}$

$= -\frac{ab}{2}(1+bx)^{-\frac{3}{2}}$

At $(1,1)$, $\frac{dy}{dx} = -\frac{3}{8}$

$\Rightarrow -\frac{ab}{2}(1+b)^{-\frac{3}{2}} = -\frac{b}{2a^2} = -\frac{3}{8}$

so $b = \frac{3a^2}{4}$

$\Rightarrow a = \sqrt{1 + \frac{3a^2}{4}} \Rightarrow a^2 = 1 + \frac{3a^2}{4}$

$\Rightarrow a = 2 \quad (a > 0) \therefore b = a^2 - 1 = 3$

4 $y = \frac{4}{(3-x)^3} = 4u^{-3}$ where $u = 3 - x$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-12u^{-4})(-1) = 12u^{-4}$

$= 12(3-x)^{-4}$

so at $x = 1$, $\frac{dy}{dx} = \frac{3}{4}$ and therefore

the normal has gradient $-\frac{4}{3}$

$\therefore y - \frac{1}{2} = -\frac{4}{3}(x-1) \Rightarrow y = -\frac{4}{3}x + \frac{11}{6}$

5 $\frac{dy}{dx} = -9x^2 + 2$

Curve horizontal when $\frac{dy}{dx} = 0$

So $x = \pm\sqrt{\frac{2}{9}} = \pm\frac{\sqrt{2}}{3}$

Exercise 5H

1 a $y = x^2(2x-1) = uv$ where $u = x^2$

and $v = 2x - 1$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2x)(2x-1) + x^2(2)$

$= 6x^2 - 2x = 2x(3x-1)$

b $y = (2x-3)(x+3)^3 = uv$ where

$u = 2x - 3$ and $v = (x+3)^3$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2)(x+3)^3 + (2x-3)\left(3(x+3)^2\right)$

$= (x+3)^2(8x-3)$

c $y = x\sqrt{2-3x} = uv$ where $u = x$

and $v = \sqrt{2-3x}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (1)\sqrt{2-3x} + x\left(-\frac{3}{2\sqrt{2-3x}}\right)$

$= \frac{4-9x}{2\sqrt{2-3x}}$

d $y = (2x+1)(x^2-x+1)^2 = uv$ where

$u = 2x + 1$ and $v = (x^2-x+1)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2)(x^2-x+1)^2$

$+ 2(2x+1)(2x-1)(x^2-x+1)$

$= 2x(5x-1)(x^2-x+1)$

e $y = (2-3x)\sqrt{x+2} = uv$

where $u = 2 - 3x$ and $v = \sqrt{x+2}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (-3)\sqrt{x+2} + (2-3x)\left(\frac{1}{2\sqrt{x+2}}\right)$

$= \frac{-10-9x}{2\sqrt{x+2}}$

2 a $y = \sqrt{x+1}(3-x)^2 = uv$ where

$u = \sqrt{x+1}$ and $v = (3-x)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \left(\frac{1}{2\sqrt{x+1}}\right)(3-x)^2 + \sqrt{x+1}\left(-2(3-x)\right)$

$\therefore \frac{dy}{dx} = \frac{(x-3)^2 + 4(x-3)(x+1)}{2\sqrt{x+1}}$

$= \frac{(x-3)(x-3+4(x+1))}{2\sqrt{x+1}}$

$= \frac{(x-3)(5x+1)}{2\sqrt{x+1}}$

b Using the result from part a,

$x - 3 = 0 \Rightarrow x = 3$

or $5x + 1 = 0 \Rightarrow x = -\frac{1}{5}$

3 $y = x(1-2x)^{-1} = uv$ where $u = x$

and $v = (1-2x)^{-1}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (1)(1-2x)^{-1} + x\left(2(1-2x)^{-2}\right) = \frac{1}{(1-2x)^2}$

so the gradient at (0,0) is 1 and the normal therefore has gradient -1

$\therefore y = -x$

Exercise 5I

1 a $y = \frac{1+3x}{5-x} = \frac{u}{v}$ where $u = 1+3x$

and $v = 5-x$, $u' = 3, v' = -1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(5-x)(3) - (1+3x)(-1)}{(5-x)^2} = \frac{16}{(5-x)^2}$

b $y = \frac{\sqrt{x}}{2-x} = \frac{u}{v}$ where $u = \sqrt{x}$

and $v = 2-x$

$u' = \frac{1}{2\sqrt{x}}, v' = -1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(2-x)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})(-1)}{(2-x)^2}$

$= \frac{2+x}{2(2-x)^2\sqrt{x}}$

c $y = \frac{1+2x}{\sqrt{1-x^2}} = \frac{u}{v}$ where $u = 1+2x$

and $v = \sqrt{1-x^2}$

$u' = 2, v' = -\frac{x}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{\sqrt{1-x^2}(2) - (1+2x)\left(-\frac{x}{\sqrt{1-x^2}}\right)}{1-x^2}$

$= \frac{x+2}{(1-x^2)^{\frac{3}{2}}}$

d $y = \frac{1+3x}{x^2+1} = \frac{u}{v}$ where $u = 1+3x$

and $v = x^2+1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x^2+1)(3) - (1+3x)(2x)}{(x^2+1)^2}$

$= \frac{3-2x-3x^2}{(x^2+1)^2}$

2 $f'(x) = \frac{-3x^2+4x+3}{(x+1)^2}$, $f'(0) = 3$

so normal at this point has gradient $-\frac{1}{3}$ and passes through $(0,-2)$

$\therefore y = -\frac{1}{3}x - 2$

3 $f(x) = \frac{x^3+x^2+x+1}{x} = \frac{u}{v}$

where $u = x^3+x^2+x+1$ and $v = x$

$f'(x) = \frac{vu' - uv'}{v^2}$

$= \frac{x(3x^2+2x+1) - (x^3+x^2+x+1)(1)}{x^2}$

$= \frac{2x^3+x^2-1}{x^2}$

$f'(x) = 1 \Rightarrow 2x^3+x^2-1 = x^2 \Rightarrow x^3 = \frac{1}{2}$

$\therefore x = \frac{1}{\sqrt[3]{2}}$

Exercise 5J

1 a

$y = (x-1)(x+3)^2 = uv$ where

$u = x-1$ and $v = (x+3)^2$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (x+3)^2 + 2(x-1)(x+3)$

$= (x+3)(x+3+2(x-1))$

$= (x+3)(3x+1)$

b Most easily done using the product (and chain) rule:

$y = (x+1)\sqrt{1-2x} = uv$

where $u = x+1$ and $v = \sqrt{1-2x}$

$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \sqrt{1-2x} - \frac{x+1}{\sqrt{1-2x}} = -\frac{3x}{\sqrt{1-2x}}$

c Most easily done using the quotient rule:

$y = \frac{x+1}{x-1} = \frac{u}{v}$ where $u = x+1$

and $v = x-1$

$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

$= \frac{(x-1)-(x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$

d Most easily done by chain rule (quotient rule also valid)

$y = 2\left(x^4 - 2x + 1\right)^{-1} = 2u^{-1}$

where $u = x^4 - 2x + 1$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(-\frac{2}{u^2}\right)\left(4x^3 - 2\right)$

$= \frac{4\left(1-2x^3\right)}{\left(x^4 - 2x + 1\right)^2}$

2 $f(x) = \frac{1+\sqrt{x}}{x-1} = \frac{u}{v}$ where $u = 1 + \sqrt{x}$

and $v = x - 1$

$f'(x) = \frac{vu' - uv'}{v^2} = \frac{\frac{x-1}{2\sqrt{x}} - \left(1+\sqrt{x}\right)}{(x-1)^2}$

$= -\frac{x + 2\sqrt{x} + 1}{2\sqrt{x}\,(x-1)^2} = -\frac{\left(\sqrt{x}+1\right)^2}{2\sqrt{x}\,(x-1)^2}$

$f'(9) = -\frac{(3+1)^2}{2(3)(9-1)^2} = -\frac{1}{24}$

Tangent: $y - \frac{1}{2} = -\frac{1}{24}(x-9)$

$\Rightarrow y = \frac{7}{8} - \frac{x}{24}$

Normal: $y - \frac{1}{2} = 24(x-9)$

$\Rightarrow y = 24x - \frac{431}{2}$

Exercise 5K

1 **a** **i** $x > 0$

ii Nowhere

b **i** $x \in (-\infty, -1) \cup (-1, 0)$ □

ii $x \in (0,1) \cup (1, \infty)$

c **i** $x \in (-\infty, -0.215) \cup (1.55, \infty)$

ii $x \in (-0.215, 1.55)$

d **i** $x \in (-\infty, -1) \cup (1, \infty)$

ii $x \in (-1, 1)$

2 **a** $f'(x) = -3x^2$

Increasing: nowhere

Decreasing: $\forall x \in$ ¡

b $f'(x) = 4x$

Increasing: $x > 0$

Decreasing: $x < 0$

c $f'(x) = -\frac{1}{2\sqrt{x-1}}$

Increasing: nowhere

(note the function is only valid here for $x > 1$)

Decreasing: $x \in (1, \infty)$

d $f'(x) = \frac{1}{2\sqrt{x}} - 2$

Increasing: $x \in \left(0, \frac{1}{16}\right)$

(note the function is only valid here for $x > 0$)

Decreasing: $x \in \left(\frac{1}{16}, \infty\right)$

Exercise 5L

1 **a** $f'(x) = 2x$, $f'(x) = 0 \Rightarrow x = 0$

$f(x)$ decreasing for $x < 0$ and increasing for $x > 0$ so this is a local minimum

$\therefore (0, -2)$ is a local minimum point

Graphically, this is a positive parabola so the turning point must be a local minimum (students should draw this)

b $f'(x) = 1 - \frac{1}{\sqrt{x}}$, $f'(x) = 0 \Rightarrow x = 1$

$f(x)$ decreasing for $0 < x < 1$ and increasing for $x > 1$ so this is a local minimum

$\therefore (1, -1)$ is a local minimum point

Graphically, the graph is continuous, begins at $(0,0)$ and $\lim_{x\to\infty} f(x) = \infty$ so the turning point at $(1, -1)$ must be a local minimum point (and in fact this case a global minimum).

(Students should draw this.)

c $f'(x) = 3x^2 - 12x = 3x(x-4)$

$f'(x) = 0 \Rightarrow x = 0$ or $x = 4$

Consider the point $(0,2)$,

$f(x)$ increasing for $x < 0$ and decreasing for $0 < x < 4$ so this is a local maximum

Consider the point $(4,-30)$

$f(x)$ decreasing for $0 < x < 4$ and increasing for $x > 4$ so this is a local minimum

$\therefore (0,2)$ is a local maximum and $(4,-30)$ is local minimum

Graphically, this is a positive cubic, so the first turning point is a maximum and the second point a minimum

(students should draw this)

2 $f'(x) = 3ax^2 + 4x = x(3ax+4)$

$f'(x) = 0$ when $x = 0$ or $x = -\frac{4}{3a}$

It is given that the turning point, away from $x = 0$, occurs at $x = 1$

$\therefore 1 = -\frac{4}{3a} \Rightarrow a = -\frac{4}{3}$

3 $p(0) = d = 1$ so $d = 1$

$p(-1) = -a + b - c + 1 = -3$

so $a - b + c = 4$

$p'(x) = 3ax^2 + 2bx + c = x(3ax+2b) + c$

$p'(0) = 3 \Rightarrow c = 3 \Rightarrow a - b = 1$

$p'(-1) = 0 \Rightarrow -(-3a+2b) + 3 = 0$

$\Rightarrow a = \frac{2b}{3} - 1$

$\therefore \frac{2b}{3} - 1 - b = 1 \Rightarrow b = -6 \Rightarrow a = -5$

so $a = -5,\ b = -6,\ c = 3,\ d = 1$

4 $\frac{dy}{dx} = 3x^2 + 2ax = x(3x+2a) = 0$

$\therefore \frac{dy}{dx} = 0$ when $x = 0$ or $x = -\frac{2a}{3}$

$\therefore -\frac{2a}{3} = 4 \Rightarrow a = -6$

$y(4) = 64 - 6(16) + b = b - 32 = -11$

$\Rightarrow b = 21$

so the local maximum is at $(0,21)$

Exercise 5M

1 $f'(x) = 5x^{\frac{3}{2}}$ $\therefore f''(x) = \frac{15}{2}x^{\frac{1}{2}}$

2 $f'(x) = \frac{x^3}{3} - 4x + 5$

$\therefore f''(x) = x^2 - 4$

$f''(x) = x^2 - 4 = 0 \Rightarrow x = \pm 2$

3 $f'(x) = -2(5-4x)^{-\frac{1}{2}}$

$\therefore f''(x) = -4(5-4x)^{-\frac{3}{2}}$

4 $\frac{d^2y}{dx^2} = 2a^2$

(terms of order x and constants disapper upon differentiating twice)

$\therefore 2a^2 = 8 \Rightarrow a = \pm 2$

Exercise 5N

1 a $\frac{dy}{dx} = 3x^2 - 1$

$\frac{d^2y}{dx^2} = 6x \Rightarrow 0$ at $x = 0$

Coordinates of point of inflexion are $(0,0)$

b $\frac{d^2y}{dx^2} = 6x > 0 \Rightarrow x > 0$

Function concave up on $]0,\infty[$

c $\frac{d^2y}{dx^2} = 6x < 0 \Rightarrow x < 0$

Function concave down on $]-\infty,0[$

2 a $\frac{dy}{dx} = 4x^3 - 3$

$\frac{d^2y}{dx^2} = 12x^2 > 0$

There are no points of inflexion

b $\frac{d^2y}{dx^2} = 12x^2 > 0$

Functions is concave up throughout its domain

c Function is never concave down

3 a $\frac{dy}{dx} = 3x^2 - 12x - 12$

$\frac{d^2y}{dx^2} = 6x - 12 = 0$ at $x = 2$

Coordinates of point of inflexion are $(2,-38)$

b $\frac{d^2y}{dx^2} = 6x - 12 > 0 \Rightarrow x > 2$

Function is concave up on $]2,\infty[$

c $\frac{d^2y}{dx^2} = 6x - 12 < 0 \Rightarrow x < 2$

Function is concave down on $]-\infty, 2[$

4 a $\frac{dy}{dx} = 3x^2 + 2x$

$\frac{d^2y}{dx^2} = 6x + 2 = 0 \text{ at } x = -\frac{1}{3}$

Coordinates of point of inflexion are $\left(-\frac{1}{3}, -\frac{25}{27}\right)$

b $\frac{d^2y}{dx^2} = 6x + 2 > 0 \Rightarrow x > -\frac{1}{3}$

Function is concave up on $]-\frac{1}{3}, \infty[$

c $\frac{d^2y}{dx^2} = 6x + 2 < 0 \Rightarrow x < -\frac{1}{3}$

Function is concave up on $]-\infty, -\frac{1}{3}[$

5 a $\frac{dy}{dx} = 12x^2 - 4x^3$

$\frac{d^2y}{dx^2} = 24x - 12x^2 = 0 \text{ at } x = 0, 2$

Coordinates of point of inflexion are $(0,0), (2,16)$

b $\frac{d^2y}{dx^2} = 24x - 12x^2 > 0 \Rightarrow 0 < x < 2$

Function is concave up for $0 < x < 2$

c $\frac{d^2y}{dx^2} = 24x - 12x^2 < 0 \Rightarrow 0 > x, x > 2$

Function is concave down for $x < 0$; $x > 2$

6 a $\frac{dy}{dx} = 3x^2 - 6x + 3$

$\frac{d^2y}{dx^2} = 6x - 6 = 0 \text{ at } x = 1$

Coordinates of point of inflexion are $(1,0)$

b $\frac{d^2y}{dx^2} = 6x - 6 > 0 \Rightarrow x > 1$

Function is concave up on $]1, \infty[$

c $\frac{d^2y}{dx^2} = 6x - 6 < 0 \Rightarrow x < 1$

Function is concave down on $]-\infty, 1[$

7 a $\frac{dy}{dx} = 8x^3 + 3x^2$

$\frac{d^2y}{dx^2} = 24x^2 + 6x = 0 \text{ at } x = 0, -0.25$

Coordinates of point of inflexion are $(-0.25, 0.992), (0,1)$

b $\frac{d^2y}{dx^2} = 24x^2 + 6x > 0 \Rightarrow x > 0, x < -0.25$

Function is concave up for $x > 0$; $x < -0.25$

c $\frac{d^2y}{dx^2} = 24x^2 + 6x < 0 \Rightarrow -0.25 < x < 0$

Function is concave down for $-0.25 < x < 0$

8 a $\frac{dy}{dx} = 4x^3 - 12x^2 + 16$

$\frac{d^2y}{dx^2} = 12x^2 - 24x = 0 \text{ at } x = 0, 2$

Coordinates of point of inflexion are $(0, -16)$, $(2, 0)$

b $\frac{d^2y}{dx^2} = 12x^2 - 24 > 0 \Rightarrow x < 0, x > 2$

Function is concave up when $x < 0$, $x > 2$

c $\frac{d^2y}{dx^2} = 12x^2 - 24 < 0 \Rightarrow 0 < x < 2$

Function is concave down when $0 < x < 2$

9 a $f'(x) = 3x^2 + 4x$

$3x^2 + 4x = 0 \Rightarrow x = 0, -\frac{4}{3}$

$f''(x) = 6x + 4$

$f''(0) = 4$

$f''\left(-\frac{4}{3}\right) = -4$

$f''(x) = 0 \Rightarrow x = -\frac{2}{3}$

Non-horizontal inflexion at $\left(-\frac{2}{3}, \frac{43}{27}\right)$

b $f'(x) = 3(x-1)^2$

$3(x-1)^2 = 0 \Rightarrow x = 1$

$f''(x) = 6(x-1) = 0 \text{ at } x = 1$

$f''(1.1) = 0.6 > 0$

$f''(0.9) = -0.6 < 0$

Second derivative = 0 at $x = 1$, there is a change in concavity at $x = 1$.

Therefore there is a horizontal inflexion at (1, 0)

c $f'(x) = -12x^3 - 24x^2$

$-12x^3 - 24x^2 = 0 \Rightarrow x = 0, -2$

$f''(x) = -36x^2 - 48x = 0 \text{ at } x = 0, -\frac{4}{3}$

$f''(0.1) = -\frac{129}{25}$

$f''(-0.1) = \frac{111}{25}$

First and second derivatives = 0 at $x = 0$, and there is a change in concavity at $x = 0$.

Therefore there is a horizontal inflexion at (0, 2)

$f''(-2) = -48$

Second derivative = 0 at $x = -\frac{4}{3}$, and there is a change in concavity at $x = -\frac{4}{3}$

Therefore there is a non-horizontal point of inflexion at $\left(-\frac{4}{3}, \frac{310}{27}\right)$

d $f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$

First derivative has no roots, therefore there are no points of inflexion.

10 a i $f'(x) = 3x^2 - 6x - 6$

$3x^2 - 6x - 6 = 0 \Rightarrow x = -0.732, 2.73$

$f''(x) = 6x - 6$

$f''(-0.732) = -10.392$

$f''(2.73) = 10.38$

Therefore local max at (−0.732, 3.39) and local min at (2.73, −17.4)

ii $f''(x) = 6x - 6 = 0 \Rightarrow x = 1$

Non-horizontal inflexion at (1, −7)

iii Increasing: $x < -0.732$ or $x > 2.73$ decreasing for $-0.732 < x < 2.73$

iv Concave downward $x < 1$ and concave upward for $x > 1$

b i $f'(x) = 2(x-1) \Rightarrow x = 1$

$f''(x) = 2$

$f(1) = 2$

Therefore local min at (1, 0)

ii $f''(x) = 2$ therefore there are no inflexion points

iii Increasing for $x > 1$

decreasing for $x < 1$

iv Concave upward for $x \in$ ¡

c i $f'(x) = 12x^3 + 12x^2 = 0 \Rightarrow x = -1, 0$

$f''(x) = 36x^2 + 24x$

$f''(-1) = 60$

$f''(0) = 0$

Therefore local min at (−1, −3)

ii $f''(x) = 36x^2 + 24x = 0 \Rightarrow x = -\frac{2}{3}, 0$

Horizontal inflexion point at (0, −2)

Non-horizontal inflexion point at $\left(-\frac{2}{3}, -\frac{70}{27}\right)$

iii Increasing $x > -1$

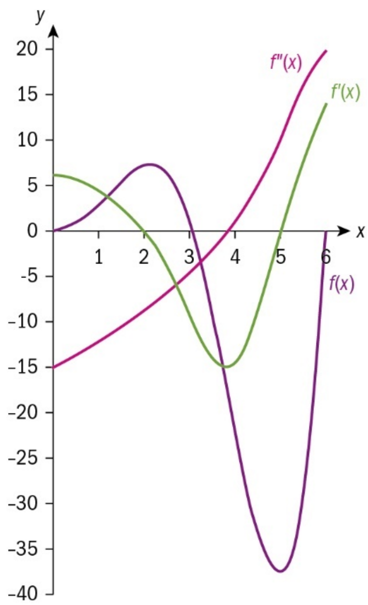
Decreasing $x < -1$

iv Concave upward $x < -\frac{2}{3}$ or $x > 0$

Concave downward $-\frac{2}{3} < x < 0$

Exercise 5O

1 a



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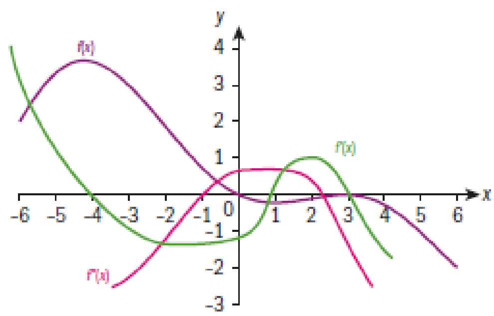
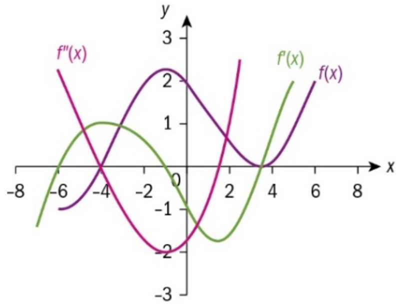
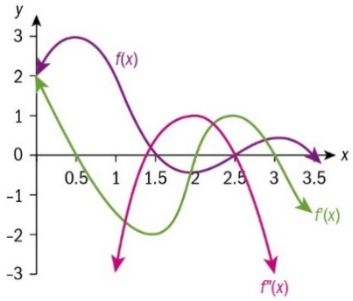


c

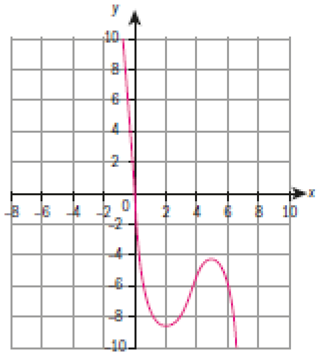


d

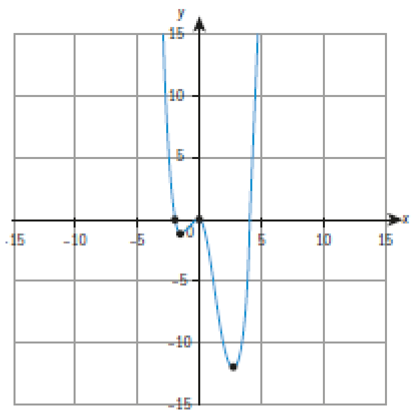


Exercise 5P

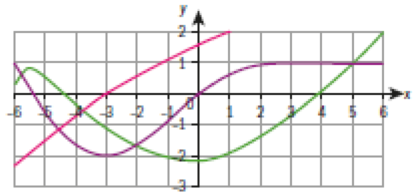
1 a



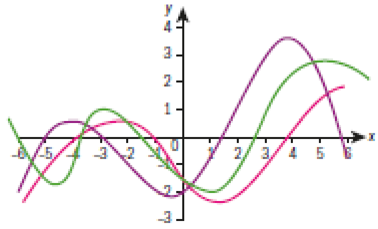
b



2 a



b



Exercise 5Q

1 a $L = \dfrac{100}{x}$

b $P = 2x + \dfrac{100}{x}$ assuming clarification to the question is made, as written in the comments

c $P'(x) = 2 - \dfrac{100}{x^2}$

$P'(x) = 0 \Rightarrow x = \sqrt{50} = 5\sqrt{2} \quad (x > 0)$

This must be a minimum because

$\lim_{x \to 0^+} P(x) = \infty$ and $\lim_{x \to \infty} P(x) = \infty$

and this is the only turning point

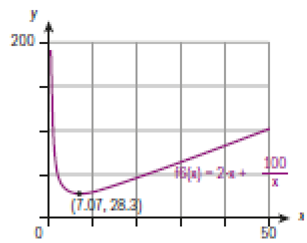
(and the function is continuous)

So $x = 5\sqrt{2} \therefore P\left(5\sqrt{2}\right) = 2\left(5\sqrt{2}\right) + \dfrac{100}{5\sqrt{2}}$

$= 10\sqrt{2} + 10\sqrt{2}$

$= 20\sqrt{2}$ (measured in metres)

d



2 $\dfrac{dy}{dx} = 600 + 30x - 3x^2$

$\dfrac{dy}{dx} = 0 \Rightarrow x^2 - 10x - 200$

$= (x - 20)(x + 10) = 0$

$\therefore x = 20$ since $x > 0$

$y(20) = 600(20) + 15(20)^2 - (20)^3 = 10000$

Therefore, maximum profit is \$10 000

3 **a** $h = \frac{216}{s^2}$

b $A = s^2 + \frac{864}{s}$

c $\frac{dA}{ds} = 2s - \frac{864}{s^2} = 0$

$\Rightarrow s^3 = 432$ so $s = \sqrt[3]{432}$

4 The length of each side of the square is s

Therefore the total length of wire used for the rectangle is $150 - 4s$

Since the length is twice the length of the width,

the length of the rectangle is $50 - \frac{4s}{3}$

and the width of the rectangle is $25 - \frac{2s}{3}$

so the total area enclosed by the square and rectangle is

$A = s^2 + \left(50 - \frac{4s}{3}\right)\left(25 - \frac{2s}{3}\right)$

$= s^2 + 2\left(25 - \frac{2s}{3}\right)^2$

$= s^2 + 2\left(625 - \frac{100s}{3} + \frac{4s^2}{9}\right)$

$= \frac{17s^2}{9} - \frac{200s}{3} + 1250$

$\frac{dA}{ds} = \frac{34}{9}s - \frac{200}{3} = 0 \Rightarrow s = \frac{300}{17}$

5 Let the length of the shorter side of the base be l, so the longer side measures $2l$

Therefore the area of the base is $2l^2$.

Let the height be h

$V = 2l^2h = 10 \Rightarrow h = \frac{5}{l^2}$

So the total cost is

$C = 2l^2(10) + 2(2l)(h)(6) + 2(l)(h)(6)$

$= 20l^2 + 36lh = 20l^2 + \frac{180}{l}$

$\frac{dC}{dl} = 40l - \frac{180}{l^2} = 0 \Rightarrow l = \left(\frac{9}{2}\right)^{\frac{1}{3}}$

$\therefore C_{min} = 20\left(\frac{9}{2}\right)^{\frac{2}{3}} + 180\left(\frac{2}{9}\right)^{\frac{1}{3}}$

$= 164$ (to nearest dollar)

Exercise 5R

1 **a** $v(t) = 3t^2 - 3,\ a(t) = 6t$

b $s(0) = 1,\ v(0) = -3,\ a(0) = 0$

At this instant, the particle is 1 metre from the origin in the positive direction, travelling towards the origin at 3m/s, and is not accelerating

c The particle is moving away from the origin at 9m/s and is accelerating away from the origin at 12m/s^2

d The change of sign of $v(t)$

occurs at $t = 1 \quad (t > 0)$

e $t > 1$

f $s(0) = 1,\ s(1) = -1$

so travels 2m in this period

$s(1) = -1,\quad s(3) = 19$

so 20m travelled in this period

$\therefore$ Altogether distance travelled is 22m

2 **a** 1m

b $s'(t) = 12 - 3t^2 = 0 \Rightarrow t = 2$

$s(2) = 17$ and this is clearly a maximum so 17m

c $v(t) = 12 - 3t^2$

$v(0) = 12,\ v(1) = 9,\ v(3) = -15$

d $16 + 17 = 33$ so 33m

3 $s'(t) = 15 - 10t \Rightarrow t = \frac{3}{2}$

clearly attains maximum here as the function is a negative parabola

$s_{max} = s\left(\frac{3}{2}\right) = 15\left(\frac{3}{2}\right) - 5\left(\frac{3}{2}\right)^2 = \frac{45}{4}$

4 **a** $s(0) = 10$ so 10m

b $s(t) = 0 \Rightarrow t^2 - 5t - 10 = 0$

$\therefore t = \frac{5 + \sqrt{65}}{2} \approx 6.53 \ (t > 0)$

c $v(t) = s'(t) = 5 - 2t$

$v\left(\frac{5 + \sqrt{65}}{2}\right) = -8.06\,\text{ms}^{-1}$

$a(t) = v'(t) = -2\,\text{ms}^{-2}$

As both the velocity and acceleration are negative, the diver is speeding up as he/she hit the water.

5 $h_0 = 0,\ v_0 = 50\ \therefore h(t) = 50t - 4.9t^2$

$h'(t) = 50 - 9.8t = 0 \Rightarrow t = \frac{50}{9.8}$

(clearly maximum here)

$h_{max} = h\left(\frac{50}{9.8}\right) = 127.551$

so maximum height is 127.6m to 1d.p.

$t_{ground} = \frac{50}{4.9} = 10.2041$

so hits ground after 10.2s to 1d.p.

6 a $t = 0,\ t = 3,\ t = 6,\ t = 11$

b i Eastward is positive → $0 < t < 3$; $6 < t < 11$

ii Westward is negative → $3 < t < 6$

c i $t = 1.5$ **ii** $t = 4.5$

d $t = 1.5$ and $t = 4.5$

e Speeding up: $t \in (0, 1.5)$;

$t \in (3, 4.5); t \in (6, 9)$

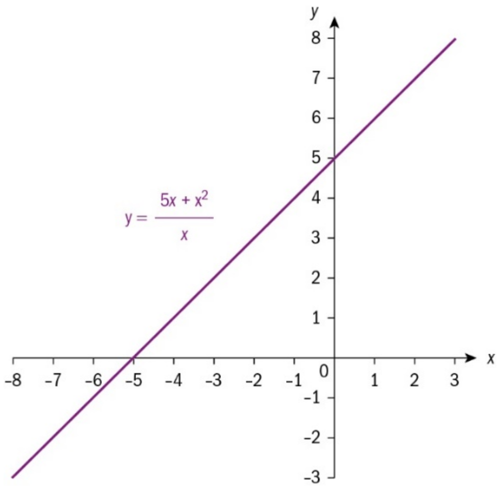
Slowing down: $t \in (1.5, 3)$;

$t \in (4.5, 6); t \in (9, 11)$

Chapter Review

1 a $y = 2$ **b** $a = 2$

2 a



b $\lim_{x \to 0^-} \frac{5x + x^2}{x} = \lim_{x \to 0^+} \frac{5x + x^2}{x} = 5$

3 a $y = 6,\ x = \pm 3$ **b** $y = 0,\ x = -3$

4 a Using the product rule,

$\frac{dy}{dx} = -10(1 - 2x)^4 (3x - 2)^6$

$+ 18(1 - 2x)^5 (3x - 2)^5$

$= (1 - 2x)^4 (3x - 2)^5$

$(-10(3x - 2) + 18(1 - 2x))$

$= 2(1 - 2x)^4 (3x - 2)^5 (19 - 33x)$

b $y = \frac{x - 3}{x(x - 3)} = \frac{1}{x}$ so $\frac{dy}{dx} = -\frac{1}{x^2}$

c $\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} - \frac{4}{3} x^{-\frac{2}{3}}$

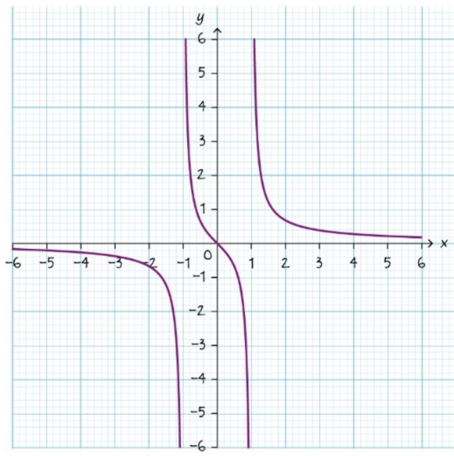
5 a $y = 0,\ x = \pm 1$

b Using the quotient rule,

$\frac{dy}{dx} = \frac{(x^2 - 1) - x(2x)}{(x^2 - 1)^2}$

$= -\frac{x^2 + 1}{(x^2 - 1)^2} < 0$ for all $x \in$ ¡

c



6 $\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x - 3)(x + 1)$

$\therefore \frac{dy}{dx} = 0 \Rightarrow x = -1$ or $x = 3$

$y(-1) = -1 - 3 + 9 + 2 = 7$

$y(3) = 27 - 27 - 27 + 2 = -25$

So $y = -25$ and $y = 7$

7 Using the quotient rule,

$\frac{dy}{dx} = \frac{2(x - 1) - 2x}{(x - 1)^2} = -\frac{2}{(x - 1)^2}$

$\therefore$ at $(2, 4)$ the gradient is -2

So the tangent at this point is

$y-4=-2(x-2) \Rightarrow y=-2x+8$

The gradient at $(3,3)$ is $-\frac{1}{2}$ so the gradient of the normal at this point is 2

Therefore the normal at this point is

$y-3=-\frac{1}{2}(x-3) \Rightarrow y=-\frac{x}{2}+\frac{9}{2}$

$\therefore -\frac{x}{2}+\frac{9}{2}=-2x+8 \Rightarrow x=\frac{7}{3}$

$\Rightarrow y=-2\left(\frac{7}{3}\right)+8=\frac{10}{3} \therefore P\left(\frac{7}{3},\frac{10}{3}\right)$

8 $f'(x)=6x^2-3$

$f'(1)=3$

So the normal to the curve at this point has gradient $-\frac{1}{3}$

$\therefore y-0=-\frac{1}{3}(x-1)=-\frac{x}{3}+\frac{1}{3}$

9 $\frac{\mathrm{d}y}{\mathrm{d}x}=-3x^2+4x$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-4 \Rightarrow 3x^2-4x-4=0$

$\Rightarrow (3x+2)(x-2)=0$

$\therefore x=-\frac{2}{3}$ or $x=2$

$y\left(-\frac{2}{3}\right)=\frac{8}{27}+2\left(\frac{4}{9}\right)+1=\frac{59}{27}$

$y(2)=-8+8+1=1$

$\therefore \left(-\frac{2}{3},\frac{59}{27}\right)$ and $(2,1)$

10 Using the quotient rule,

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x(x+1)-x^2}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}=\frac{x(x+2)}{(x+1)^2}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow x=0$ or $x=-2$

Students may either use first derivative or second derivative test here

e.g. second derivative test:

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{2(x+1)^3-2(x^2+2x)(x+1)}{(x+1)^4}$

$=\frac{2}{(x+1)^3}$

The second derivative is negative at $x=-2$ and positive at $x=0$

$\therefore (-2,-4)$ is a local maximum and $(0,0)$ is a local minimum

11 a $f(x)=0 \Rightarrow x=-1$ **b** $y=0,\ x=0$

c $f(x)=9\left(\frac{1}{x}+\frac{1}{x^2}\right)$

$\Rightarrow f'(x)=9\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)$

$\therefore f'(x)=0 \Rightarrow x=-2$

$y=f(-2)=-\frac{9}{4}$

$f''(x)=9\left(\frac{2}{x^3}+\frac{6}{x^4}\right)$

$f''(-2)=9\left(-\frac{1}{4}+\frac{3}{8}\right)=\frac{9}{8}>0$

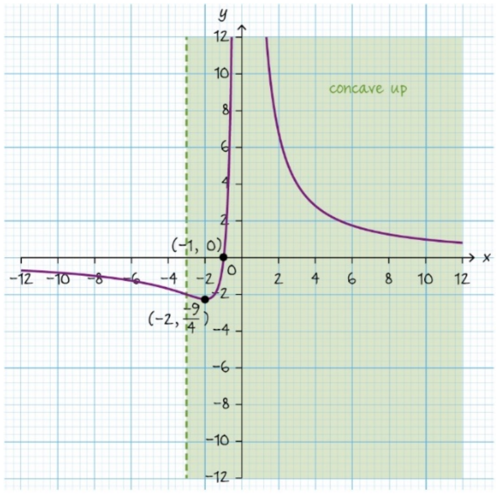
so $\left(-2,-\frac{9}{4}\right)$ is a local minimum

d For $f(x)$ to be concave up $f''(x)>0$

$9\left(\frac{2}{x^3}+\frac{6}{x^4}\right)>0 \Rightarrow 2x+6>0$

$x>-3$

e



12 a $f(x)=0 \Rightarrow \sqrt{x}\left(\sqrt{x}-b\right)=0$

$\Rightarrow x=0$ or $x=b^2$

b $f'(x)=1-\frac{b}{2\sqrt{x}}$

i $f'(x)>0$ when $x>\frac{b^2}{4}$

ii $f'(x)<0$ when $0<x<\frac{b^2}{4}$

c $f''(x)=\frac{b}{4x^{\frac{3}{2}}}$

i $f''(x)>0$ when $b>0$

ii $f''(x)<0$ when $b<0$

13 a $v(t)=49-4.9t$

b $v(t)=0 \Rightarrow t=10$

$h(10) = 49(10) - 2.45(10)^2$

$= 490 - 245 = 245$

So 245m

14 a $v(0) = -2$

b $v(t) = 0 \Rightarrow (1+t)^2 = 4t + 9$

$t^2 - 2t - 8 = (t-4)(t+2) = 0$

So $t = 4$

c $a(t) = 1 - \dfrac{2}{\sqrt{4t+9}}$

$a(4) = 1 - \dfrac{2}{\sqrt{25}} = \dfrac{3}{5}$

d Always speeding up since acceleration is always positive

15 a $f'(x) = 4x^3 - 6x^2 - 2x + 3$ A1

b $g'(x) = \dfrac{-4(x^2+1) - (-4x)\cdot 2x}{(x^2+1)^2}$ M1A1

$g'(x) = \dfrac{4x^2 - 4}{(x^2+1)^2}$ A1

c $h'(x) = 1\cdot(x-7) + (x+2)\cdot 1$ M1

$h'(x) = 2x - 5$ A1

d $i'(x) = 3\cdot 2\cdot(2x+3)^2$ M1

$i'(x) = 6(2x+3)^2$ A1

16 a Graph 1 A1

as the gradient of the tangent at any point is non-positive and therefore different from 1. R1

b Graph 2 A1

as y increases as x increases R1

c Graph 3 A1

as the other two functions are not defined at infinity R1

d Graph 1 A1

as the function is decreasing. R1

17 a i $0 \le t \le 2$, $4.6 \le t \le 5$ A1A1

and $8.5 \le t \le 10$ A1

ii $2 \le t \le 4$ and $5 \le t \le 7$ A1A1

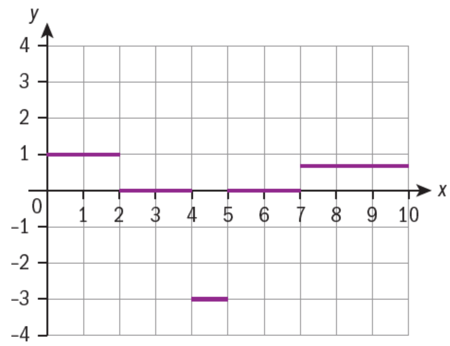
iii $4.6 \le t \le 8.5$ A1

b $f(t) = 2t$, $g(t) = 2$

$h(t) = -3t + 14$, $i(t) = -1$

$f(t) = \dfrac{1}{3}(2t - 17)$ A4

c Up to two correct branches correct A1; all branches correct A2; all branches correct and labels and scale also correct A3



18 a Letting x represent the number of \$10 increases above \$320. Then rental income is

$R(x) = (320 + 10x)(200 - 5x)$ A1

$R'(x) = 400 - 100x = 0$ M1

$x = 4$ A1

Which corresponds to \$360 rent R1

b i $200 - 5 \times 4 = 180$ A1

ii $360 \times 180 = \$64800$ M1A1

19 a $h(4) = 370$

and $h(5) = 438$ (3 s.f.) A1A1

b $v(t) = h'(t) = 112 - 9.8t$ M1A1

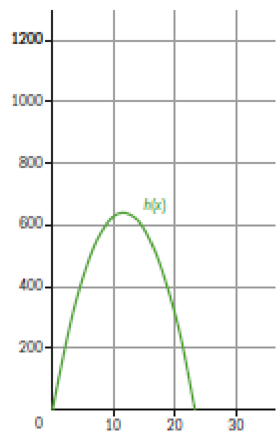
c $v(t) = 0 \Rightarrow 112 - 9.8t = 0$ M1

$t = 11.4$ (3 s.f.) A1

d Double x-coordinate of maximum or determine zero M1

22.8 (3 s.f.) A1

e



Shape A1

Domain $0 \le x \le 22.9$ (3 sf) A1

Maximum 640 (3 sf) A1

f $v(22.9) = -112\,\text{ms}^{-1}$ M1A1

g $a(t) = v'(t) = -9.8$ M1A1

which is constant A1AG

20 a i $\left(\frac{f}{g}\right)'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2}$

M1

$= \frac{10 \times 4 - 9 \times \left(-\frac{4}{3}\right)}{4^2}$ A1

$= \frac{52}{16}\left(= \frac{13}{4} = 3.25\right)$ A1

ii $(g \circ f)'(1) = g'(f(1))f'(1)$ M1

$= -\frac{4}{3} \times 4 = -\frac{16}{3}$ A1

b i False A1

as derivative changes sign. R1

b False A1

as the derivatives at these points are not negative reciprocals.

R1

21 a $\frac{N(3) - N(1)}{3 - 1} = 1410$ M1A1

$\frac{N(5) - N(4)}{5 - 4} = 2220$ A1

the first period the number of cases is increasing in average 1410 per day; in the second period it increases in average 2220 per day.

b $\frac{dN}{dt} = 900t - 90t^2$ M1A1

c After 10 days (reaches 15 000 cases)

M1A1

d $\frac{d^2N}{dt^2} = 900 - 180t$ M1A1

which gives the variation of the rate at which the spread of the disease spreads. R1

22 a $x = \frac{y+2}{y-1}$ M1

$x(y-1) = y + 2$ M1

$xy - y = x + 2$ A1

$g^{-1}(x) = \frac{x+2}{x-1} = g(x)$ A1AG

b $(h \circ g^{-1})'(x) = h'(g^{-1}(x))(g^{-1})'(x)$

M1A1

$= 2 \cdot \frac{x+2}{x-1} \cdot \left(-\frac{3}{(x-1)^2}\right) = \left(-\frac{6(x+2)}{(x-1)^3}\right)$

A1

$(h' \circ g^{-1})(x) = h'(g^{-1}(x))$ M1

$= 2 \cdot \frac{x+2}{x-1}$ A1

$(h \circ g^{-1})'(x) \neq (h' \circ g^{-1})(x)$ AG

6 Representing data: statistics for univariate data

Skills check

1 **a** Mean $= \dfrac{2+3+4+5+6}{5} = 4$

b Mean $= \dfrac{13+9+7+12+15+19+2}{7} = 11$

2 **a** The number that occurs most often is 5

b The numbers that occur most often are 1 and 7. The data is bimodal

3 **a** The median is the middle number, 6

b Arrange the data in order of size. $2,3,5,7,8,9$.

The median is in between 5 and 7.

$\frac{1}{2}(5+7) = 6$

Exercise 6A

1 **a** Discrete **b** Continuous

c Continuous

d Discrete

2 **a** Stratified sampling

b Systematic sampling

c Simple random sampling

d Quota sampling

3 **a** Stratified sampling

b Stratified sampling

c Systematic sampling

d Simple random sampling

e Quota sampling

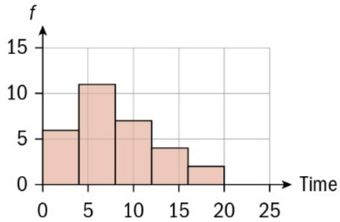
Exercise 6B

1 **a** Continuous

b The frequency table is given here (note difference between this one and the one in the solutions provided)

Time (t)	f
$0 \le t \le 4$	6
$4 < t \le 8$	11
$8 < t \le 12$	7
$12 < t \le 16$	4
$16 < t \le 20$	2

c The histogram is given here (note difference between this one and the one in the solutions provided)



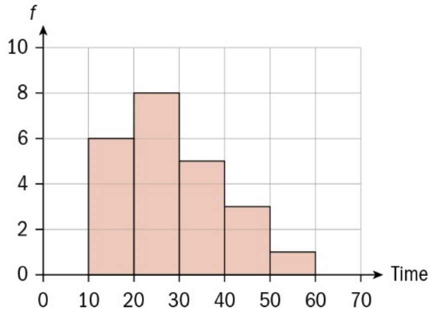
d The data is right or positively skewed

2 **a** Continuous

b The frequency table is given here (note difference between this one and the one in the solutions provided)

Time (t)	f
$10 \le t \le 20$	6
$20 < t \le 30$	8
$30 < t \le 40$	5
$40 < t \le 50$	3
$50 < t \le 60$	1

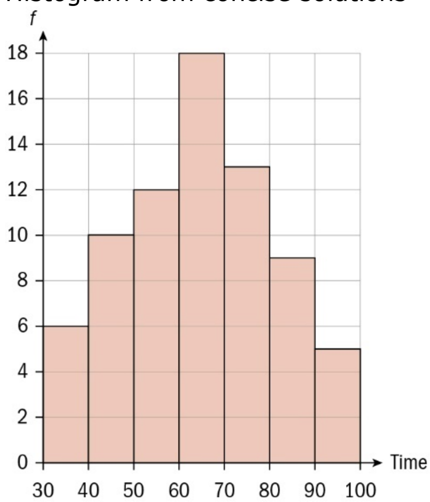
c The histogram is given here (note difference between this one and the one in the solutions provided)



d The data is right or positively skewed

3 **a** Continuous

b Histogram from concise solutions



c The data is neither right nor left skewed, it has normal distribution

4 a Frequency table from concise solutions

b The data is left or negatively skewed

5 a The frequency table is given here (note difference between this one and the one in the solutions provided)

Hours	Days
$0 < h \le 1$	1
$1 < h \le 2$	2
$2 < h \le 3$	3
$3 < h \le 4$	4
$4 < h \le 5$	6
$5 < h \le 6$	8
$6 < h \le 7$	6

b The data is left or negatively skewed

Exercise 6C

1 a The number that occurs most often is 8

b The number that occurs most often is 4

c The number that occurs most often is 13

d Each number occurs only once, so there is no mode

e The numbers that occur most often are 2 and 4. The data is bimodal

2 a The shoe size with the highest frequency is 10

b The modal mark range is $60 < y \le 80$

3 a i The mode is 3

ii The modal range is $30 < x \le 35$

b i Discrete data, since the scale on the x-axis is given as discrete values.

ii Continuous data, since there is a continuous scale of values on the x-axis.

Exercise 6D

1 **a** The mean is 3.65

b The mean is 12.8056

c The mean is 3.35

2 a

x	f	Mid value (m)	fm
$0 \le x \le 10$	18	5	90
$10 < x \le 20$	14	15	210
$20 < x \le 30$	12	25	300
$30 < x \le 40$	9	35	315
$40 < x \le 50$	7	45	315
	$\Sigma f = 60$		$\Sigma fm = 1230$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{1230}{60} = 20.5$$

b

x	f	Mid value (m)	fm
$0 \le x \le 12$	4	6	24
$12 < x \le 24$	0	18	0
$24 < x \le 36$	8	30	240
$36 < x \le 48$	15	42	630
$48 < x \le 60$	13	54	702
$60 < x \le 72$	7	66	462
	$\Sigma f = 47$		$\Sigma fm = 2058$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{2058}{47} = 43.7872 \approx 43.8$$

c

x	f	Mid value (m)	fm
$1 \le x \le 1.5$	4	1.25	5
$1.5 < x \le 2$	6	1.75	10.5
$2 < x \le 2.5$	7	2.25	15.75
$2.5 < x \le 3$	7	2.75	19.25
$3 < x \le 3.5$	5	3.25	16.25
	$\Sigma f = 29$		$\Sigma fm = 66.75$

$$\bar{x} = \frac{\Sigma fm}{\Sigma f} = \frac{66.75}{29} = 2.30172 \approx 2.30$$

3 Mean

$$= \frac{0 \times 6 + 1 \times 5 + 2 \times 4 + 3 \times 7 + 4 \times 10 + 5 \times 4}{6 + 5 + 4 + 7 + 10 + 4}$$

$$= \frac{94}{36} = 2.6111 \approx 2.6 \text{ cups of coffee}$$

4 a Phil had $2 + 4 + 4 + 6 + 10 + 15 + 4 + 5$

$= 50$ tomato plants

b The modal number of tomatoes per plant was 8

c Mean =

$$\frac{3 \times 2 + 4 \times 4 + 5 \times 4 + 6 \times 6 + 7 \times 10 + 8 \times 15 + 9 \times 4 + 10 \times 5}{50}$$

$$= \frac{177}{25} = 7.08$$

5 The mean number of fish caught per day was

$$\frac{0\times1+1\times5+2\times4+3\times2+4\times3+5\times5+6\times3+7\times2+8\times3+9\times1+10\times2}{1+5+4+2+3+5+3+2+3+1+2}$$

$$=\frac{141}{31}\approx 4.55$$

6 The mean amount received per day is

$$\frac{5\times6+15\times14+25\times15+35\times8+45\times2}{6+14+15+8+2}$$

$$=\frac{197}{9}=\$21.89$$

7 **a** There are

$7+12+10+9+7+6+6+3$

$=60$ families represented

b The data is right or positively skewed

c The mode of the data is 2 children per family

d The mean number of children is

$$\frac{7\times1+12\times2+10\times3+9\times4+7\times5+6\times6+6\times7+3\times8}{60}$$

$$=\frac{39}{10}=3.9$$

8 **a** There are $40+60+80+30+10$

$=220$ people in the village

b The modal class is $40<a\le 60$

c The mean age of the villagers is

$$\frac{40\times10+60\times30+80\times50+30\times70+10\times90}{220}$$

$$=\frac{460}{11}\approx 41.8$$

9 In the set of numbers, each appears only once, so therefore for 2 to be the mode, $a=2$. Given that the mean is 5, we have

$$5=\frac{1+2+2+4+5+6+b+8+10}{9},$$

$5\times9=38+b$, $45=38+b$, $b=45-38$, $b=7$

10 Given that the mean of the numbers is 23, we have to find x

$$23=\frac{8+x+17+(2x+3)+45}{5}$$

$23\times5=73+3x$

$115=73+3x$

$115-73=3x$

$42=3x$

$x=\frac{42}{3},$

$x=14$

11 Starting with 2 and 3, we know that as 4 is the mode, it must occur at least twice, start by assuming that there are two 4s then x and y are the remaining two numbers, the mean is 4, so

$$4=\frac{2+3+4+4+x+y}{6}$$

$6\times4=13+x+y$

$24-13=x+y$

$11=x+y$

As x and y are positive integers less than 8, the only possible solution is if $x=5$ and $y=6$ (or $y=5$ and $x=6$), so the numbers are $2,3,4,4,5,6$

12 The mean mass of the students is

$$\frac{52\times8+44\times12}{20}=\frac{236}{5}=47.2\text{ kg}$$

Exercise 6E

1 **a** The median is the middle number, 18

b The middle number lies between 18 and 19, $\frac{18+19}{2}=\frac{37}{2}=18.5$

c Arranging the numbers in size order $1,2,4,5,5$, the middle number is 4

d The numbers are already in size order (reversed), so the middle number is the median, the middle number lies between 3 and 4, $\frac{3+4}{2}=\frac{7}{2}=3.5$

e $2,4,5,5,6,7,7,8,8,10$. The middle number is between 6 and 7,

$$\frac{6+7}{2}=\frac{13}{2}=6.5$$

f $2,3,5,5,6,8$. The median is 5

2 Total number of days

$=2+4+3+7+11+18+6+2=53$

Median $=\left(\frac{n+1}{2}\right)^{th}=\left(\frac{53+1}{2}\right)^{th}=27^{th}=9$

3 **a** Mean

$$=\frac{2000000\times1+1000\times10+600\times14+200\times25}{1+10+14+25}$$

$$=\frac{2023400}{50}=\$40468$$

b The mode is the most common number, 200

c The median number is the number in the $\left(\frac{50+1}{2}\right)^{th}=25.5^{th}$ position, that is the number between 200 and 600,

$$\frac{200+600}{2}=400$$

Exercise 6F

1 **a** The median is the middle number, 8

b $Q_1 = \left(\frac{n+1}{4}\right)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd} = 7$

c $Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} = \left(\frac{3(11+1)}{4}\right)^{th}$

$= 9^{th} = 12$

d $\text{IQR} = Q_3 - Q_1 = 12 - 7 = 5$

e Range = largest – smallest

= 15 – 3 = 12

2 In ascending order, the numbers are

$2, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15$

a Median

$= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{12+1}{2}\right)^{th} = 6.5^{th} = 6$

b Q_1 is the median of the lower half of the numbers, 5

c Q_3 is the median of the upper half of the numbers, it lies between 8 and 9, 8.5

d $\text{IQR} = Q_3 - Q_1 = 8.5 - 5 = 3.5$

e $\text{Range} = \text{largest} - \text{smallest} = 15 - 2 = 13$

3 Sorting the number of sit-ups into ascending order,

$2, 10, 10, 12, 14, 16, 16, 20, 25, 25,$

$28, 30, 37, 40, 45, 50$

a Median

$= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{16+1}{2}\right)^{th} = 8.5^{th}$

$= \frac{20+25}{2} = 22.5$

b Q_1 is the median of the lower half of the numbers, $\frac{12+14}{2} = 13$

Q_3 is the median of the upper half of the numbers, $\frac{30+37}{2} = 33.5$

$\text{IQR} = Q_3 - Q_1 = 33.5 - 13 = 20.5$

c On 8 of the 16 days, Lincy did more than 22.5 sit-ups.

d The 'middle half' of the number of sit-ups Lincy did was between 13 and 33.5.

e On 4 of the 16 days, Lincy did more than 33.5 sit-ups.

4 Sorting the number of cars into ascending order,

$20, 20, 25, 30, 35, 35, 35, 35, 45, 45, 50.$

$\text{IQR} = Q_3 - Q_1.$

$Q_1 = \left(\frac{n+1}{4}\right)^{th} = \left(\frac{11+1}{4}\right)^{th} = 3^{rd} = 25$

$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th} = \left(\frac{3(11+1)}{4}\right)^{th} = 9^{th} = 45$

$\text{IQR} = Q_3 - Q_1 = 45 - 25 = 20$

5 Q_1 is the median of the lower half of the numbers, 2

Q_3 is the median of the upper half of the numbers, it lies between 4 and 5, 4.5

$\text{IQR} = Q_3 - Q_1 = 4.5 - 2 = 2.5$

6 **a** **i** Median $= \left(\frac{n+1}{2}\right)^{th} = \left(\frac{12+1}{2}\right)^{th} = 6.5^{th}$

$\Rightarrow \frac{9+r}{2} = 9.5$

$9.5 = \frac{9+r}{2}$

$9.5 \times 2 = 9 + r$

$19 - 9 = r$

$r = 10$

ii Q_3 is the median of the upper half of the numbers, it is between s and 13

$13 = \frac{s+13}{2}$

$21 \times 3 = s + 13$

$26 - 13 = s$

$s = 13$

b The value of t can be found as follows

$10 = \frac{5+6+7+7+9+9+10+10+13+13+13+t}{12},$

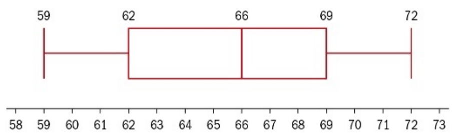
$10 \times 12 = 102 + t$

$120 - 102 = t$

$t = 18$

Exercise 6G

1



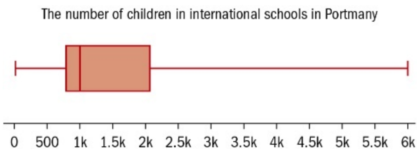
2 **a** The minimum time was 30.1

b The maximum time was 35

c The median time was 32.5

d The IQR was $33.1 - 31.9 = 1.2$

3 a



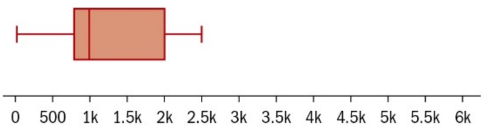
b $Q_3 + 1.5(\text{IQR})$

$= 2067.5 + 1.5 \times 1272.5$

$= 3976.25 < 6000$

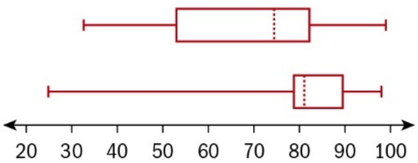
so it is an outlier

c



d The outlier was removed because it distorted the analysis

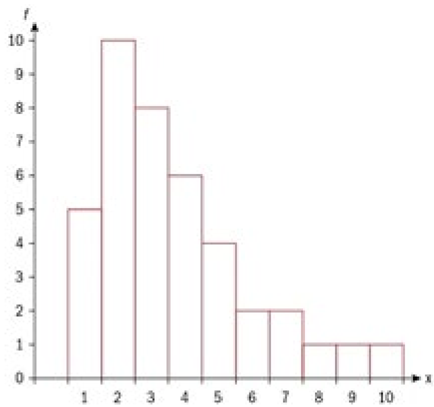
4 a



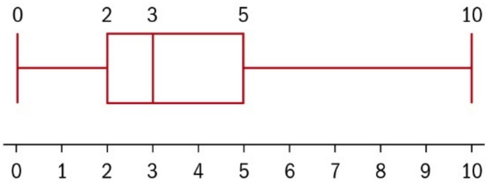
b The morning exam

c This means that there is a bigger difference between the 25% and the 75% of the scores

5 a



b



c The data is right or positively skewed

6 1A, 2C, 3B

Exercise 6H

1 a The longest time taken was 18 minutes

b The median is 11 minutes

c IQR = 13.6 – 8.2 = 5.4 minutes

d k = 15.6 minutes

2 a The median is 40 minutes

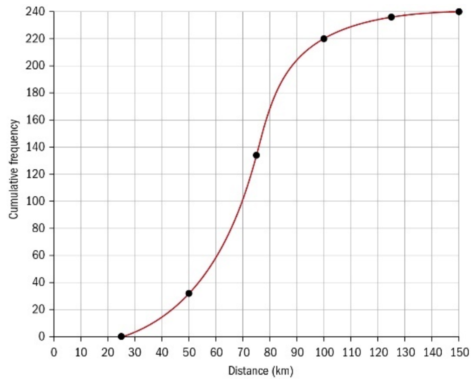
b IQR = 50 – 30 = 20

c 53

3 a

Distance	$0 \le d \le 25$	$25 < d \le 50$	$50 < d \le 75$	$75 < d \le 100$	$100 < d \le 125$	$125 < d \le 150$
CF	0	32	134	220	236	240

b



c The median is 73 km

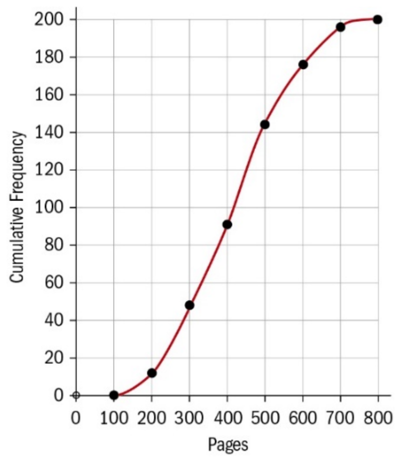
d IQR = 82 – 60 = 22 km

e 3 cars

4 a

Pages	*CF*
$100 \le p \le 200$	12
$200 < p \le 300$	48
$300 < p \le 400$	90
$400 < p \le 500$	143
$500 < p \le 600$	176
$600 < p \le 700$	196
$700 < p \le 800$	200

b



c The median is 420

d IQR = 510 – 300 = 210

e 80 students

5 1C, 2B, 3A

Exercise 6I

1 a $\bar{x} = \frac{\Sigma x}{n} = \frac{4+6+7+7+5+1+2+3}{8}$

$= 4.375$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{4^2+6^2+7^2+7^2+5^2+1^2+2^2+3^2}{8} - 4.375^2$

$= 23.625 - 19.1406 \approx 4.48$

$\sigma = \sqrt{\sigma^2} \approx 2.12$

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{2+5+8+7+1+3+9+11+4+2}{10}$

$= 5.2$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{2^2+5^2+8^2+7^2+1^2+3^2+9^2+11^2+4^2+2^2}{10} - 5.2^2$

$= 37.4 - 37.04 \approx 10.4$

$\sigma = \sqrt{\sigma^2} \approx 3.22$

c $\bar{x} = \frac{\Sigma x}{n} = \frac{-4+-2+0+3+-5}{5} = -1.6$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{(-4)^2+(-2)^2+0^2+3^2+(-5)^2}{5} - 5.2^2$

$= 10.8 - 2.56 = 8.24$

$\sigma = \sqrt{\sigma^2} \approx 2.87$

d $\bar{x} = \frac{\Sigma x}{n} = \frac{1+2+3+4+5}{5} = 3$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2+2^2+3^2+4^2+5^2}{5} - 3^2$

$= 11 - 9 = 2$

$\sigma = \sqrt{\sigma^2} \approx 1.41$

e $\bar{x} = \frac{\Sigma x}{n} = \frac{1+2+3+4+5+500}{6} \approx 85.8$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2+2^2+3^2+4^2+5^2+500^2}{6} - 85.833^2$

$= 41675.833 - 7367.36 \approx 34308$

$\sigma = \sqrt{\sigma^2} \approx 185.2$

2 a $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{1\times3+2\times8+3\times6+4\times6+5\times7}{3+8+6+6+7} = 3.2$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2\times3+2^2\times8+3^2\times6+4^2\times6+5^2\times7}{3+8+6+6+7} - 3.2^2$

$= 12 - 10.24 = 1.76$

$\sigma = \sqrt{\sigma^2} \approx 1.33$

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{1\times5+3\times12+5\times16+7\times22+9\times27+11\times30+13\times18}{5+12+16+22+27+30+18}$

≈ 8.32

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2\times5+3^2\times12+5^2\times16+7^2\times22+9^2\times27+11^2\times30+13^2\times18}{5+12+16+22+27+30+18} - 8.323^2$

$= 80.385 - 69.273 = 11.111$

$\sigma = \sqrt{\sigma^2} = 3.33$

c $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{5\times18+15\times14+25\times13+35\times11+45\times6}{18+14+13+11+6}$

≈ 20.6

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{5^2\times18+15^2\times14+25^2\times13+35^2\times11+45^2\times6}{18+14+13+11+6} - 20.645^2$

$= 602.419 - 426.223 = 176.197$

$\sigma = \sqrt{\sigma^2} \approx 13.3$

3 a $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{1\times2+2\times2+3\times4+4\times10+5\times12+6\times2+7\times2+18\times1}{2+2+4+10+12+2+2+1}$

$= 4.63$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2\times2+2^2\times2+3^2\times4+4^2\times10+5^2\times12+6^2\times2+7^2\times2+18^2\times1}{2+2+4+10+12+2+2+1} - 4.629^2$

$= 28.571 - 21.424 = 7.148$

$\sigma = \sqrt{\sigma^2} = 2.67$

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{1\times2+2\times2+3\times4+4\times10+5\times12+6\times2+7\times2}{2+2+4+10+12+2+2}$

$= 4.24$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2\times2+2^2\times2+3^2\times4+4^2\times10+5^2\times12+6^2\times2+7^2\times2}{2+2+4+10+12+2+2} - 4.235^2$

$= 19.882 - 17.938 = 1.945$

$\sigma = \sqrt{\sigma^2} = 1.40$

4 $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{93 + 86.2 + 80 + 64 + 60.6 + 50 + 50 + 47.3 + 46.6 + 46}{10}$

$= 62.37$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{93^2 + 86.2^2 + 80^2 + 64^2 + 60.6^2 + 50^2 + 50^2 + 47.3^2 + 46.6^2 + 46^2}{10} - 62.37^2$

$= 4177.27 - 3890.02 = 287.248$

$\sigma = \sqrt{\sigma^2} = 16.9$

5 $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{150 \times 3 + 250 \times 6 + 350 \times 11 + 450 \times 5}{3 + 6 + 11 + 5} = 322$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{150^2 \times 3 + 250^2 \times 6 + 350^2 \times 11 + 450^2 \times 5}{3 + 6 + 11 + 5} - 322^2$

$= 112100 - 103684 = 8416$

$\sigma = \sqrt{\sigma^2} = 91.7$

6 **a** $6 + 8 + 6 + 3 + 1 = 24$ months

b The modal range is 20 to 30 hours

c $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{15 \times 6 + 25 \times 8 + 35 \times 6 + 45 \times 3 + 55 \times 1}{6 + 8 + 6 + 3 + 1}$

$= 28.75$

d $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{15^2 \times 6 + 25^2 \times 8 + 35^2 \times 6 + 45^2 \times 3 + 55^2 \times 1}{6 + 8 + 6 + 3 + 1} - 28.75^2$

$= 950 - 826.563 = 123.438$

$\sigma = \sqrt{\sigma^2} = 11.1$

Exercise 6J

1 **a** mean $= \frac{1 + 3 + 5 + 5 + 8}{5} = 4.4$

median $= 5$

mode $= 5$

b mean $= \frac{5 + 7 + 9 + 9 + 12}{5} = 8.4$

median $= 9$

mode $= 9$

c Adds 4 to mean, median and mode

2 **a** $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{7 + 9 + 3 + 0 + 1 + 8 + 6 + 4 + 10 + 5 + 5}{11} = 5.2727$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{7^2 + 9^2 + 3^2 + 0^2 + 1^2 + 8^2 + 6^2 + 4^2 + 10^2 + 5^2 + 5^2}{11} - 5.2727^2$

$= 36.909 - 27.802 = 9.11$

$\sigma = \sqrt{\sigma^2} = 3.02$

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{21 + 27 + 9 + 0 + 9 + 24 + 18 + 12 + 30 + 15 + 15}{11}$

$= 15.8182$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{21^2 + 27^2 + 9^2 + 0^2 + 9^2 + 24^2 + 18^2 + 12^2 + 30^2 + 15^2 + 15^2}{11} - 15.8182^2$

$= 332.182 - 250.215 ; \ 82.0$

$\sigma = \sqrt{\sigma^2} = 9.054$

c The mean is multiplied by 3, and since the variance is multiplied by 9, standard deviation (which is square root of variance) is multiplied by 3.

3 mean $= 17.2 + 4 = 21.2$

median $= 17 + 4 = 21$

standard deviation $= 0.5$

4 The mean, median and standard deviation will double

5 The new variance is $9^2 x = 81x$

Chapter Review

1 **a** mode $= 1$

b median $= \left(\frac{10 + 1}{2}\right)^{th} = 5.5^{th} = \frac{4 + 2}{2} = 3$

c mean $= \frac{2 + 8 + 1 + 5 + 0 + 4 + 4 + 1 + 1 + 6}{10}$

$= \frac{16}{5} = 3.2$

d range $= 8 - 0 = 8$

2 $\frac{9 \times 420 + 3 \times 740}{12} = 500$

3 **a** mode $= 3$

b median $= \left(\frac{50 + 1}{2}\right)^{th} = 25.5^{th} = 3$

c mean $= \frac{0 \times 4 + 1 \times 8 + 2 \times 10 + 3 \times 20 + 4 \times 4 + 5 \times 3 + 6 \times 1}{50}$

$= \frac{5}{2} = 2.5$

4 The mean will increase by 4 and the standard deviation will stay the same; $\text{mean} = 21.9$, standard deviation $= 1.1$

5 a $\text{mean} = \frac{736}{23} = 32$

b $\text{mean} = \frac{736 + 24 + 15}{23 + 2} = 31$

6 a The mean will increase by 10 and the standard deviation will stay the same; $\text{mean} = 58$, standard deviation $= 5$

b The mean will increase by a factor of 10 and the variance will increase by a factor of 10^2; $\text{mean} = 480$,

$\text{variance} = 5^2 \times 100 = 2500$

7 a 40

b 60

c $50 = c - 40 \Rightarrow c = 50 + 40 = 90$

d $\text{IQR} = 24 = 74 - d \Rightarrow d = 74 - 24 = 50$

8 a 800 students

b 65 marks

c $\text{IQR} = 75 - 55 = 20$

d 100 students

e No, because there are 100 students who scored more than 80 marks, this is not 10%

f $k = 40$

9 1B, 2C, 3A

10 a $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{15+12+22+30+25+7+19+33+19+41+53+12+3+8+6+17}{16}$

$= 20.125$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{15^2+12^2+22^2+30^2+25^2+7^2+19^2+33^2+19^2+41^2+53^2+12^2+3^2+8^2+6^2+17^2}{16}$

$- 20.125^2$

$= 579.375 - 405.016 = 174.359$

$\sigma = \sqrt{\sigma^2} = 13.2$

b Write the numbers in size order;

3, 6, 7, 8, 12, 12, 15, 17, 19, 19, 22, 25, 30, 33, 41, 53

then find Q_1 as the median of the first half of the list,

$Q_1 = \left(\frac{8+1}{2}\right)^{th} = 4.5^{th} = \frac{8+12}{2} = 10$ and

find Q_2 as the median of the second half of the list,

$Q_1 = \left(8 + \frac{8+1}{2}\right)^{th} = 12.5^{th} = \frac{25+30}{2} = 27.5$

, so $\text{IQR} = 27.5 - 10 = 17.5$

11 mode = 4

mean

$= \frac{2\times3+3\times4+4\times10+5\times3+6\times2+7\times2}{3+4+10+3+2+2}$

$= 4.125$

$\text{median} = \left(\frac{24+1}{2}\right)^{th} = 12.5^{th} = 4$

$\sigma^2 = \frac{2^2\times3+3^2\times4+4^2\times10+5^2\times3+6^2\times2+7^2\times2}{3+4+10+3+2+2} - 4.125^2$

$= 18.875 - 17.0156 = 1.8593$

$\sigma = 1.36$

12 a $\text{mean} = \frac{2.5\times15+7.5\times11+12.5\times9+17.5\times12+22.5\times6}{15+11+9+12+6}$

≈ 10.9

$\text{median} = \left(\frac{53+1}{2}\right)^{th} = 27^{th} = 12.5$

$\sigma^2 = \frac{2.5^2\times15+7.5^2\times11+12.5^2\times9+17.5^2\times12+22.5^2\times6}{15+11+9+12+6} - 10.896^2$

$= 166.627 - 118.728 = 47.8996$

$\sigma = 6.92$

b Because we are using the midpoint of each range, as opposed to the actual original data, which assumes that the number of items is equally spread throughout the class interval.

13 Given that the mean number of watches is 2.5, we have to find k

$2.5 = \frac{0\times11+1\times7+2\times6+3\times k+4\times8+5\times10}{11+7+6+k+8+10}$,

$2.5\times(42+k) = 101 + 3k$

$105 + 2.5k = 101 + 3k$

$105 - 101 = (3 - 2.5)k$

$4 = 0.5k$

$k = \frac{4}{0.5}$

$k = 8$

14 a 80 bats

b 50 grams

c $\frac{5}{80} = 0.0625 = 6.25\%$

d $a = 10$, $c = 80 - 75 = 5$

e $b = 75 - 55 = 20$

f $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{15\times10+45\times45+75\times20+105\times5}{10+45+20+5}$

$= 52.5$

$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{15^2\times10+45^2\times45+75^2\times20+105^2\times5}{10+45+20+5} - 52.5^2$

$= 3262.5 - 2756.25 = 506.25$

$\sigma = \sqrt{\sigma^2} = 22.5$

15 a $50 = 3 + 11 + 16 + m + 8$

$\Rightarrow m = 50 - 38 = 12$

$n = 14 + 16 = 30$

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{10 \times 3 + 15 \times 11 + 20 \times 16 + 25 \times 12 + 30 \times 8}{3 + 11 + 16 + 12 + 8}$

$= 21.1$

c $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{10^2 \times 3 + 15^2 \times 11 + 20^2 \times 16 + 25^2 \times 12 + 30^2 \times 8}{3 + 11 + 16 + 12 + 8} - 21.1^2$

$= 477.5 - 445.21 = 32.3$

16 a Discrete

b $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{1 \times 41 + 2 \times 60 + 3 \times 52 + 4 \times 32 + 5 \times 15 + 6 \times 8}{41 + 60 + 52 + 32 + 15 + 8}$

≈ 2.73

c $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$= \frac{1^2 \times 41 + 2^2 \times 60 + 3^2 \times 52 + 4^2 \times 32 + 5^2 \times 15 + 6^2 \times 8}{41 + 60 + 52 + 32 + 15 + 8} - 2.731^2$

$= 9.25 - 7.4571 = 1.7929$

$\sigma = \sqrt{\sigma^2} = 1.34$

d 1 standard deviation above the mean is $2.731 + 1.339 = 4.07$, so $15 + 8 = 23$ families have more than one standard deviation above the mean mobile devices

17 a Discrete A1

b Continuous A1

c Continuous A1

d Discrete A1

18 a As the mode is 5 there must be at least another 5. R1

So we have 1, 3, 5, 5, 6 with another number to be placed in order R1

The median will be the average of the 3rd and 4th pieces of data. R1

For this to be 4.5 the missing piece of data must be a 4.

Thus a=5, b=4 A1 A1

b $\bar{x} = \frac{1 + 3 + 4 + 5 + 5 + 6}{6} = \frac{24}{6} = 4$

M1 A1

19 a An outlier is further than 1.5 times the IQR below the lower quartile or above the upper quartile. A1

b **i** mode = 8 A1

ii median = 7 A1

iii lower quartile = 3 A1

iv upper quartile = 9 A1

c IQR = 6

$1.5 \times$ IQR = 9

19 – 9 = 10 M1

19 is the (only) outlier A1

20 a $\frac{\sum x}{10} = 70 \Rightarrow \sum x = 700$ A1

Let the new student's mass be s.

$\frac{\sum x + s}{11} = 72$ M1

$700 + s = 792$ A1

So s = 92kg A1

b IQR = 10 A1

$76 + 1.5 \times IQR = 76 + 15 = 91$ M1

So new student's mass of 92 is an outlier R1

21 a 200 A1

b 35 A1

c Using mid-points 5, 15, 25... as estimates for each interval M1

i Estimate for mean is 22.25 A2

ii Estimate for standard deviation is 11.6 (3sf) A2

d Median is approximately the 100th piece of data which lies in the interval $20 < h \le 30$. A1

Will be 15 pieces of data into this interval

Estimate is $20 + \frac{15}{50} \times 10 = 23$ M1A1

22 a Discrete A1

b 5 A1

c **i** 4.79 (3sf) A2

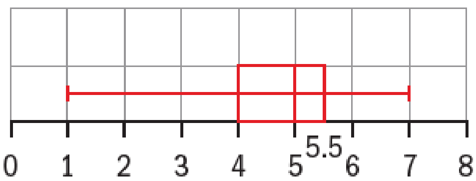
ii 1.62 (3sf) A2

d **i** 5 A1

ii 4 A1

iii 5.5 A1

e



A1 general shape

A1 median

A1 quartiles

f IQR=1.5

$1.5 \times 1.5 = 2.25$ (A1)

5.5 + 2.25 = 7.75

4 - 2.25 = 1.75 M1

So the 2 (unhappy) candidates with grade 1 are outliers A1

23 a

x	Frequency	Cumulative frequency
0	10	10
1	7	17
2	11	28
3	13	41
4	15	56
5	15	71
6	12	83
7	10	93
8	4	97
9	2	99
10	1	100

A4 for 6 correct
A3 for 4 or 5 correct
A2 for 2 or 3 correct
A1 for 1 correct

b **i** 4
ii 2
iii 6 A1A1A1

c **i** 4.05 (3sf)
ii $(2.4140...)^2 = 5.83$ (3sf) A1(M1)A1

d No. It is bimodal at *x*= 4 and 5. 24
A1R1

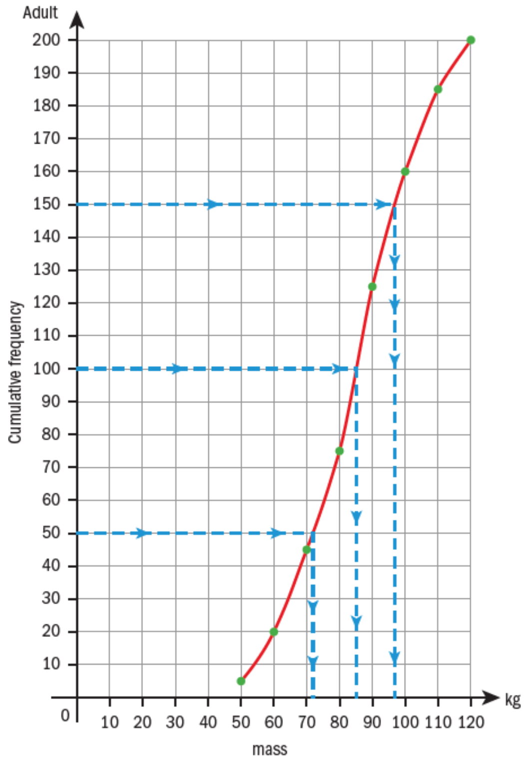
24 a $80 < w \le 90$ A1

b

mass	cumulative frequency
$40 < w \le 50$	5
$50 < w \le 60$	20
$60 < w \le 70$	45
$70 < w \le 80$	75
$80 < w \le 90$	125
$90 < w \le 100$	160
$100 < w \le 110$	185
$110 < w \le 120$	200

A2 numbers A1 labelling

c



A1A1scales A3 points and curve

d **i** 85
ii 73
iii 97 A1A1A1
M1 lines

25 a **i** 7.5
ii 6.125 A1A2

b **i** 6
ii 6.9 A1A2

c Sally's had the greater median R1

d Rob's had the greater mean R1

26 a

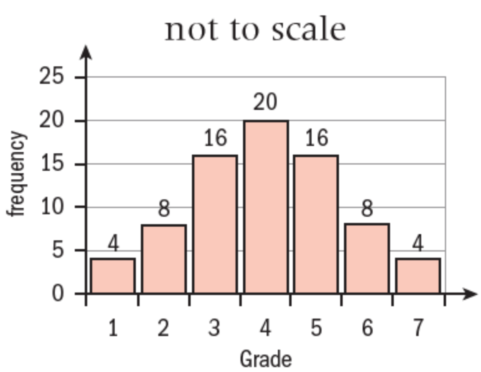


chart A1; scaleA2

b **i** 4
ii 4
iii 4 A1A1A1

c The values of the median and the mean are the same due to the symmetry of the bar chart. A1R1