11 Relationships in space: geometry and trigonometry in 2D and 3D

Skills check

1 **a** $5\sqrt{10} \approx 15.8$ **b** $2\sqrt{6} \approx 4.9$

2 **a** **i** $12600\,\text{cm}^2$ **ii** $1.26\,\text{m}^2$

b $4\pi \approx 12.6\,\text{m}^3 \approx 12566\,l$

Exercise 11A

1 **a** $(3,0,0)$ **b** $(3,4,0)$

c $(3,0,2)$ **d** $(3,4,2)$

e Midpoint of OF

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{0+3}{2},\frac{0+4}{2},\frac{0+2}{2}\right)=(1.5,2,1)$$

f Distance of OF

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(3-0)^2+(4-0)^2+(2-0)^2}$$
$$=\sqrt{9+16+4}=\sqrt{29}\approx 5.4$$

2 **a** $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-4+5}{2},\frac{4-1}{2},\frac{3+3}{2}\right)=(0.5,1.5,3)$$

b $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-4-2}{2},\frac{4+2}{2},\frac{5+9}{2}\right)=(-3,3,7)$$

c $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{5-4}{2},\frac{2-3}{2},\frac{-4-8}{2}\right)=(0.5,-0.5,-6)$$

d $$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$
$$=\left(\frac{-5.1+1.4}{2},\frac{-2+1.7}{2},\frac{9+11}{2}\right)$$
$$=(-1.85,-0.15,10)$$

3 **a** $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(4-2)^2+(3-3)^2+(1-5)^2}$$
$$=\sqrt{4+0+16}=\sqrt{20}\approx 4.47$$

b $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(2+3)^2+(4-7)^2+(-1-2)^2}$$
$$=\sqrt{25+9+9}=\sqrt{43}\approx 6.56$$

c $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1+1)^2+(-3-3)^2+(4+4)^2}$$
$$=\sqrt{4+36+64}=\sqrt{104}\approx 10.2$$

d $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(-2-2)^2+(1+1)^2+(3-3)^2}$$
$$=\sqrt{16+4+0}=\sqrt{20}\approx 4.47$$

4 **a** $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(-5-1)^2+(-6-2)^2+(-7-3)^2}$$
$$=\sqrt{36+64+100}=\sqrt{200}\approx 14.1$$

b $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(4-0)^2+(0+4)^2+(5-2)^2}$$
$$=\sqrt{16+16+9}=\sqrt{41}\approx 6.4$$

c $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1+1)^2+(2+1)^2+(3+1)^2}$$
$$=\sqrt{4+9+16}=\sqrt{29}\approx 5.39$$

d $$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$
$$=\sqrt{(1-4)^2+(1-1)^2+(1+3)^2}$$
$$=\sqrt{9+0+16}=\sqrt{25}=5$$

Exercise 11B

1 **a** $\text{SA}=x^2+2xl=20^2+2\times 20\times 26$
$=1440\,\text{cm}^2$

b $\text{SA}=x^2+2xl=4^2+2\times 4\times 6.3=66.4\,\text{cm}^2$

c $\text{SA}=x^2+2xl=5^2+2\times 5\times 13=155\,\text{cm}^2$

d $\text{SA}=\pi r^2+\pi rl=\pi\times 5^2+\pi\times 5\times 13$
$=283\,\text{cm}^2$

e $\text{SA}=\pi r^2+\pi rl=\pi\times 6^2+\pi\times 6\times 14$
$=377\,\text{cm}^2$

f $\text{SA}=\pi r^2+\pi rl=\pi\times 4^2+\pi\times 4\times 12$
$\approx 201\,\text{cm}^2$

2 **a** $SA = 4\pi r^2 = 4\pi \times 5^2 \approx 314\text{cm}^2$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 5^3 = 524\text{cm}^3$

b $SA = 4\pi r^2 = 4\pi \times \left(\frac{3}{2}\right)^2 = 28.3\text{cm}^2$

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{3}{2}\right)^3 = 14.1\text{cm}^3$

3 **a** $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}(4 \times 4 \times 12) = 64\text{cm}^3$

b $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}\left(\frac{10 \times 13.1}{2} \times 11\right) = 240\text{cm}^3$

c $V = \frac{1}{3}(\text{base area} \times \text{height})$

$= \frac{1}{3}(9 \times 7 \times 5) = 105\text{cm}^3$

4 SA_c = curved surface area,

SA_T = total surface area

a $l = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

$SA_c = \pi rl = \pi \times 5 \times 13 \approx 204\text{cm}^2$

b $SA_T = \pi r^2 + \pi rl = \pi \times 5^2 + 204.2$

$\approx 283\text{cm}^2$

c $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 5^2 \times 12 \approx 314\text{cm}^2$

5 **a** $SA_c = \frac{4\pi r^2}{2} = \frac{4\pi \times 3^2}{2} = 56.5\text{cm}^2$

b $SA_T = \frac{4\pi r^2}{2} + \pi r^2 = 56.5 + \pi \times 3^2$

$= 84.8\text{cm}^2$

c $V = \frac{4}{6}\pi r^3 = \frac{2}{3}\pi \times 3^3 = 56.5\text{cm}^3$

6 **a** $V_T = V_C + V_{HS} = \frac{1}{3}\pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$= \frac{1}{3}\pi \times 4^2 \times 10 + \frac{1}{2}\left(\frac{4}{3}\pi \times 4^3\right) = 302\text{cm}^3$

b Sum of the curved surface area of the cone, SA_{cc}, and curved surface area of the hemisphere, SA_{CHS}

$l = \sqrt{10^2 + 4^2} \approx 10.8cm$

$SA_T = SA_{CC} + SA_{CHS} = \pi rl + 2\pi r^2$

$= \pi \times 4 \times 10.8 + 2\pi \times 4^2 = 236\text{cm}^2$

7 Volume of the water tank, V_T

$V_T = \pi r^2 h_{cyl} + \frac{1}{3}\pi r^2 h_{cone}$

$= \pi \times 1^2 \times (13 - 2) + \frac{1}{3}\pi \times 1^2 \times 2$

$= 36.7\text{m}^3$

Conversion to litres

$36.7\text{m}^3 \times \frac{1000000\text{cm}^3}{1\text{m}^3} \times \frac{1\text{L}}{1000\text{cm}^3}$

$= 36700\text{L}$

8 Volume of each ball,

$V_{ball} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times (3.35)^3 = 157.5\ \text{cm}^3$.

$h_{cyl} = 3 \times 6.7 = 20.1\text{cm}$

$V_{cyl} = \pi r^2 h = \pi \times 3.35^2 \times 20.1 = 708.7\text{cm}^3$

total free space is

$V_T = V_{cyl} - 3V_B \approx 708.7 - 3 \times 157.5 = 236\ \text{cm}^3$

Exercise 11C

1 **a** $\sin\theta = \frac{19}{27}, \theta = \sin^{-1}\frac{19}{27} = 44.7°$

b $\tan\theta = \frac{33}{56}, \theta = \tan^{-1}\frac{33}{56} = 30.5°$

c $\tan\theta = \frac{12}{5}, \theta = \tan^{-1}\frac{12}{5} = 67.4°$

d $\cos\theta = \frac{11}{20}, \theta = \cos^{-1}\frac{11}{20} = 56.6°$

2 **a** $\cos 22° = \frac{x}{27}, x = 27\cos 22° = 25.0$

b $\tan 46° = \frac{44}{x}, x = \frac{44}{\tan 46°} = 42.5$

c $\tan 46° = \frac{7}{x}, x = \frac{7}{\tan 46°} = 6.76$

d $\sin 43° = \frac{x}{22}, x = 22\sin 43° = 15.0$

3 $\tan 25° = \frac{h}{12}, h = 12\tan 25° = 5.60$

$\tan a = \frac{5.6}{10}, a = \tan^{-1}\frac{5.6}{10} = 29.2°$

4 $\tan 30° = \frac{q}{55}$,

$q = 55\tan 30° = \frac{55}{\sqrt{3}} = 31.75\ldots \approx 31.8$

$\tan 50° = \frac{q}{(55 - p)} = \frac{31.75}{(55 - p)}$

$(55 - p)\tan 50° = 31.75$

$(55 - p) = \frac{31.75}{\tan 50°}$

$p = 55 - \frac{31.75}{\tan 50°} \approx 28.4$

5 $\cos 20° = \frac{7}{x}$, $x = \frac{7}{\cos 20°} = 7.45$

6 $\tan 30° = \frac{6}{h}$, $h = \frac{6}{\tan 30°} = 10.4$

7 **a** $\frac{1}{3} \times 20 \times 15 \times 30 = 3000\text{cm}^3$

b $AC = 2NC = \sqrt{20^2 + 15^2} = 25$. Then $NC = 12.5\text{cm}$

Then $TC = \sqrt{30^2 + NC^2} = \sqrt{30^2 + 12.5^2}$
$= \sqrt{1056.25} = 32.5\text{cm}$

c $\sin\theta = \frac{30}{32.5}$, $\theta = \sin^{-1}\frac{30}{32.5} = 67.4°$

d $\text{TM} = \sqrt{\text{TC}^2 - \text{MC}^2} = \sqrt{32.5^2 - 7.5^2} = 31.6$

e $\sin \text{TMN} = \frac{30}{\text{TM}} = \frac{30}{31.6}$,

$\text{TMN} = \sin^{-1}\frac{30}{31.6} = 71.7°$

8 **a** $V = \frac{1}{3} \times 230.4 \times 230.4 \times 146.5$

$= 2592276.5\text{cm}^3$

b $AB = 2OM$,

$EM = \sqrt{146.5^2 + OM^2}$

$= \sqrt{146.5^2 + 115.2^2} \approx 186\text{cm}$

c $\tan EMO = \frac{EO}{OM} = \frac{146.5}{115.2}$,

$EMO = \tan^{-1}\frac{146.5}{115.2} = 51.8° \approx 52°$

d $A_T = 4 \times A_F + A_B$, $A_F = \frac{1}{2} \times AB \times EM$

$A_T = 2 \times 230.4 \times 186.4 + 230.4 \times 230.4$

$= 139000\text{cm}^2$

e $EB = \sqrt{EM^2 + BM^2}$

$= \sqrt{186.4^2 + 115.2^2} = 219.1$

$\sin EBO = \frac{EO}{EB} = \frac{146.5}{219.1}$

$\sin^{-1}\frac{146.5}{219.1} = 42.0°$

Exercise 11D

1 $\tan 25° = \frac{h}{12}$

$h = 12\tan 25° = 5.60\text{m}$

2 $\sin 55° = \frac{h}{50}$

$h = 50\sin 55° = 41\text{m}$

3 **a** $\sin\theta = \frac{0.8}{3}$

$\theta = \sin^{-1}\frac{0.8}{3} = 15.5°$

b $h = \sqrt{3^2 - 0.8^2} = 2.89\text{m}$

4 $\tan 40° = \frac{81.5}{d}$

$d = \frac{81.5}{\tan 40°} = 97.1\text{m}$

5 R and J are the same height, so it cancels out. Then the calculation is

$\tan 70° = \frac{H}{3}$

$H = 3\tan 70° = 8.24\text{m}$

6 **a** $\cos 36° = \frac{N}{25}$

$N = 25\cos 36° = 20.2\text{km}$

b $\sin 36° = \frac{W}{25}$

$W = 25\sin 36° = 14.7\,km$

7 $\sin 68° = \frac{W}{51}$

$W = 51\sin 68° = 47.3\text{km}$

8 $H = H_E + H_D$

$\tan 23° = \frac{H_E}{300}$

$H_E = 300\ \tan 23° = 127.3\text{m}$

$\tan 30° = \frac{H_D}{300}$

$H_D = 300\tan 30° = 173.2\text{m}$

$H = H_E + H_D = 301\text{m}$

9 Let C be the bottom of the Eiffel tower. Then

$\text{AB} = \text{ABC} - \text{BC}$

$40° + 32° + \theta = 90°$

$\theta = 18°$

$\tan(32° + 18°) = \frac{\text{ABC}}{300}$

$\text{ABC} = 300\tan 50° = 357.53$

$\tan 18° = \frac{\text{BC}}{300}$

$\text{BC} = 300\tan 18° = 97.48$

$\text{AB} = \text{ABC} - \text{BC} = 260\text{m}$

10 $\tan 75° = \frac{\text{D} + 1.5}{498}$

$D = 498\tan 75° - 1.5 = 1857\text{m}$

Exercise 11 E

1 $\text{Area} = \frac{1}{2}ab\sin C$

a $\text{Area} = \frac{1}{2} \times 8 \times 6 \times \sin 80°$

$\text{Area} = 24\sin 80° = 23.6\text{cm}^2$

b $\text{Area} = \frac{1}{2} \times 10 \times 15 \times \sin 125°$

$\text{Area} = 75\sin 125° = 61.4\text{cm}^2$

c $\text{Area} = \frac{1}{2} \times 2.5 \times 3.9 \times \sin 34°$

$\text{Area} = 4.875\sin 34° = 2.73\text{cm}^2$

d $\text{Area} = \frac{1}{2} \times 4 \times 7 \times \sin 96°$

$\text{Area} = 14\sin 96° = 13.9\text{cm}^2$

e $\text{Area} = \frac{1}{2} \times 12 \times 20 \times \sin(180° - 80° - 40°)$

$\text{Area} = 120\sin 60° = 104\text{cm}^2$

f $\text{Area} = \frac{1}{2} \times 14 \times 18 \times \sin(180° - 78° - 60°)$

$\text{Area} = 126\sin 42° = 84.3\text{cm}^2$

g $\text{Area} = \frac{1}{2} \times 12 \times 8 \times \sin(180° - 30° - 67°)$

$\text{Area} = 48\sin 83° = 47.6\text{cm}^2$

2 $\text{Area} = \frac{1}{2}ab\sin C$

a $\text{Area} = \frac{1}{2} \times 8 \times 5 \times \sin 39°$

$\text{Area} = 20\sin 39° = 12.6\text{cm}^2$

b $16 = \frac{1}{2} \times 8 \times 8 \times \sin C$

$\sin C = \frac{1}{2}$

$C = \sin^{-1}\frac{1}{2} = 30°$

3 $\text{Area} = 2 \times \frac{1}{2}ab\sin C$

$\text{Area} = 20 \times 12 \times \sin 60°$

$\text{Area} = 240\sin 60° = 208\text{cm}^2$

4 4 faces, so area is multiplied by 4

3 angles in an equilateral triangle are $60°$

$\text{Area} = 4 \times \frac{1}{2}ab\sin C$

$\text{Area} = 2 \times 10 \times 10 \times \sin 60°$

$\text{Area} = 173\text{cm}^2$

5 $\text{Area} = 5 \times \frac{1}{2}ab\sin C$

$\text{Area} = 5 \times \frac{1}{2} \times 4 \times 4 \times \sin 72° = 38.0\text{m}^2$

6 $\frac{1}{2}(x+2)(2x+1)\sin 60° = 5\sqrt{3}$

$(x+2)(2x+1) = \frac{10\sqrt{3}}{\frac{\sqrt{3}}{2}} = 20$

$2x^2 + x + 4x + 2 = 20$

$2x^2 + 5x - 18 = 0$

Either factorise, or use the quadratic formula:

$$x_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4 \times 2 \times (-18)}}{2 \times 2}$$

$$= \frac{-5 \pm \sqrt{25 + 144}}{4}$$

$$x_{1,2} = \frac{-5 \pm 13}{4}$$

We take the positive value, as distances cannot be negative.

Then $x = 2$

Exercise 11F

1 a $\frac{\sin\theta}{23} = \frac{\sin 35°}{45}$

$\sin\theta = \frac{23\sin 35°}{45}$

$\theta = \sin^{-1}\frac{23\sin 35°}{45} = 17.0°$

b $\frac{\sin\theta}{4} = \frac{\sin 66°}{8}$

$\sin\theta = \frac{4\sin 66°}{8}$

$\theta = \sin^{-1}\frac{4\sin 66°}{8} = 27.2°$

c $\frac{\sin\theta}{6} = \frac{\sin 75°}{18}$

$\sin\theta = \frac{6\sin 75°}{18}$

$\theta = \sin^{-1}\frac{6\sin 75°}{18} = 18.8°$

d $\frac{\sin\theta}{22} = \frac{\sin 48°}{63}$

$\sin\theta = \frac{22\sin 48°}{63}$

$$\theta = \sin^{-1}\frac{22\sin 48^\circ}{63} = 15.0^\circ$$

e $\dfrac{\sin\theta}{20} = \dfrac{\sin 82^\circ}{29}$

$$\sin\theta = \frac{20\sin 82^\circ}{29}$$

$$\theta = \sin^{-1}\frac{20\sin 82^\circ}{29} = 43.1^\circ$$

f $\dfrac{\sin\theta}{18} = \dfrac{\sin 78^\circ}{34}$

$$\sin\theta = \frac{18\sin 78^\circ}{34}$$

$$\theta = \sin^{-1}\frac{18\sin 78^\circ}{34} = 31.2^\circ$$

2 a $\dfrac{\sin 99^\circ}{37} = \dfrac{\sin(180^\circ - 99^\circ - 18^\circ)}{x}$

$$x = 37\frac{\sin 63^\circ}{\sin 99^\circ} = 33.4\,\text{cm}$$

b $\dfrac{\sin 53^\circ}{x} = \dfrac{\sin 44^\circ}{7}$

$$x = 7\frac{\sin 53^\circ}{\sin 44^\circ} = 8.05$$

c $\dfrac{\sin 23^\circ}{x} = \dfrac{\sin 77^\circ}{15}$

$$x = 15\frac{\sin 23^\circ}{\sin 77^\circ} = 6.02$$

d $\dfrac{\sin 33^\circ}{x} = \dfrac{\sin 108^\circ}{24}$

$$x = 24\frac{\sin 33^\circ}{\sin 108^\circ} = 13.7$$

e $\dfrac{\sin(180^\circ - 100^\circ - 22^\circ)}{x} = \dfrac{\sin 22^\circ}{10}$

$$x = 10\frac{\sin 58^\circ}{\sin 22^\circ} = 22.6$$

f $\dfrac{\sin(180^\circ - 52^\circ - 56^\circ)}{x} = \dfrac{\sin 52^\circ}{6}$

$$x = 6\frac{\sin 72^\circ}{\sin 52^\circ} = 7.24$$

3 base $= 15 + x$

We have that $\dfrac{\sin 70^\circ}{h} = \dfrac{\sin 20^\circ}{x}$

and $\dfrac{\sin 40^\circ}{h} = \dfrac{\sin 50^\circ}{15 + x}$

Then $15 + x = \dfrac{\sin 50^\circ}{\sin 40^\circ}h$

$$x = \frac{\sin 50^\circ}{\sin 40^\circ}h - 15$$

We substitute back into the first equation

$$h = \frac{\sin 70^\circ}{\sin 20^\circ}x$$

so $h = \dfrac{\sin 70^\circ}{\sin 20^\circ}\left(\dfrac{\sin 50^\circ}{\sin 40^\circ}h - 15\right)$

$$\left(1 - \frac{\sin 70^\circ}{\sin 20^\circ}\left(\frac{\sin 50^\circ}{\sin 40^\circ}\right)\right)h = -15 \times \frac{\sin 70^\circ}{\sin 20^\circ}$$

$$h = \frac{-15 \times \dfrac{\sin 70^\circ}{\sin 20^\circ}}{1 - \dfrac{\sin 70^\circ}{\sin 20^\circ}\left(\dfrac{\sin 50^\circ}{\sin 40^\circ}\right)} = 18.1$$

4 $\dfrac{\sin 74^\circ}{10} = \dfrac{\sin 64^\circ}{x}$

$$x = \frac{10\sin 64^\circ}{\sin 74^\circ} = 9.35\text{km}$$

5 We have $\dfrac{\sin 49^\circ}{d} = \dfrac{\sin 41^\circ}{20 - b}$

$$20 - b = d\frac{\sin 41^\circ}{\sin 49^\circ}$$

$$b = 20 - d\frac{\sin 41^\circ}{\sin 49^\circ}$$

and $\dfrac{\sin 38^\circ}{d} = \dfrac{\sin 52^\circ}{b}$

$$b = d\frac{\sin 52^\circ}{\sin 38^\circ}$$

We equate both expressions for b

$$20 - d\frac{\sin 41^\circ}{\sin 49^\circ} = d\frac{\sin 52^\circ}{\sin 38^\circ}$$

$$20 = d\left(\frac{\sin 41^\circ}{\sin 49^\circ} + \frac{\sin 52^\circ}{\sin 38^\circ}\right)$$

$$d = \frac{20}{\dfrac{\sin 41^\circ}{\sin 49^\circ} + \dfrac{\sin 52^\circ}{\sin 38^\circ}} = 9.31\text{km}$$

6 We have $\dfrac{\sin 74^\circ}{16 + AD} = \dfrac{\sin 16^\circ}{DC}$

$$16 + AD = DC\frac{\sin 74^\circ}{\sin 16^\circ}$$

$$AD = DC\frac{\sin 74^\circ}{\sin 16^\circ} - 16$$

and $\dfrac{\sin 62^\circ}{AD} = \dfrac{\sin 28^\circ}{DC}$

$$AD = DC\frac{\sin 62^\circ}{\sin 28^\circ}$$

We equate both expressions for AD

$$DC\frac{\sin 74^\circ}{\sin 16^\circ} - 16 = DC\frac{\sin 62^\circ}{\sin 28^\circ}$$

$$DC\left(\frac{\sin 74^\circ}{\sin 16^\circ} - \frac{\sin 62^\circ}{\sin 28^\circ}\right) = 16$$

$$DC = \frac{16}{\frac{\sin 74^\circ}{\sin 16^\circ} - \frac{\sin 62^\circ}{\sin 28^\circ}} = 9.96\text{km}$$

7 We have $\frac{\sin 15^\circ}{h} = \frac{\sin 75^\circ}{10+d}$

$10 + d = h\frac{\sin 75^\circ}{\sin 15^\circ} = 3.73h$

$d = 3.73h - 10$

and $\frac{\sin 18^\circ}{h} = \frac{\sin 72^\circ}{d}$

$d = h\frac{\sin 72^\circ}{\sin 18^\circ} = 3.08h$

Then $3.08h = 3.73h - 10$

$3.73h - 3.08h = 10$

$0.65h = 10$

$h = 15.3$

8 We have $\frac{\sin 55^\circ}{h} = \frac{\sin 90^\circ}{4}$

$h = 4\sin 55^\circ = 3.27660..\text{mm}$

and $\frac{\sin 78^\circ}{4} = \frac{\sin 47^\circ}{b}$

$b = 4\frac{\sin 47^\circ}{\sin 78^\circ} = 2.99077..\text{mm}$

Then

$$Area = \frac{1}{2}bh = \frac{1}{2} \times 3.27660 \times 2.99077 = 4.90 \text{ mm}^2$$

Exercise 11G

1 $\frac{\sin 64^\circ}{10} = \frac{\sin A}{8}$

$\sin A = \frac{8}{10}\sin 64^\circ$

$A = \sin^{-1}\left(\frac{8}{10}\sin 64^\circ\right) = 46.0^\circ$

and $180^\circ - 46.0^\circ = 134^\circ$

2 $\frac{\sin 20^\circ}{3} = \frac{\sin A}{5}$

$\sin A = \frac{5}{3}\sin 20^\circ$

$C = \sin^{-1}\left(\frac{5}{3}\sin 20^\circ\right) = 34.8^\circ$

and $180^\circ - 34.8^\circ = 145.2^\circ$

3 $\frac{\sin 45^\circ}{8} = \frac{\sin B}{10}$

$\sin B = \frac{10}{8}\sin 45^\circ$

$B = \sin^{-1}\left(\frac{10}{8}\sin 45^\circ\right) = 62.1^\circ$

and $180^\circ - 62.1^\circ = 117.9^\circ$

4 $\frac{\sin 40^\circ}{24} = \frac{\sin C}{30}$

$\sin C = \frac{30}{24}\sin 40^\circ$

$C = \sin^{-1}\left(\frac{30}{24}\sin 40^\circ\right) = 53.5^\circ$

and $180^\circ - 53.5^\circ = 126.5^\circ$

5 $Area = \frac{1}{2} \times AB \times BC \times \sin B = 20$

$\frac{1}{2} \times 8 \times 10 \times \sin B = 20$

$\sin B = \frac{20}{40}$

$B = \sin^{-1}\frac{20}{40} = 30^\circ$

The obtuse angle is $180^\circ - 30^\circ = 150^\circ$

Exercise 11H

1 a $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 12^2 + 9^2 - 2 \times 12 \times 9 \times \cos 62^\circ$

$a = 11.1\text{cm}$

b $b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 15^2 + 28^2 - 2 \times 15 \times 28 \times \cos 112^\circ$

$b = 36.4\text{cm}$

c $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 14^2 + 22^2 - 2 \times 14 \times 22 \times \cos 80^\circ$

$a = 23.9\text{m}$

d $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 10^2 + 9^2 - 2 \times 10 \times 9 \times \cos 66^\circ$

$c = 10.4\text{m}$

e $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 40^2 + 25^2 - 2 \times 40 \times 25 \times \cos 20^\circ$

$a = 18.6\text{cm}$

f $a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 21^2 + 30^2 - 2 \times 21 \times 30 \times \cos 123^\circ$

$a = 45.0\text{cm}$

2 a $\cos\theta = \frac{10.4^2 + 18^2 - 21.9^2}{2 \times 10.4 \times 18}$

$\theta = \cos^{-1}(-0.1267) = 97.3^\circ$

b $\cos\theta = \frac{8.6^2 + 3.1^2 - 9.7^2}{2 \times 8.6 \times 3.1}$

$\theta = \cos^{-1}(-0.197299) = 101°$

c $\cos\theta = \dfrac{65^2 + 55^2 - 118^2}{2 \times 65 \times 55}$

$\theta = \cos^{-1}\dfrac{-3337}{3575} = 159°$

d $\cos\theta = \dfrac{5^2 + 5^2 - 3^2}{2 \times 5 \times 5}$

$\theta = \cos^{-1} 0.82 = 34.9°$

e $\cos\theta = \dfrac{24^2 + 22^2 - 20^2}{2 \times 24 \times 22}$

$\theta = \cos^{-1} 0.625 = 51.3°$

f $\cos\theta = \dfrac{3.8^2 + 7^2 - 4^2}{2 \times 3.8 \times 7}$

$\theta = \cos^{-1} 0.891729 = 26.9°$

3 a $\cos\theta = \dfrac{9^2 + 12^2 - 6^2}{2 \times 9 \times 12}$

$\theta = \cos^{-1} 0.875 = 29.0°$

b $A = \dfrac{1}{2}ab\sin C$

$A = \dfrac{1}{2} \times 12 \times 9 \times \sin 29.0° = 26.1\text{cm}^2$

4 $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 60^2 + 30^2 - 2 \times 60 \times 30 \times \cos 160°$

$c = 89\text{km}$

5 a $\tan 33° = \dfrac{h_1}{46}$

$h_1 = 46\tan 33° = 29.9\text{m}$

and $\tan 17° = \dfrac{h_2}{28}$

$h_2 = 28\tan 17° = 8.56\text{m}$

b $A = 180° - 33° - 17° = 130°$

$b = \sqrt{46^2 + 29.87^2} = 54.9$

$c = \sqrt{28^2 + 8.56^2} = 29.3$

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 54.9^2 + 29.3^2$
$- 2 \times 54.9 \times 29.3 \times \cos 130°$

$a = 77.0\text{m}$

6 $C = 210° - 70° = 140°$

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 9^2 + 15^2 - 2 \times 9 \times 15 \times \cos 140°$

$c = 22.6\text{km}$

Exercise 11I

1 $\cos A = \dfrac{20^2 + 12^2 - 14^2}{2 \times 20 \times 12}$

$A = \cos^{-1} 0.725 = 43.53°$

$Area = \dfrac{1}{2}bc\sin A$

$= \dfrac{1}{2} \times 20 \times 12 \times \sin 43.5° = 82.6\text{cm}^2$

2 $\sin 7° = \dfrac{150 \times 10^6}{d}$

$d = \dfrac{150 \times 10^6}{\sin 7°} = 1230.8$ million km

3 a $PR^2 = PS^2 + RS^2 - 2 \times PS \times RS \times \cos S$

$PR^2 = 14^2 + 11^2 - 2 \times 14 \times 11 \times \cos 55°$

$PR = 11.8\text{m}$

b $\dfrac{\sin PSR}{PR} = \dfrac{\sin PRS}{PS}$

$\dfrac{\sin 55°}{11.84..} = \dfrac{\sin PRS}{14}$

$\sin PRS = 14 \times \dfrac{\sin 55°}{11.84..}$

$P\hat{R}S = \sin^{-1}\left(14 \times \dfrac{\sin 55°}{11.84..}\right) = 75.4809..°$

$P\hat{R}Q = 180° - 75.4809..° = 104.519...°$

$Q\hat{P}R = 180° - 50° - 104.519..° = 25.4809...°$

$\dfrac{\sin P\hat{Q}R}{PR} = \dfrac{\sin Q\hat{P}R}{QR}$

$\dfrac{\sin 50°}{11.84...} = \dfrac{\sin 25.4809..°}{QR}$

$QR = 11.84.. \times \dfrac{\sin 25.4809..°}{\sin 50°} = 6.65\text{m}$

c $A = \dfrac{1}{2} \times QS \times PS \times \sin S$

$= \dfrac{1}{2} \times (11 + 6.68) \times 14 \times \sin 50° = 94.81\text{m}^2$

4 a $\cos ADB = \dfrac{DB^2 + DA^2 - BA^2}{2 \times DB \times DA}$

$\cos ADB = \dfrac{12^2 + 20^2 - 28^2}{2 \times 12 \times 20}$

$ADB = \cos^{-1} - 0.5 = 120°$

b $Area = \dfrac{1}{2} \times BD \times DA \times \sin ADB$

$Area = \dfrac{1}{2} \times 12 \times 20 \times \sin 120° = 104m^2$.

c $\dfrac{\sin DCB}{BD} = \dfrac{\sin BDC}{BC}$

$$\frac{\sin DCB}{12} = \frac{\sin 60^\circ}{13}$$

$$\sin DCB = \frac{12}{13}\sin 60^\circ$$

$$DCB = \sin^{-1}\frac{12}{13}\sin 60^\circ = 53.1^\circ$$

d $CBD = 180^\circ - BCD - BDC$

$$= 180^\circ - 53.1^\circ - 60^\circ = 66.9^\circ$$

$$\frac{\sin BAD}{BD} = \frac{\sin ADB}{AB}$$

$$\frac{\sin BAD}{12} = \frac{\sin 120^\circ}{28}$$

$$\sin BAD = \frac{12}{28}\sin 120^\circ$$

$$BAD = \sin^{-1}\frac{12}{28}\sin 120^\circ = 21.79^\circ$$

Then

$ABD = 180^\circ - ADB - BAD$

$= 180^\circ - 120^\circ - 21.79^\circ = 38.2^\circ$

and so

$ABC = CBD + ABD = 66.9^\circ + 38.2^\circ$

$= 105.1^\circ \neq 90^\circ$

5 a $$\frac{\sin ABC}{AC} = \frac{\sin ACB}{AB}$$

$$\frac{\sin 46^\circ}{48} = \frac{\sin ACB}{22.5}$$

$$\sin ACB = \frac{22.5}{48}\sin 46^\circ$$

$$ACB = \sin^{-1}\frac{22.5}{48}\sin 46^\circ = 19.7^\circ$$

b $BAC = 180^\circ - ABC - ACB$

$= 180^\circ - 46^\circ - 19.71^\circ = 114.3^\circ$

$BC^2 = AC^2 + AB^2$
$\quad - 2 \times AC \times AB \times \cos BAC$

$BC^2 = 22.5^2 + 48^2$
$\quad - 2 \times 22.5 \times 48 \times \cos 114.3^\circ$

$BC = 60.8\text{m}$

6 a Let $AC=b$

$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 5^2 + 4^2 - 2 \times 5 \times 4 \times \cos 95^\circ$

$b = 6.67\text{cm}$

b Let BAC = A. Then

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin 95^\circ}{6.67} = \frac{\sin A}{5}$$

$$\sin A = \frac{5}{6.67}\sin 95^\circ$$

$$A = \sin^{-1}\left(\frac{5}{6.67}\sin 95^\circ\right) = 48.3^\circ$$

c $ACD = 180^\circ - 32^\circ - 48.3^\circ = 99.7^\circ$

d Let C=AD. Then

$$\frac{\sin D}{d} = \frac{\sin C}{c}$$

$$\frac{\sin 48.3..^\circ}{6.67..} = \frac{\sin 99.7..^\circ}{c}$$

$$c = 6.67..\frac{\sin 99.7..^\circ}{\sin 48.3..^\circ} = 8.8039..$$

e $$A_{ABC} = \frac{1}{2} \times 4 \times 5 \times \sin 95^\circ = 9.96\text{cm}^2$$

$$A_{ACD} = \frac{1}{2} \times 6.67.. \times 8.80.. \times \sin 32^\circ = 15.55..\text{cm}^2$$

$$A_{ABCD} = A_{ABC} + A_{ACD} = 9.96.. + 15.55.. = 25.5\text{cm}^2$$
(3 s.f.)

7 $$\tan 50^\circ = \frac{h}{x}$$

$$x = \frac{h}{\tan 50^\circ}$$

and $\tan 60^\circ = \dfrac{h}{10 - x}$

$$(10 - x)\tan 60^\circ = h$$

$$10 - x = \frac{h}{\tan 60^\circ}$$

$$x = 10 - \frac{h}{\tan 60^\circ}$$

We equate both expressions for x and get

$$\frac{h}{\tan 50^\circ} = 10 - \frac{h}{\tan 60^\circ}$$

$$h\left(\frac{1}{\tan 50^\circ} + \frac{1}{\tan 60^\circ}\right) = 10$$

$$h = \frac{10}{\left(\frac{1}{\tan 50^\circ} + \frac{1}{\tan 60^\circ}\right)} = 7.06\text{m}$$

8 a $180^\circ - 67^\circ = 113^\circ$

$113^\circ + 123^\circ + ABC = 360^\circ$

$ABC = 124^\circ$

b $b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = 80^2 + 120^2 - 2 \times 80 \times 120 \times \cos 124^\circ$

$b = 178\text{km}$

c $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{120^2 + 178^2 - 80^2}{2 \times 120 \times 178} = 0.9289$

$\cos^{-1} 0.9289 = 21.7°$

complement to $123°$ is $57°$, then

$360° - 57° - 21.73° = 281°$

9 We use lowercase letters for sides opposite capital letter angles. Find p

Complementary angle to $84°$:

$180° - 84° = 96°$.

Then $H\hat{P}Q = 360° - 210° - 96° = 54°$

We have two sides and one angle

a $p^2 = q^2 + h^2 - 2hq \cos H\hat{P}Q$

$p^2 = 340^2 + 160^2 - 2 \times 340 \times 160 \times \cos 54°$

$p = 278\text{km}$

b We find H as

$\cos H = \dfrac{p^2 + q^2 - h^2}{2pq}$

$= \dfrac{278^2 + 340^2 - 160^2}{2 \times 278 \times 340}$

$= 0.884913$

$H = \cos^{-1} 0.884913 = 27.8°$

Then $84° + 27.8° = 111.8°$ is B^c, the complement of the angle complementary to the bearing B.

$B^c = 180° - 111.8° = 68.2°$

Then $B = 360° - 68.2° = 292°$

10 Triangle ABC with sides a, b, c

$B = 360° - (180° - 30°) - 100° = 110°$

$b^2 = a^2 + c^2 - 2ac \cos B$

$b^2 = 320^2 + 500^2 - 2 \times 320 \times 500 \times \cos 110°$

$b = 680\text{km}$

Then

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$= \dfrac{680^2 + 500^2 - 320^2}{2 \times 680 \times 500}$

$= 0.897$

$A = \cos^{-1} 0.897 = 26.2°$

Bearing: $360° - (180° - 30° - A) = 236°$

11 a Let M be the midpoint of AD. Then triangle OMD is right-angled with OM=5, MD=5, then

$OD = \sqrt{OM^2 + MD^2} = 7.07\text{cm}$

Then

$VD = \sqrt{OD^2 + VO^2}$

$= \sqrt{7.07^2 + 20^2} = 21.2\text{cm}$

b $\tan\alpha = \dfrac{VO}{OM}$

$\alpha = \tan^{-1}\dfrac{20}{5} = 76.0°$

c Let K be the point that connects A with OA perpendicularly. Let M denote the midpoint of BA

$\cos OAM = \dfrac{OA^2 + MA^2 - OM^2}{2 \times OA \times MA}$

$= \dfrac{21.2^2 + 5^2 - 20.6^2}{2 \times 21.2 \times 5}$

$OAM = \cos^{-1} 0.236226 = 76.3°$

Then $\sin BAK = \dfrac{BK}{AB}$

$BK = AB \sin BAK = 10 \times \sin 76.3° = 9.72$

The angle between two sloping edges, β is formed by two sides of length BK and the diagonal of the base

$\cos\beta = \dfrac{9.72^2 + 9.72^2 - 14.4^2}{2 \times 9.72 \times 9.72}$

$\beta = \cos^{-1}\dfrac{9.72^2 + 9.72^2 - 14.4^2}{2 \times 9.72 \times 9.72} = 95.6°$

12 a $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{6^2 + 8^2 - 12^2}{2 \times 6 \times 8} = \dfrac{-11}{24}$

b The cosine of the angle is negative, so $ABC > 90°$, i.e. we have an obtuse angle.

Chapter Review

1 $a = \dfrac{1}{3}(\textit{base area} \times \textit{height})$

$= \dfrac{1}{3}(8 \times 8 \times 3) = 64 \text{ m}^3$

Slant height l is the hypotenuse of the triangle formed by the pyramid height and the distance from the origin O to the midpoint of a side of the base.

$l = \sqrt{3^2 + 4^2} = 5$

$b = x^2 + 2xl = 8^2 + 2 \times 8 \times 5 = 144\text{m}^2$

2 $a = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi \times 6^2 \times 8 = 96\pi\,\text{cm}^3$

slant height l is the hypotenuse of the triangle formed by the cone height and the cone radius

$l = \sqrt{8^2 + 6^2} = 10$

$b = \pi r^2 + \pi r l = \pi \times 6^2 + \pi \times 6 \times 10 = 96\pi\,\text{cm}^2$

3 $V = \frac{4}{3}\pi r^3 = \frac{32}{3}\pi$

$\frac{4}{3}r^3 = \frac{32}{3}$

$4r^3 = 32$

$r^3 = 8$

$r = 2$

Then $SA = 4\pi r^2 = 4\pi \times 2^2 = 16\pi \text{m}^2$

4 a $V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times 4^2 \times 10$

$= \frac{160\pi}{3} = 168\text{cm}^3$

b $V_{tr} = V_{cone} - V_{cut}$

$V_{cut} = \frac{1}{3}\pi r_{cut}^2 h_{cut} = \frac{1}{3}\pi \times 2^2 \times (10-6)$

$= \frac{16\pi}{3} = 16.8\text{cm}^3$

$V_{tr} = 168 - 16.8 = 151\text{cm}^3$

5 a $d = 2r$

$65 = 2r$

$r = \frac{65}{2}$

$r = 32.5\text{mm} = 3.25 \text{ cm}$

$V = \pi r^2 h = \pi \times 3.25^2 \times 39 = 1294.14\text{cm}^3$

b Each ball has a diameter of $2 \times 3.25 = 6.5\text{cm}$

$\frac{h}{6.5} = \frac{39}{6.5} = 6$

6 tennis balls fit in the cylinder

c $V_{air} = V_{cyl} - 6V_{ball}$

$V_{air} = 1294.14 - 6\left(\frac{4}{3}\pi \times 3.25^3\right)$

$= 431.38.. = 431\text{cm}^3$ (3 s.f.)

d $431.38..\text{cm}^3 \times \frac{1\text{m}^3}{1000000\text{cm}^3}$

$= 0.431 \times 10^{-3} \text{ m}^3$

6 a $V_T = V_{cyl} + V_{sph}$

$r = \frac{d}{2} = \frac{3}{2} = 1.5\text{m}$

$V_T = \pi r^2 h + \frac{4}{3}\pi r^3$

$= \pi \times 1.5^2 \times 8.5 + \frac{4}{3}\pi \times 1.5^3 = 74.2\text{m}^3$

b $SA = h \times 2\pi r + 4\pi r^2$

$= 8.5 \times 2\pi \times 1.5 + 4\pi \times 1.5^2 = 108\text{cm}^2$

7 a $V = \frac{1}{3}(base\,area \times height)$

$= \frac{1}{3} \times 6 \times 6 \times 3 = 36 \text{ cm}^3$

b $W = 12 \times V = 12 \times 36 = 432\text{grams}$

c $OC = \frac{1}{2}AC$

$AC = \sqrt{AB^2 + BC^2} = \sqrt{6^2 + 6^2} = 8.49\text{cm}$

Then $OC = \frac{1}{2}AC = \frac{1}{2} \times 8.49 = 4.24\text{cm}$

Then

$VC = \sqrt{VO^2 + OC^2}$

$= \sqrt{3^2 + 4.24^2} = 5.20\text{cm}$

as required.

d We split the triangle BVC into two right-angles triangles, BVM and MVC, M is the midpoint of BC.

Then $\frac{\sin BMV}{VB} = \frac{\sin BVM}{BM}$

$\frac{\sin 90°}{5.20} = \frac{\sin BVM}{3}$

$\sin BVM = 3\frac{\sin 90°}{5.20}$

$BVM = \sin^{-1} 3\frac{\sin 90°}{5.20} = 35.2°$

$2BMV = BVC$

so $BVC = 2 \times 35.2 = 70.5°$

e Slant height

$VM = \sqrt{VB^2 - BM^2}$

$= \sqrt{5.20^2 - 3^2} = 4.25\text{cm}$

$SA = x^2 + 2xl$

$= 6^2 + 2 \times 6 \times 4.25 = 87.0\text{cm}^2$

8 a $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$d = \sqrt{(1-1)^2 + (5-0)^2 + (3-3)^2}$

$d = 5$

b Midpoint

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{1+1}{2}, \frac{5+0}{2}, \frac{3+3}{2}\right) = (1{,}2.5{,}3)$

c $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$d = \sqrt{(7-1)^2 + (\sqrt{15} - 0)^2 + (10-3)^2}$

$d = \sqrt{6^2 + 15 + 7^2} = 10$

9 $Area = \frac{1}{2}ab\sin C$

$Area = \frac{1}{2} \times 6 \times 4\sin 30°$

$= 12\sin 30° = \frac{12}{2} = 6\text{cm}^2$

10 a $r = \frac{d}{2} = \frac{16}{2} = 8\text{cm}$

$V_B = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times 8^3 = 2144.66\text{cm}^3$

Then

$V_T = 80V_B = 80 \times 2144.66$

$= 171573\text{cm}^3$

b $V_{cone} = \frac{1}{3}\pi r^2 h = 171573\text{cm}^3$

Then $\frac{1}{3}\pi \times 40^2 h = 171573$

$h = 3\frac{171573}{1600\pi} = 102\text{cm}$

11 a $\tan 60° = \frac{CD}{10}$

$CD = 10\tan 60° = 10\sqrt{3}$ m

b $\tan 30° = \frac{CD}{OA}$

$OA = \frac{10\sqrt{3}}{\tan 30°} = 30\text{m}$

$AB = OA - OB = 30 - 10 = 20\text{m}$

12 4 faces, equilateral triangles

$h = \sqrt{6^2 - 3^2} = 3\sqrt{3}$

$A = \frac{1}{2}bh = \frac{1}{2} \times 6 \times 3\sqrt{3} = 9\sqrt{3}$

$A_T = 4 \times 9\sqrt{3} = 36\sqrt{3}\text{cm}^2$

13 a $A = 2(x)(3x) + 2xh + 2(3x)h$

$= 6x^2 + 8xh$

as required.

b $6x^2 + 8xh = 600$

$8xh = 600 - 6x^2$

$h = \frac{600 - 6x^2}{8x} = \frac{300 - 3x^2}{4x}$

c $V = base\,area \times height = (3x)(x)h$

$V = 3x^2\frac{300 - 3x^2}{4x}$

$= \frac{3}{4}x(300 - 3x^2)$

$= \frac{9}{4}x(100 - x^2)$

14 Distance at angle $45°$

$\tan 45° = \frac{70}{OA}$

$OA = \frac{70}{\tan 45°} = 70\text{m}$

distance at angle $10°$

$\tan 10° = \frac{70}{OB}$

$OB = \frac{70}{\tan 10°} = 397\text{m}$

$\frac{\Delta d}{\Delta t} = \frac{397 - 70}{5} = 65.4\text{ m / min}$

$65.4\frac{\text{m}}{\text{min}} \times \frac{1\text{ km}}{1000\text{ m}} \times \frac{60\text{ min}}{1\text{ h}} = 3.92\text{ km / h}$

15 We can form a triangle with the bottom of the building, O

$ACO = 90° - 32° = 58°$

$\tan 15° = \frac{x}{CO}$

$CO = \frac{x}{\tan 15°}$

and $\tan 58° = \frac{CO}{80 + x}$

$\tan 58° = \frac{\frac{x}{\tan 15°}}{80 + x} = \frac{x}{(80 + x)\tan 15°}$

$\tan 58°\tan 15°(80 + x) = x$

$x(\tan 58°\tan 15° - 1) = -80\tan 58°\tan 15°$

$x = \frac{-80\tan 58°\tan 15°}{\tan 58°\tan 15° - 1} = 60.1\text{m}$

16 a $\cos\theta = \frac{8^2 + 7^2 - 6^2}{2 \times 8 \times 7} = 0.6875$

$\theta = \cos^{-1} 0.6875 = 46.6°$

b $A = \frac{1}{2}ab\sin C = \frac{1}{2} \times 8 \times 7 \times \sin 46.6°$

$= 20.3\text{cm}^2$

17 a $CB^2 = AC^2 + AB^2 - 2 \times AC \times AB \times \cos BAC$

$CB^2 = 15^2 + 34^2 - 2 \times 15 \times 34 \times \cos 25°$

$CB = 21.4$ m

b $ACB = 180° - 25° - 85° = 70°$

Then $\dfrac{\sin ACB}{AB} = \dfrac{\sin ABC}{AC}$

$\dfrac{\sin 70°}{34} = \dfrac{\sin 85°}{AC}$

$AC = 34\dfrac{\sin 85°}{\sin 70°} = 36.0\text{m}$

c $A = \dfrac{1}{2} \times AB \times AC \times \sin BAC$

$= \dfrac{1}{2} \times 34 \times 36.0 \times \sin 25° = 259\text{m}^2$

d $ACB = 180° - 70° = 110°$

and so $ABC = 180° - 110° - 25° = 45°$

$\dfrac{\sin ACB}{AB} = \dfrac{\sin ABC}{AC}$

$\dfrac{\sin 110°}{34} = \dfrac{\sin 45°}{AC}$

$AC = 34\dfrac{\sin 45°}{\sin 110°} = 25.6\text{m}$

18 a $\dfrac{\sin QRS}{QS} = \dfrac{\sin QSR}{QR}$

$\dfrac{\sin 42°}{QS} = \dfrac{\sin 85°}{30}$

$QS = 30\dfrac{\sin 42°}{\sin 85°} = 20.2\text{m}$

b $SQR = 180° - 85° - 42° = 53°$

$Area = \dfrac{1}{2} \times QS \times QR \times \sin SQR$

$Area = \dfrac{1}{2} \times 20.2 \times 30 \times \sin 53° = 242\text{m}^2$

c $Area = \dfrac{1}{2} \times PQ \times QS \sin\theta = 141$

$\sin\theta = \dfrac{242}{12 \times 20.2}$

$\theta = \sin^{-1}\dfrac{242}{12 \times 20.2} = 86.7°$

and the obtuse angle,
$180° - 86.7° = 93.3°$

d We choose the obtuse angle, $\theta = 93.3°$

$PS^2 = PQ^2 + QS^2 - 2 \times PQ \times QS \times \cos\theta$

$PS^2 = 12^2 + 20.2^2$
$- 2 \times 12 \times 20.2 \times \cos 93.3°$

$PS = 24.1\text{m}$

19 a $ABC = 360° - (180° - 100°) - 150°$

$= 130°$

b $\dfrac{\sin ABC}{AC} = \dfrac{\sin BCA}{AB}$

$\dfrac{\sin 130°}{100} = \dfrac{\sin BCA}{70}$

$\sin BCA = 70\dfrac{\sin 130°}{100}$

$BCA = \sin^{-1} 70\dfrac{\sin 130°}{100} = 32.4°$

Then the bearing is given by

$360° - 32.4° - (180° - 150°) = 298°$

c $CAB = 180° - 130° - 32.4° = 17.6°$

$BC^2 = AC^2 + AB^2 - 2 \times AC \times AB \times \cos CAB$

$BC^2 = 100^2 + 70^2$
$- 2 \times 100 \times 70 \times \cos 17.6°$

$BC = 39.4\text{km}$

20 a $PQA = 180° - 95° = 85°$

$QPA = 180° - 85° - 26.5° = 68.5°$

b $\dfrac{\sin PAQ}{PQ} = \dfrac{\sin QPA}{QA}$

$\dfrac{\sin 26.5°}{PQ} = \dfrac{\sin 68.5°}{119}$

$PQ = 119\dfrac{\sin 26.5°}{\sin 68.5°} = 57.1\text{m}$

c $\dfrac{\sin QPA}{QA} = \dfrac{\sin PQA}{PA}$

$\dfrac{\sin 68.5°}{119} = \dfrac{\sin 85°}{PA}$

$PA = 119\dfrac{\sin 85°}{\sin 68.5°} = 127.4\text{m}$

$\sin PAG = \dfrac{PG}{PA}$

$\sin 26.5° = \dfrac{PG}{127.4}$

$PG = 127.4 \sin 26.5° = 56.8\text{m}$

21 a $S = 4 \times \dfrac{5 \times 7}{2} + 5^2 = 95 \text{ cm}^2$ M1A1

b $h = \sqrt{7^2 - 2.5^2} = 6.54 \text{ cm}$ M1A1

$V = \dfrac{1}{3} \times 5^2 \times 6.538\ldots = 54.5 \text{ cm}^3$

M1A1

22 a $l = \sqrt{10^2 - 3^2} = 9.54\ m$ M1A1

b $S = 4 \times \dfrac{6 \times 9.538\ldots}{2}$

$= 114.47\ldots = 114 \text{ m}^2$ M1A1

c $h = 42 + \sqrt{9.538\ldots^2 - 3^2}$

$= 51.1 \text{ m}$ M1A1

d $\arccos\left(\dfrac{3}{10}\right) = 72.5^{\text{o}}$ M1A1

e $CP = 6\tan 60^o = 10.4$ m M1A1

23 a $l = \sqrt{5^2+3^2} = 5.83$ cm M1A1

$S = 2\times(\pi\times 3\times 5.83...) = 110\text{ cm}^2$

M1A1

b $\dfrac{2\times\frac{1}{3}\times\pi\times 3^2\times 5}{\pi\times 3.05^2\times 10.1}\times 100\% = 31.9\%$

M1A1

24 a $x = \dfrac{22-12}{2} = 5$ A1

$h = \sqrt{13^2-5^2} = 12$ M1A1

b $A = \dfrac{22+12}{2}\times 12 = 204\text{ cm}^2$ M1A1

c $C = 90^o + \sin^{-1}\left(\dfrac{5}{13}\right) = 112.6^o$ M1A1

d $AC = \sqrt{17^2+12^2} = 20.8$ cm M1A1

25 a $A\hat{B}C = 135^o$ A1

$AC = \sqrt{20^2+25^2-2\times 20\times 25\times\cos 135^o}$

M1A1

$AC = 41.6$ km A1

b $\dfrac{\sin\hat{C}}{20} = \dfrac{\sin 135^o}{41.61...}$ M1

$\hat{C} = 19.9^o$ A1

Therefore the bearing of A from point C is $360-105-19.9 = 235.1°$ M1A1

26 a $FC = \sqrt{8^2+10^2+6^2} = 10\sqrt{2}$

M1A1AG

b $M\left(\dfrac{0+8}{2},\dfrac{0+0}{2},\dfrac{6+6}{2}\right) = (4,0,6)$ M1A1

c $FM = \sqrt{4^2+0^2+6^2} = 2\sqrt{13}$ M1A1

d $CM = \sqrt{4^2+10^2+0^2} = 2\sqrt{29}$ A1

$p = 2\sqrt{13}+2\sqrt{29}+10\sqrt{2}$ cm M1A1

e $\tan M = \frac{6}{4} = \frac{3}{2}$ M1

$\cos M = \dfrac{1}{\sqrt{\frac{9}{4}+1}} = \dfrac{2\sqrt{13}}{13}$ M1A1AG

27 a $A = \dfrac{1}{2}\times 5\times 10\sin 30^o = \dfrac{25}{2}$ M1A1

b $BD^2 = 5^2+10^2-2\times 5\times 10\cos 30^o$

M1A1

$BD = \sqrt{125-50\sqrt{3}}$ A1

$BD = \sqrt{25(5-2\sqrt{3})}$ M1

$BD = 5\sqrt{5-2\sqrt{3}}$ AG

c $\dfrac{\sin C\hat{D}B}{13} = \dfrac{\sin 45^o}{5\sqrt{5-2\sqrt{3}}}$ M1A1

$\sin C\hat{D}B = \dfrac{13\sqrt{2}}{10\sqrt{5-2\sqrt{3}}}$ A1

d The angle $C\hat{D}B$ can either be acute or obtuse A1

and the two possible values add up to 180°. A1

28 a $V = \dfrac{1}{2}\times\dfrac{4\pi}{3}\times 3^3 + \pi\times 3^2\times 7$ M1A1A1

$= 81\pi = 254\text{ cm}^3$ (3 s.f.) A1

b $S = \dfrac{1}{2}\times 2\pi\times 3^2 + 2\pi\times 3\times 7 + \pi\times 3^2$

M1A1

$= 60\pi = 188\text{ cm}^2$ (3 s.f.) A1

29 a $\tan 32^o = \dfrac{30}{x}$

$\Rightarrow x = \dfrac{30}{\tan 32^o} = 48.0$ metres M1A1A1

b $AB = y = \sqrt{(3+48.0)^2+30^2}$

$= 59.2$ metres M1A1

c $arctan\left(\frac{30}{51.0}\right) = 30.5^o$ M1A1

30 a $BC = \sqrt{48^2+57^2-2\times 48\times 57\cos 117^o}$

$= 89.7$ metres M1A1A1

b $A = \dfrac{1}{2}\times 48\times 57\sin(117^o)$

$= 1219$ sq metres (3 s.f.) M1A1

c $\frac{\sin B}{48} = \frac{\sin 117^o}{89.7} \Rightarrow B = 28.5^o$ M1A1A1

12 Periodic relationships: trigonometric functions

Skills check

1 **a** $\frac{\sqrt{2}}{2}$ **b** $\sqrt{3}$ **c** $\frac{\sqrt{3}}{2}$

2 **a** $(-0.618, 0), (1, 0), (1.62, 0)$

b $(0.633, 0)$

3 **a** $(-1.61, 0.199)$

b $(2.21, 0.792)$

Exercise 12A

1 **a** $45° = \frac{45\pi}{180} = \frac{\pi}{4}$

b $60° = \frac{60\pi}{180} = \frac{\pi}{3}$

c $270° = \frac{270\pi}{180} = \frac{3\pi}{2}$

d $360° = \frac{360\pi}{180} = 2\pi$

e $18° = \frac{18\pi}{180} = \frac{\pi}{10}$

f $225° = \frac{225\pi}{180} = \frac{5\pi}{4}$

g $80° = \frac{80\pi}{180} = \frac{4\pi}{9}$

h $200° = \frac{200\pi}{180} = \frac{10\pi}{9}$

i $120° = \frac{120\pi}{180} = \frac{2\pi}{3}$

j $135° = \frac{135\pi}{180} = \frac{3\pi}{4}$

2 **a** $\frac{\pi}{6} = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$

b $\frac{\pi}{10} = \frac{\pi}{10} \times \frac{180°}{\pi} = 18°$

c $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$

d $3\pi = 3\pi \times \frac{180°}{\pi} = 540°$

e $\frac{7\pi}{20} = \frac{7\pi}{20} \times \frac{180°}{\pi} = 63°$

f $\frac{4\pi}{5} = \frac{4\pi}{5} \times \frac{180°}{\pi} = 144°$

g $\frac{7\pi}{4} = \frac{7\pi}{4} \times \frac{180°}{\pi} = 315°$

h $\frac{14\pi}{9} = \frac{14\pi}{9} \times \frac{180°}{\pi} = 280°$

i $\frac{5\pi}{3} = \frac{5\pi}{3} \times \frac{180°}{\pi} = 300°$

j $\frac{13\pi}{4} = \frac{13\pi}{4} \times \frac{180°}{\pi} = 585°$

3 **a** $10° = \frac{10\pi}{180} \approx 0.175$

b $40° = \frac{40\pi}{180} \approx 0.698$

c $25° = \frac{25\pi}{180} \approx 0.436$

d $300° = \frac{300\pi}{180} \approx 5.24$

e $110° = \frac{110\pi}{180} \approx 1.92$

f $75° = \frac{75\pi}{180} \approx 1.31$

g $85° = \frac{85\pi}{180} \approx 1.48$

h $12.8° = \frac{12.8\pi}{180} \approx 0.233$

i $37.5° = \frac{37.5\pi}{180} \approx 0.654$

j $1° = \frac{\pi}{180} \approx 0.0175$

4 **a** $1^c = 1 \times \frac{180°}{\pi} = 57.3°$

b $2^c = 2 \times \frac{180°}{\pi} = 115°$

c $0.63^c = 0.63 \times \frac{180°}{\pi} = 36.1°$

d $1.41^c = 1.41 \times \frac{180°}{\pi} = 80.8°$

e $1.55^c = 1.55 \times \frac{180°}{\pi} = 88.8°$

f $3^c = 3 \times \frac{180°}{\pi} = 172°$

g $0.36^c = 0.36 \times \frac{180°}{\pi} = 20.6°$

h $1.28^c = 1.28 \times \frac{180°}{\pi} = 73.3°$

i $0.01^c = 0.01 \times \frac{180°}{\pi} = 0.573°$

j $2.15^c = 2.15 \times \frac{180°}{\pi} = 123°$

Exercise 12B

1 a i $l = r\theta = 14 \times \frac{\pi}{2} = 7\pi\,\text{cm}$

ii $l = r\theta = 12 \times \frac{3\pi}{4} = 9\pi\,\text{cm}$

iii $l = r\theta = 3 \times \frac{5\pi}{6} = \frac{5\pi}{2}\text{cm}$

iv $l = r\theta = 15 \times \frac{14\pi}{9} = \frac{70\pi}{3}\text{cm}$

b i $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 14^2 \times \frac{\pi}{2} = 49\pi\,\text{cm}^2$

ii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 12^2 \times \frac{3\pi}{4} = 54\pi\,\text{cm}^2$

iii $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{5\pi}{6} = \frac{15\pi}{4}\text{cm}^2$

iv $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 15^2 \times \frac{14\pi}{9} = 175\pi\,\text{cm}^2$

2 $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times r^2 \times \frac{\pi}{12} = 3\pi$

$r^2 = 3\pi \times 2 \times \frac{12}{\pi} = 72$

$r = \sqrt{72} = 6\sqrt{2}\,\text{cm}$

3 a $A = \frac{1}{2}r^2\theta = 36\pi\,\text{cm}^2$

$\frac{1}{2} \times 12^2 \times \theta = 36\pi$

$\theta = \frac{36\pi \times 2}{144} = \frac{\pi}{2}$

b $l = r\theta = 12 \times \frac{\pi}{2} = 6\pi$

Then $P = 2r + l = 2 \times 12 + 6\pi = 42.8\,\text{m}$

4 Area of sector:

$A_s = \frac{1}{2}r^2\theta = \frac{1}{2} \times 10^2 \times 1.5 = 75$

Area of triangle:

$A_t = \frac{1}{2}r^2 \sin\theta = \frac{1}{2}10^2 \sin 1.5 = 49.9$

note that the angle is in radians

Area of shaded region:

$A = A_s - A_t = 75 - 49.9 = 25.1\,\text{units}^2$

5 l per second: $l = r\theta = 4 \times \frac{\pi}{12} = \frac{\pi}{3}$

60 times in a minute:

$60l = 20\pi\,\text{m}$

6 We have $\frac{1}{60}^\circ$ per minute, which in radians is $\frac{1}{60}^\circ = \frac{\pi}{60 \times 180} = \frac{\pi}{10800}$

Then $l = r\theta = 6371 \times \frac{\pi}{10800} = 1.85\,\text{km}$

Exercise 12C

1 a 130° is obtuse, hence we have a negative cosine

b 320° is obtuse, hence we have a negative sine

c $\tan 225^\circ = \frac{\sin 225^\circ}{\cos 225^\circ}$ is negative sine divided by negative cosine hence we have a positive tangent

2 a $\sin 36^\circ = \sin(180^\circ - 36^\circ) = \sin 144^\circ$

b $\sin 50^\circ = \sin(180^\circ - 50^\circ) = \sin 130^\circ$

c $\sin 85^\circ = \sin(180^\circ - 85^\circ) = \sin 95^\circ$

d $\sin 460^\circ = \sin(460^\circ - 360^\circ) = \sin 100^\circ$

e $\sin\frac{\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{2\pi}{3}$

f $\sin\frac{\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{4\pi}{5}$

g $\sin\frac{2\pi}{7} = \sin\left(\pi - \frac{2\pi}{7}\right) = \sin\frac{5\pi}{7}$

h $\sin\frac{8\pi}{3} = \sin\left(\frac{8\pi}{3} - 2\pi\right) = \sin\frac{2\pi}{3}$

3 a $\cos 40^\circ = \cos(360^\circ - 40^\circ) = \cos 320^\circ$

b $\cos 110^\circ = \cos(360^\circ - 110^\circ) = \cos 250^\circ$

c $\cos 300^\circ = \cos(360^\circ - 300^\circ) = \cos 60^\circ$

d $\cos 500^\circ = \cos(500^\circ - 360^\circ) = \cos 140^\circ$

e $\cos\frac{\pi}{8} = \cos\left(2\pi - \frac{\pi}{8}\right) = \cos\frac{15\pi}{8}$

f $\cos\frac{\pi}{10} = \cos\left(2\pi - \frac{\pi}{10}\right) = \cos\frac{19\pi}{10}$

g $\cos\frac{3\pi}{2} = \cos\left(2\pi - \frac{3\pi}{2}\right) = \cos\frac{\pi}{2}$

h $\cos\frac{9\pi}{4} = \cos\left(\frac{9\pi}{4} - 2\pi\right) = \cos\frac{\pi}{4}$

4 a $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$\sin\frac{\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{5\pi}{6}$

angles $\frac{\pi}{6}, \frac{5\pi}{6}$

b $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$\cos\frac{\pi}{4} = \cos\left(2\pi - \frac{\pi}{4}\right) = \cos\frac{7\pi}{4}$

angles $\frac{\pi}{4}, \frac{7\pi}{4}$

c $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\tan\frac{\pi}{3} = \tan\left(\frac{\pi}{3} + \pi\right) = \tan\frac{4\pi}{3}$

angles $\frac{\pi}{3}, \frac{4\pi}{3}$

5 a $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \left(\frac{8}{17}\right)^2$

$\sin\theta = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \pm\frac{15}{17}$

We take the positive value for θ acute

b $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{15}{17}}{\frac{8}{17}} = \frac{15}{8}$

Exercise 12D

1 a $\theta = \cos^{-1}0.6 = 53.1°$

and $360° - 53.1° = 306.9°$

b $\theta = \sin^{-1}0.15 = 8.63°$

and $180° - 8.63° = 171.4°$

c $\theta = \tan^{-1}0.2 = 11.3°$

and $180° + 11.3° = 191.3°$

d $\theta = \tan^{-1} - 0.76 = 322.8°$

and $322.8° - 180° = 142.8°$

e $\theta = \cos^{-1} - 0.43 = 115.5°$

and $360° - 115.5° = 244.5°$

2 a $\theta = \sin^{-1}0.82 = 0.96$

and $\pi - 0.96 = 2.18$

b $\theta = \tan^{-1} - 0.94 = 5.53$

and $5.53 - \pi = 2.39$

c $\theta = \cos^{-1} - 0.94 = 2.79$

and $2\pi - 2.79 = 3.49$

d $\theta = \cos^{-1}0.77 = 0.69$

and $2\pi - 0.69 = 5.59$

e $\theta = \sin^{-1} - 0.23 = 6.05$

and $\pi - 6.05 = -2.91 = 2\pi - 2.91 = 3.37$

3 $2\sin^2\theta + 5\sin\theta = 3$

$(\sin\theta + 3)(2\sin\theta - 1) = 0$

Then $\sin\theta + 3 = 0$

$\sin\theta = -3$

$\theta = \sin^{-1} - 3$

This is outside of the domain for the sine function. Second equation gives us

$2\sin\theta - 1 = 0$

$\sin\theta = \frac{1}{2}$

$\theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

4 a $4\cos x = 3\sin x$

$\frac{\sin x}{\cos x} = \frac{4}{3}$

$\tan x = \frac{4}{3}$

$x = \tan^{-1}\frac{4}{3} = 0.93$

and $0.93 + \pi = 4.07$

b $2\sin x + \cos x = 0$

$2\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -\frac{1}{2}$

$\tan x = -\frac{1}{2}$

$x = \tan^{-1} - \frac{1}{2} = 5.82$ and 2.68

c $\tan^2 x - \tan x - 2 = 0$

$(\tan x - 2)(\tan x + 1) = 0$

$\tan x = 2$ and $\tan x = -1$

Then $x = \tan^{-1}2 = 1.107.. = 1.11$ (3 s.f.)

and $1.107 + \pi = 4.249.. = 4.25$ (3 s.f.)

and $x = \tan^{-1}(-1) = 5.50$

and $5.50 - \pi = 2.36$

d $2\cos^2 x + \sin x = 1$

$2(1 - \sin^2 x) + \sin x = 1$

$-2\sin^2 x + \sin x + 1 = 0$

$-(\sin x - 1)(2\sin x + 1) = 0$

so $\sin x = 1$, then $x = 1.57$

and $\sin x = -\frac{1}{2}$, then $x = 5.76$ and 3.67

5 a $\cos\theta = 0.3$

$\theta = \cos^{-1}0.3 = 1.27$

and $2\pi - 1.27 = 5.02$

For $2\pi \le \theta \le 3\pi$

$2\pi + 1.27 = 7.55$

For $-\pi \le \theta \le 0$

$5.02 - 2\pi = -1.27$

b $\tan\theta = 1.61$

$\theta = \tan^{-1}1.61 = 1.01$

and $1.01 + \pi = 4.16$

For $2\pi \le \theta \le 4\pi$

$2\pi + 1.01 = 7.30$

$4.16 + 2\pi = 10.4$

c $\sin\theta = -2\cos\theta$

$\dfrac{\sin\theta}{\cos\theta} = -2$

$\tan\theta = -2$

$\theta = \tan^{-1} - 2 = -1.11$

$\pi - 1.11 = 2.03$

d $2\tan^2\theta + 5\tan\theta = 3$

$(\tan\theta + 3)(2\tan\theta - 1) = 0$

$\tan\theta = -3 \text{ and } \tan\theta = \dfrac{1}{2}$

$\theta = \tan^{-1} - 3 \text{ and } \theta = \tan^{-1}\dfrac{1}{2}$

$\theta = -1.25$

and $\theta = 0.46 = 0.46 - 2\pi = -5.82$

as well as $-1.25 - \pi = -4.39$

and $-5.82 + \pi = -2.68$

6 $3\cos x = 5\sin x$

$\dfrac{\sin x}{\cos x} = \dfrac{3}{5}$

$\tan x = \dfrac{3}{5}$

$x = \tan^{-1}\dfrac{3}{5} = 31°$

as well as $31° + 180° = 211°$

Exercise 12E

1 a $\cos^{-1}\dfrac{\sqrt{3}}{2} = 2x$

Because we have $2x$, and $-180° \le x \le 180°$, then we use $-360° \le 2x \le 360°$.

$2x = 30°, -30°, 330°, -330°$

Then $x = 15°, -15°, 165°, -165°$

b $\cos^{-1}\dfrac{\sqrt{3}}{2} = 3x$

Because we have $3x$, and $-180° \le x \le 180°$, then we use $-540° \le 3x \le 540°$.

$3x = 30°, -30°, 330°, -330°, 390°, -390°$

Then $x = 10°, -10°, 110°, -110°,$
$130°, -130°$

c $2\cos 3x - 1 = 0$

$\cos 3x = \dfrac{1}{2}$

$3x = \cos^{-1}\dfrac{1}{2}$

Because we have $3x$, and $-180° \le x \le 180°$, then we use $-540° \le 3x \le 540°$.

$3x = 60°, -60°, 300°, -300°, 420°, -420°$

Then $x = 20°, -20°, 100°, -100°,$
$140°, -140°$

d $3\tan\dfrac{x}{2} + 3 = 0$

$\tan\dfrac{x}{2} = -\dfrac{3}{3} = -1$

Because we have $\dfrac{x}{2}$, and $-180° \le x \le 180°$, then we use $-90° \le \dfrac{x}{2} \le 90°$.

Then $\dfrac{x}{2} = -45°$ and so $x = -90°$

2 a $\sin^{-1}\dfrac{\sqrt{3}}{2} = 3\theta$

Because we have 3θ, and $0 \le \theta \le 2\pi$, then we use $0 \le 3\theta \le 6\pi$.

$3\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}, \dfrac{8\pi}{3}, \dfrac{13\pi}{3}, \dfrac{14\pi}{3}$

then $\theta = \dfrac{\pi}{9}, \dfrac{2\pi}{9}, \dfrac{7\pi}{9}, \dfrac{8\pi}{9}, \dfrac{13\pi}{9}, \dfrac{14\pi}{9}$

b $\cos 3\theta - 1 = 0$

$3\theta = \cos^{-1}1$

Because we have 3θ, and $0 \le \theta \le 2\pi$, then we use $0 \le 3\theta \le 6\pi$.

$3\theta = 0, 2\pi, 4\pi, 6\pi$ then $\theta = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 2\pi$

c $\sin\dfrac{\theta}{2} = \dfrac{\sqrt{2}}{2}$

$\dfrac{\theta}{2} = \sin^{-1}\dfrac{\sqrt{2}}{2}$

Because we have $\dfrac{\theta}{2}$, and $0 \le \theta \le 2\pi$,

then we use $0 \le \dfrac{\theta}{2} \le \pi$

$\dfrac{\theta}{2} = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

Then $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

d $\sin^2 \frac{2\theta}{3} - 1 = 0$

$\sin \frac{2\theta}{3} = \pm 1$

Because we have $\frac{2\theta}{3}$, and $0 \le \theta \le 2\pi$,

then we use $0 \le \frac{2\theta}{3} \le \frac{4\pi}{3}$.

$\frac{2\theta}{3} = \sin^{-1} \pm 1 = \frac{\pi}{2}$

Then $\theta = \frac{\pi}{2} \times \frac{3}{2} = \frac{3\pi}{4}$

Exercise 12F

1 a $2\sin 5\cos 5 = \sin 2 \times 5 = \sin 10$ by the double angle formula

b $2\sin\frac{\pi}{2}\cos\frac{\pi}{2} = \sin 2 \times \frac{\pi}{2} = \sin \pi$ by the double angle formula

c $2\sin 4\pi \cos 4\pi = \sin 2 \times 4\pi = \sin 8\pi$ by the double angle formula

d $\cos^2 0.4 - \sin^2 0.4 = \cos 2 \times 0.4 = \cos 0.8$ by the double angle formula

e $2\cos^2 6 - 1 = \cos 2 \times 6 = \cos 12$ by the double angle formula

f $1 - 2\sin^2 \frac{\pi}{4} = \cos 2 \times \frac{\pi}{4} = \cos\frac{\pi}{2}$ by the double angle formula

2 a We use the Pythagorean identity

$\sin^2 \theta + \cos^2 \theta = 1$

$\left(\frac{1}{3}\right)^2 + \cos^2 \theta = 1$

$\cos^2 \theta = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$

We take only the positive value as θ is acute $\cos\theta = \frac{2\sqrt{2}}{3}$

b $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{1}{3} \times \frac{2\sqrt{2}}{3} = \frac{4\sqrt{2}}{9}$

c $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{8}{9} - \left(\frac{1}{3}\right)^2 = \frac{7}{9}$

d $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{4\sqrt{2}}{9}}{\frac{7}{9}} = \frac{4\sqrt{2}}{7}$

3 a $\sin^2 \theta + \cos^2 \theta = 1$

$\sin^2 \theta + \left(-\frac{1}{2}\right)^2 = 1$

$\sin^2 \theta = 1 - \frac{1}{4} = \frac{3}{4}$

$\sin\theta = \frac{\sqrt{3}}{2}$

b $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{\sqrt{3}}{2} \times -\frac{1}{2} = -\frac{\sqrt{3}}{2}$

c $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(-\frac{1}{2}\right)^2 - \frac{3}{4} = -\frac{1}{2}$

d $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

4 a θ is obtuse, so we take the negative value of the cosine

$\sin^2 \theta + \cos^2 \theta = 1$

$\left(-\frac{1}{8}\right)^2 + \cos^2 \theta = 1$

$\cos^2 \theta = 1 - \frac{1}{64} = \frac{63}{64}$

$\cos\theta = -\frac{\sqrt{63}}{8}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times -\frac{1}{8} \times -\frac{\sqrt{63}}{8} = \frac{\sqrt{63}}{32}$

b $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$= \left(-\frac{\sqrt{63}}{8}\right)^2 - \left(-\frac{1}{8}\right)^2$

$= \frac{63}{64} - \frac{1}{64} = \frac{62}{64} = \frac{31}{32}$

c $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{\sqrt{63}}{32}}{\frac{31}{32}} = \frac{\sqrt{63}}{31}$

d $\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2 \times \frac{\sqrt{63}}{32} \times \frac{31}{32} = \frac{31\sqrt{63}}{512}$

5 a $\sin 2\theta = \sin\theta$

this is true for $\theta = 0, 2\pi, \pi$

divide by $\sin\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}, \frac{5\pi}{3}$

b $\cos 2\theta + \sin\theta = 0$

$1 - 2\sin^2\theta + \sin\theta = 0$

$-(\sin\theta - 1)(2\sin\theta + 1) = 0$

$\sin\theta = 1$

$\theta = \sin^{-1} 1 = \frac{\pi}{2}$

and $\sin\theta = -\frac{1}{2}$

$\theta = \sin^{-1} - \frac{1}{2} = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

and $\theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

c $\sin 2\theta = \sqrt{3}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{3}\cos\theta$

this is true for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

divide by $\cos\theta$

$2\sin\theta = \sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{2}$

then $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

d $\cos\theta = \sin\theta \sin 2\theta$

$\cos\theta = \sin\theta 2\sin\theta\cos\theta$

true for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

divide by $\cos\theta$

$2\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$

Then $\theta = \sin^{-1} \pm 1/\sqrt{2} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

e $\cos 2\theta = \cos\theta$

$2\cos^2\theta - 1 = \cos\theta$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(\cos\theta - 1)(2\cos\theta + 1) = 0$

$\cos\theta = 1$

$\cos^{-1} 1 = \theta = 0, 2\pi$

$2\cos\theta + 1 = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1} - \frac{1}{2} = \frac{2\pi}{3}, \frac{4\pi}{3}$

6 a $32\sin x\cos x$

$= 2 \times 16 \times \sin x\cos x = 16\sin 2x$

Then a=16, b=2

b $16\sin 2x = 8$

so $\sin 2x = \frac{8}{16} = \frac{1}{2}$

Note that $0 \le 2x \le 2\pi$, so

$\sin^{-1}\frac{1}{2} = 2x = \frac{\pi}{6}, \frac{5\pi}{6}$

and so $x = \frac{\pi}{12}, \frac{5\pi}{12}$

7 $Area = \frac{1}{2} \times \sqrt{15}x\sin 2\theta = 10$

$\frac{1}{2}\sqrt{15}x \times 2 \times \sin\theta\cos\theta = 10$

$\sqrt{15}x\sin\theta\cos\theta = 10$

$\sqrt{15}x\sin\theta\left(\sqrt{1 - \sin^2\theta}\right) = 10$

with $\sin\theta = \frac{1}{4}$ we have that

$\sqrt{15}x \times \frac{1}{4} \times \left(\sqrt{1 - \left(\frac{1}{4}\right)^2}\right) = 10$

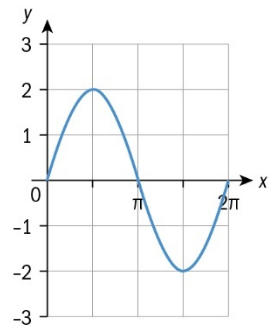
$\sqrt{15}x \times \frac{1}{4} \times \frac{\sqrt{15}}{4} = 10$

$\frac{15}{16}x = 10$

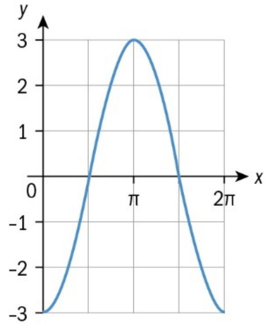
$x = \frac{160}{15} = \frac{32}{3}$

Exercise 12G

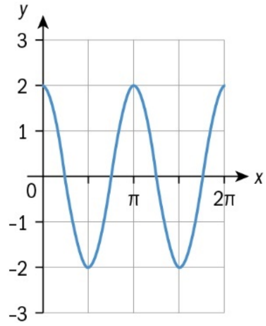
1 a



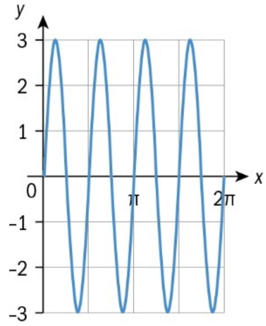
b



c



d



2 a The amplitude is $|1| = 1$ and the period is $\frac{2\pi}{3}$.

b The amplitude is $|0.5| = 0.5$ and the period is $\frac{2\pi}{2} = \pi$.

c The amplitude is $|-4| = 4$ and the period is $\frac{2\pi}{3}$.

d The amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$ and the period is $\frac{2\pi}{\frac{1}{3}} = 6\pi$.

3 a period of π, amplitude of 1, then $y = \sin 2x$

b period of π, amplitude of 3, then $y = 3\cos 2x$

c period of 2π, amplitude of 2, then $y = -2\cos x$

d period of 2π, amplitude of 2, then $y = 2\sin(-x)$

4 a The amplitude is $|6| = 6$

b The period is $\frac{2\pi}{\pi/2} = 4$

Exercise 12H

1 a amplitude 3, period $\frac{2\pi}{2} = \pi$. Option iv

b amplitude 2, period 2π and vertical shift $+1$. Option ii

c amplitude $\frac{1}{2}$, horizontal shift $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ units to the right. Option i

d amplitude 1, period $\frac{2\pi}{1/2} = 4\pi$, vertical shift 2. Option iii

2 a period 2π, amplitude $\frac{max - min}{2} = \frac{5-1}{2} = 2$, vertical shift $+3$

, $y = 2\sin x + 3$

b period 2π, amplitude $\frac{max - min}{2} = \frac{2-0}{2} = 1$, vertical shift $+1$

horizontal shift π, $y = \cos(x - \pi) + 1$

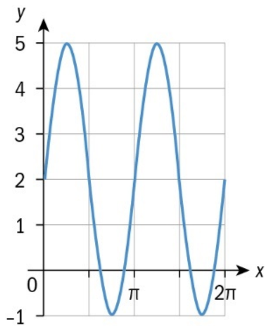
c period 2π, amplitude $\frac{max - min}{2} = \frac{0+4}{2} = 2$, vertical shift -2

$y = 2\cos x - 2$

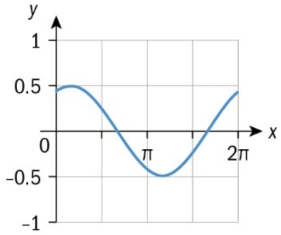
d period $\frac{2\pi}{3}$, amplitude $\frac{max - min}{2} = \frac{-0.5+1.5}{2} = 0.5$,

vertical shift -1, $y = 0.5\sin 3x - 1$

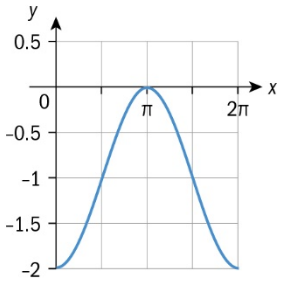
3 a vertical shift 2, amplitude 3, period π, plot given



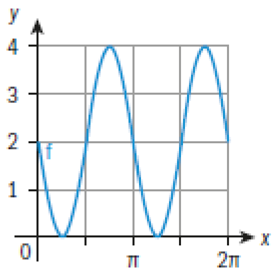
b horizontal shift $-\frac{\pi}{3}$, amplitude 0.5, period 2π, plot given



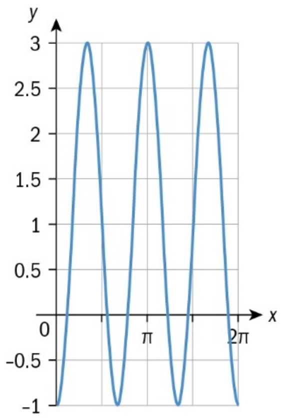
c vertical shift -1, amplitude 1, period 2π, horizontal shift $-\pi$, plot given



d vertical shift 2, amplitude 2, period π, plot given



e vertical shift 1, amplitude 2, period $\frac{2\pi}{3}$, horizontal shift $-\pi$, plot given



4 $(-0.824, 0), (0.824, 0)$

5 Plot $2\sin x - x - 1 = y$. Find zero at $(-2.38, 0)$

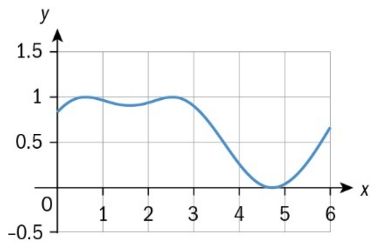
6 Plot $\sin x - \frac{1}{2} = y$, find zero at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

7 Plot $e^x - \cos x = y$ between -2 and -1. Find zero at $(-1.29, 0)$

8 a c is the horizontal shift: $\frac{\pi}{2}$

b The graph of $y = \cos x$ may be translated $\frac{\pi}{2}$ horizontally to the right to become the graph of $y = \sin x$

9 a



b $0.6075, 1.571, 2.534, 4.712$

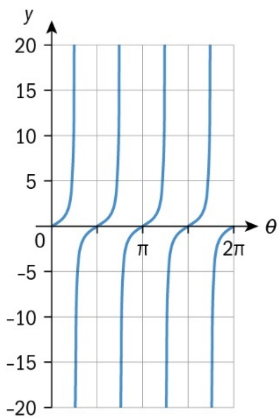
10 a $(2.36, -1)$

b $\frac{2\pi}{\frac{1}{2}} = 4\pi$

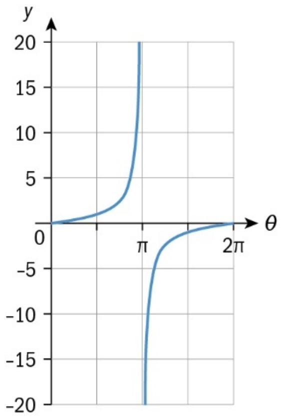
c $x = 0, 1.26, 3.77, 4.19$

Exercise 12I

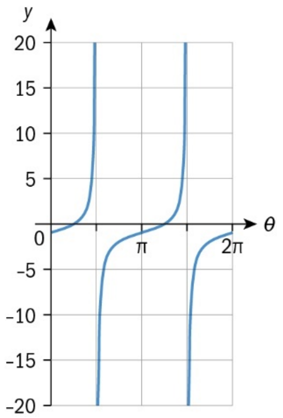
1 a



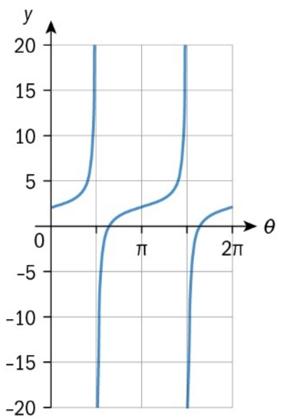
b



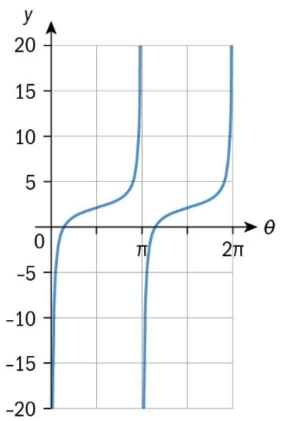
c



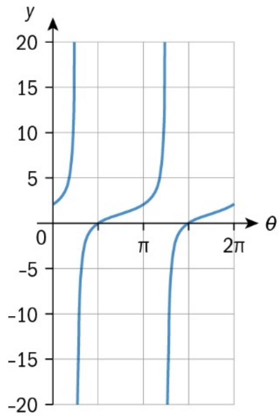
d



e



f



2 **a** $1.25, 4.39$

b $1.13, 4.53$

c $-\pi, 0, \pi$

d $-0.903, 0.677, 1.98, 2.61$

Exercise 12J

1 **a** Minimum when $\sin\left(\frac{\pi}{6}(t-5)\right) = -1$, i.e.

$h(t) = -5 + 7 = 2$ m

Maximum when $\sin\left(\frac{\pi}{6}(t-5)\right) = 1$, i.e.

$h(t) = 5 + 7 = 12$ m

b $\sin\left(\frac{\pi}{6}(t-5)\right) = 1$

$\left(\frac{\pi}{6}(t-5)\right) = \frac{\pi}{2} \Rightarrow t = 8$

High tide at 8 am

c $\sin\left(\frac{\pi}{6}(t-5)\right) = -1$

$\left(\frac{\pi}{6}(t-5)\right) = -\frac{\pi}{2} \Rightarrow t = 2$

Low tide at 2 am

d $h(9) = 5\sin\left(\frac{\pi}{6}(9-5)\right) + 7$

$= 5\sin\frac{2\pi}{3} + 7$

$= \frac{5\sqrt{3}}{2} + 7 = 11.3$ m

e $h(t) = 5\sin\left(\frac{\pi}{6}(t-5)\right) + 7 = 3$

$\frac{\pi}{6}(t-5) = \sin^{-1}\left(-\frac{4}{5}\right)$

$\frac{\pi}{6}(t-5) = -\pi + \sin^{-1}\left(\frac{4}{5}\right), -\sin^{-1}\left(\frac{4}{5}\right),$

$\pi + \sin^{-1}\left(\frac{4}{5}\right), 2\pi - \sin^{-1}\left(\frac{4}{5}\right)$

$t = 0.771, 3.229, 12.771, 15.229$

0:46 am, 3:14 am, 12:46 am, 3:14 pm (multiply decimals by 60 to convert the decimal number of hours into minutes)

2 a 13000 on February 1st

b 7000 on August 9th

c $3000\cos(0.5(4-1)) + 10000 = 10212$

3 a 20

b 10

c amplitude $\frac{20-0}{2} = 10$, vertical shift 10

period $\approx 4\pi$. Then $y = 10\sin 0.5x + 10$

d 16 fish

4 a 35 m

b 5 m

c $\frac{35-5}{2} = 15\text{m}$

d $a = 15, c = \textit{vertical shift} = 20$

e horizontal distance between maximums: $4 - 0 = 4$ (period)

f b is calculated as $b = \frac{2\pi}{4} = \frac{\pi}{2}$

g plot $y = 15\cos\frac{\pi}{2}t + 20$. Find $y = 30$ at

$t = 0.535$

5 a radius=12 m, so diameter=24 m
maximum is at 2m + 24 m = 26 m above the ground
minimum is at 2 m above the ground

amplitude $\frac{26-2}{2} = 12$, increases then decreases, so $a = -12$

$c = \frac{\max + \min}{2} = \frac{26+2}{2} = 14\text{m}$

$b = \frac{2\pi}{40} = \frac{\pi}{20}$

$h(t) = -12\cos\frac{\pi}{20}x + 14$

b At $t = p$, $-12\cos\frac{\pi}{20}p + 14 = 20$

so $-12\cos\frac{\pi}{20}p = 6$, i.e. $\cos\frac{\pi}{20}p = -\frac{1}{2}$.

Then the required angle is $\theta = \frac{2\pi}{3}$

c $\frac{\pi}{20}p = \frac{2\pi}{3}$ then $p = \frac{40}{3} = 13.3\text{s}$

6 a $\max = 20 \times 2 + 1 = 41\text{m}$, $\min = 1\text{m}$.
Then amplitude is 20, $a = -20$

vertical shift =

$\frac{\max + \min}{2} = \frac{41+1}{2} = 21\text{m}$

period $\frac{2\pi}{40} = \frac{\pi}{20}$

$h(t) = -20\cos\frac{\pi}{20}x + 21$

b for $h = 23m = -20\cos\frac{\pi}{20}x + 21$,

$\cos\frac{\pi}{20}x = \frac{23-21}{-20} = -\frac{1}{10}$

$\frac{20}{\pi}\cos^{-1} - \frac{1}{10} = x = 10.64s$

7 a amplitude $\frac{60-40}{2} = 10$

period $2 \times (1.8 - 0.3) = 3$, then $b = \frac{2\pi}{3}$

vertical shift $\frac{60+40}{2} = 50$

maximum at 0.3 instead of 0, so horizontal shift of 0.3

$y = 50 + 10\cos\left(\frac{2\pi}{3}(t-0.3)\right)$

b

$y = 50 + 10\cos\left(\frac{2\pi}{3}(17.2-0.3)\right) = 43.3\text{m}$

c Plotting the function and the line $y = 59$ gives $t = 0.0847$ s

Chapter review

1 a $30 \times \frac{\pi}{180} = \frac{\pi}{6}$

b $150 \times \frac{\pi}{180} = \frac{5\pi}{6}$

c $315 \times \frac{\pi}{180} = \frac{7\pi}{4}$

d $120 \times \frac{\pi}{180} = \frac{2\pi}{3}$

e $-20 \times \frac{\pi}{180} = -\frac{\pi}{9}$

f $-240 \times \frac{\pi}{180} = -\frac{4\pi}{3}$

g $-270 \times \frac{\pi}{180} = -\frac{3\pi}{2}$

h $144 \times \frac{\pi}{180} = \frac{4\pi}{5}$

2 a $\frac{3\pi}{2} \times \frac{180}{\pi} = 270°$

b $\frac{7\pi}{6} \times \frac{180}{\pi} = 210°$

c $-\frac{7\pi}{12} \times \frac{180}{\pi} = -105°$

d $\frac{\pi}{9} \times \frac{180}{\pi} = 20°$

e $\frac{7\pi}{3} \times \frac{180}{\pi} = 420°$

f $-\frac{11\pi}{30} \times \frac{180}{\pi} = -66°$

g $\frac{11\pi}{6} \times \frac{180}{\pi} = 330°$

h $\frac{34\pi}{15} \times \frac{180}{\pi} = 408°$

3 a We have that $r\theta = l$ so

$6 = 5 \times \theta$

$\theta = \frac{6}{5}$ rad

b $A = \frac{1}{2}r^2\theta = \frac{1}{2} \times 25 \times \frac{6}{5} = 15$

4 a We use the Pythagorean identity

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + 0.6^2 = 1$

$\cos^2\theta = 1 - 0.36 = 0.64$

$\cos\theta = \pm 0.8$

We take the positive value for acute θ

$\cos\theta = 0.8$

and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75$

b $2\sin x = \tan x$

holds for $x = 0$. We divide by $\tan x$ to get

$2\cos x = 1$

$\cos x = \frac{1}{2}$

holds for $x = \frac{\pi}{3}$ and $x = -\frac{\pi}{3}$

5 a $\sin^{-1} - \frac{1}{2} = \pi + \frac{\pi}{6}$ and $2\pi - \frac{\pi}{6}$

So $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

b $\cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ and $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

c $\tan^{-1} 1 = \frac{\pi}{4}$ and $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

6 a $8\sin x\cos x = 4 \times 2\sin x\cos x = 4\sin 2x$

so $a = 4$ and $b = 2$

b $4\sin 2x = 2$

$2\sin 2x = 1$

$\sin 2x = \frac{1}{2}$

$\sin^{-1}\frac{1}{2} = 2x$

Note that $0 \le 2x \le 4\pi$, so

$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

so $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

7 a the angle is obtuse, so we need a positive sine

$\left(-\frac{12}{13}\right)^2 + \sin^2\theta = 1$

$\sin^2\theta = 1 - \frac{144}{169} = \frac{25}{169}$

$\sin\theta = \frac{5}{13}$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{119}{169}$

c $\sin(\theta + \pi) = -\sin\theta = -\frac{5}{13}$

8 a $2\sin^2\theta + \sin\theta - 1 = 0$

$(\sin\theta + 1)(2\sin\theta - 1)$

Then $\sin\theta = -1$

and $\sin\theta = \frac{1}{2}$

b $\sin^{-1} - 1 = \frac{3\pi}{2}$

and $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}, \frac{5\pi}{6}$

9 a $\left(\frac{3}{4}\right)^2 + \cos^2\theta = 1$

$\cos^2\theta = 1 - \frac{9}{16} = \frac{7}{16}$

Obtuse angle, so we take the negative cosine $\cos\theta = \frac{-\sqrt{7}}{4}$

b $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(-\frac{\sqrt{7}}{4}\right)^2 - \left(\frac{3}{4}\right)^2 = -\frac{1}{8}$

10 period: $\frac{\pi}{2}$, amplitude $= \frac{30+30}{2} = 30$.

Then $b = \frac{2\pi}{\frac{\pi}{2}} = 4$ and $a = 30$

11 a period is $\pi - 0 = \pi$

b 3

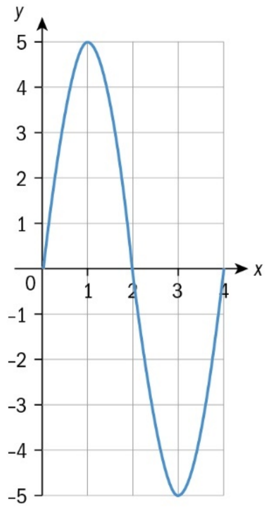
c 3

d $b = \frac{2\pi}{\pi} = 2$

12 a 5

b $\frac{2\pi}{\frac{\pi}{2}} = 4$

c



13 $A_{shaded} = A_{sector} - A_{triangle}$

$A_{shaded} = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$A_{shaded} = \frac{1}{2}\times 8^2 \times \frac{\pi}{6} - \frac{1}{2}\times 8^2 \times \sin\frac{\pi}{6}$

$= 0.755$

14 a $A = \frac{1}{2}\sqrt{2}\times 4 \times \sin\theta = 2$

$2\sqrt{2}\sin\theta = 2$

$\sin\theta = \frac{1}{\sqrt{2}} = \frac{3\pi}{4}$

for obtuse θ

b Then $CBD = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

and so $A_{BDC} = \frac{1}{2}\times 4^2 \times \frac{\pi}{4} = 2\pi$

15 a Plot $y = \cos x - x^2$ and find zeros for $0 \le x \le \frac{\pi}{2}$, then $x = 0.824$

b Plot $y = 4\sin\pi x - 4e^{-x} + 3$ for $0.5 \le x \le 1.5$ and find zero at $x = 1.14$

16 $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}\times 10 \times 8 \times \sin\theta = 10$

$\sin\theta = \frac{1}{4}$

$\theta = \sin^{-1}\frac{1}{4} = 0.25$

Obtuse angle and positive sine,

$\pi - 0.25 = 2.89$

17 Area of the sector is

$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2}\times 10^2 \times 0.8 = 40$

Then $O\hat{A}N = \pi - 0.8 = 0.77$

Using the sine rule $\frac{\sin A\hat{N}O}{AO} = \frac{\sin O\hat{A}N}{ON}$

$\frac{\sin\frac{\pi}{2}}{10} = \frac{\sin 0.77}{ON}$

$ON = 10\sin 0.77 = 6.96$

Then $A_{triangle} = \frac{1}{2}\times 10 \times 6.96 \times \sin 0.8 = 25$

Then

$A_{shaded} = A_{sect} - A_{triangle}$

$= 40 - 25 = 15\text{cm}^2$

18 a Using the sine rule $\frac{\sin ADO}{AO} = \frac{\sin AOD}{AD}$

$AD = AO\frac{\sin AOD}{\sin ADO} = 8\frac{\sin 0.8}{\sin 0.4} = 14.7\text{cm}$

b $DAO = \pi - ADO - AOD$

$= \pi - 0.8 - 0.4 = 1.94$

$\frac{\sin DAO}{OD} = \frac{\sin ADO}{AO}$

$OD = AO\frac{\sin DAO}{\sin ADO} = 8\frac{\sin 1.94..}{\sin 0.4} = 19.1\text{cm}$

c $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 8^2 \times 0.8 = 25.6\text{cm}^2$

d $A_{triangle} = \frac{1}{2}\times AD \times OD \times \sin ADO$

$= \frac{1}{2}\times 14.7 \times 19.1 \times \sin 0.4 = 54.9\text{cm}^2$

$A_{ABCD} = A_{triangle} - A_{sector}$

$= 54.9 - 25.6 = 29.3\text{cm}^2$

19 a $f(x) = 2\times 10\times \sin 3x\cos 3x = 10\sin 6x$

b $10\sin 6x = 0$

$\sin 6x = 0$

for $0 \le 6x \le 2\pi$. Then $6x = 0,\pi,2\pi$

and $x = 0,\frac{\pi}{6},\frac{\pi}{3}$

20 a $f(\theta) = \left(2\cos^2\theta - 1\right) + \cos\theta + 1$

$= 2\cos^2\theta + \cos\theta$

b This is a quadratic equation in $\cos\theta$ with positive discriminant, so there are 2 distinct values of $\cos\theta$ which satisfy it.

c $\cos\theta\left(2\cos\theta + 1\right) = 0$

$\cos\theta = 0$

$\Rightarrow \theta = -90°, 90°$

$2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = -120°, 120°$

21 a $l = r\theta = 15\times\frac{\pi}{6} = 7.85\text{m}$

b $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 15^2\times\frac{\pi}{6} = 58.9\ \text{m}^2$

c When wheel turns through $\frac{\pi}{2}$, A is 15 m above the ground.

Wheel then turns a further $\frac{\pi}{6}$ radians to complete a total turn of $\frac{2\pi}{3}$.

$h_A = 15 + 15\sin\frac{\pi}{6} = 15\times\frac{3}{2} = 22.5\ \text{m}$

d $h\left(\frac{\pi}{4}\right) = 15 - 15\cos 2\left(\frac{\pi}{4} + \frac{\pi}{8}\right) = 25.6\text{m}$

e $h(0) = 15 - 15\cos 2\left(\frac{\pi}{8}\right) = 4.39\text{m}$

f When $\cos 2\left(t + \frac{\pi}{8}\right)$ is -1, we have a maximum. Then

$\cos 2\left(t + \frac{\pi}{8}\right) = -1$

$2\left(t + \frac{\pi}{8}\right) = \pi$

$t = \frac{\pi}{2} - \frac{\pi}{8} = 1.18\text{s}$

22 a $p = 3$, $q = \dfrac{2\pi}{\frac{3\pi}{2}} = \frac{4}{3}$, $b = \frac{2\pi}{\pi} = 2$

$-d = -\frac{2}{2} = -1$, so $d = 1$

b $3\cos\frac{4}{3}x = \sin 2x - 1$

The intersection points can be found as $x = 1.30, 3.41, 6.19$

23 a 3

b $10 - 2 = 8 =$ period, $b = \frac{2\pi}{8} = \frac{\pi}{4}$

c $\frac{8+2}{2} = 5$

d $f(x) = 3\sin\frac{\pi}{4}x + 5$

translated (3,0)

$\hat{f}(x) = 3\sin\frac{\pi}{4}(x-3) + 5$

reflected in the x axis

$\hat{\hat{f}}(x) = -(3\sin\frac{\pi}{4}(x-3) + 5) = g(x)$

24 a $A(t) = (2\cos t - 1)(\cos t - 1)$ A1A1

b $(2\cos t - 1)(\cos t - 1) = 0$

$2\cos t - 1 = 0 \Rightarrow \cos t = \frac{1}{2}$

$\Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$ M1A1

$\cos t - 1 = 0 \Rightarrow \cos t = 1$

$t = 0, 2\pi$ M1A1

25 $2\cos^2 x = \sin 2x$

$\Rightarrow 2\cos^2 x - 2\sin x\cos x = 0$ M1

$2\cos x(\cos x - \sin x) = 0$ M1

$\cos x = 0, \cos x = \sin x$ M1

$x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ A1

$x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$ A1

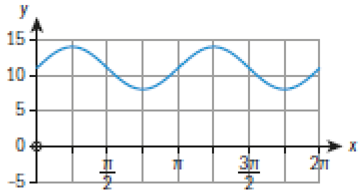
26 a Correct attempt to at least one parameter M1

$a = \frac{14-8}{2} = 3$ A1

$b = \frac{2\pi}{\pi} = 2$ A1

$c = \frac{14+8}{2} = 11$ A1

b

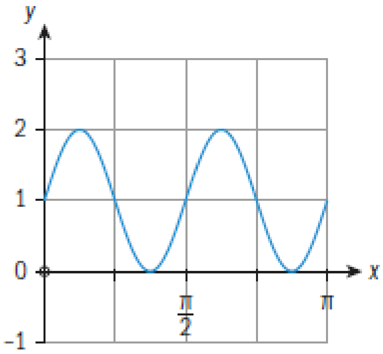


A1 for trigonometric scale and correct domain,
A1 for correct max/min,
A1 for two complete cycles

27 a $S(x) = \underbrace{\sin^2 2x + \cos^2 2x}_{} + \underbrace{2\sin 2x \cos 2x}_{}$ M1A1A1

$= 1 + \sin 4x$ AG

b A1 for correct shape, A1 for 2 cycles, A1 for correct max/min



c i $\frac{\pi}{2}$ A1

ii $0 \le y \le 2$ A1

d <Please insert the graph of $y = \cos(2x) - 1$ for $0 \le x \le 2\pi$ > A3

e i $\frac{1}{2}$ A1

ii $p = \pi$ (or any $\pi + 2n\pi, n \in$ ¢) A1

$q = -2$ A1

28 a i $A = \frac{1}{2} \times 2^2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$ A1

ii $l = 2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$ A1

b i $r\theta = \frac{\pi}{3}$ A1

$r^2 \frac{\theta}{2} = \pi$ A1

Solve simultaneously M1

$r = 6$ *cm* A1

ii $\theta = \frac{\pi}{18}$ A1

29 a i $-1 \le y \le 3$ A1

ii 2 A1

b $a = -2$ A1

$b = \frac{2\pi}{2} = \pi$ M1A1

$c = 1$ A1

c $-2\cos \pi x + 1 = 0 \Rightarrow \cos \pi x = \frac{1}{2}$ M1

$\pi x \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3},\right\}$ M1

$x \in \left\{-\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, \frac{7}{3},\right\}$ M1

30 a i $x = 0, x = \frac{\pi}{2}, x = \pi$ A1

ii $\frac{\pi}{2}$ A1

iii ¡ A1

b $b = 2$ A1

$d = 1$ A1

c The first point of inflexion occurs at $x = \frac{\pi}{4}$ R1

d $f\left(\frac{\pi}{8}\right) = -2$

$\Rightarrow f(x) = a\tan\left(2\left(\frac{\pi}{8} - \frac{\pi}{4}\right)\right) + 1 = -2$ M1

$\underbrace{a\tan\left(-\frac{\pi}{4}\right)}_{-1} = -3$ A1

$a = 3$ AG

e $\frac{\pi}{8}$ A1

$\frac{5\pi}{8}$ and $\frac{9\pi}{8}$ A1

31 a $-2\cos^2 x + \sin x + 3$

$= -2\left(1 - \sin^2 x\right) + \sin x + 3$ M1

$= 2\sin^2 x + \sin x + 1$ A1

b $-2\cos^2 x + \sin x + 3 = 2$

$\Rightarrow 2\sin^2 x + \sin x + 1 = 2$ M1

$2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$ M1

$\sin x = \frac{1}{2}, \sin x = -1$ A1

$x \in \left\{-\frac{11\pi}{6}, -\frac{\pi}{2}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$

A2

Award A1 for two correct solutions