The Solution of Cubic Equations

HL Exploration (IA) **Student Sample B**

After having used the quadratic formula numerous times, I began to wonder if a similar formula existed for cubic equations. After discovering that such a formula did exist, I became fascinated with its complexity, which led me to wonder about its origin. I am specifically intrigued by how geometry is utilized to derive the formula. It was not until the 16^{th} century when the combined efforts of numerous Italian mathematicians paid off and a general formula was uncovered, and the "primitive state of algebraic symbolism" (Dunham 142) during this time explains why geometry was relied on so heavily for this derivation. It is this geometry-based solution to cubic equations which will be explored in this essay.

Depressed cubic equations are a subset of cubic equations. While a cubic equation in standard form is written as $y = ax^3 + bx^2 + cx + d$, the depressed cubic equation is written as $y = x^3 + mx + n$, in which there are no terms containing x^2. The absence of a term with a squared variable is what makes depressed cubic equations so unique. The solution to the depressed cubic gives way to the solution to the general cubic, so the first goal is to derive a formula for the roots of a depressed cubic equation.

The roots of such an equation will occur at $y = 0$, so $x^3 + mx + n = 0$. This can be written as $x^3 + mx = n$. It may appear incorrect that the constant n remains positive despite the fact that it has been subtracted from both sides, but n will just have the same value with its sign flipped. Additionally, if a given depressed cubic equation has a coefficient of x^3 which is not equal to 1, then it must be rearranged such that this coefficient is 1. Take, for example, the equation $2x^3 + 4x = 8$. In order to express it in the form $x^3 + mx = n$, it is necessary to divide through by the leading coefficient, in this case, 2, so that $x^3 + 2x = 4$. More generally, if $Ax^3 + Bx = C$, then rearrange so that $x^3 + \frac{B}{A}x = \frac{C}{A}$, where $m = \frac{B}{A}$ and $n = \frac{C}{A}$. This ensures that the formula is in terms of two unknown constants rather than three, making it simpler and easier to work with.

The first step in deriving a formula for solving depressed cubic equations involves splitting a cube into various sections. These sections can be expressed in terms of their volume to generate a depressed cubic equation which can be solved in terms of the cube's dimensions. This method is somewhat similar to completing the square but in three dimensions (Merzbach 258).

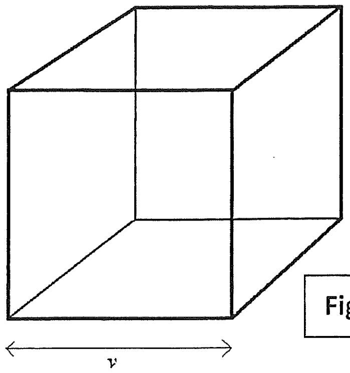

Figure 1

Imagine a cube with edges of length v.

Now picture the same cube, but with a section cut off by a plane parallel to one of the faces, for example, the right face, so that there is a separate "slice" of the cube. Let the distance between the right face of the cube and the plane be u, and assume that $u < v$.

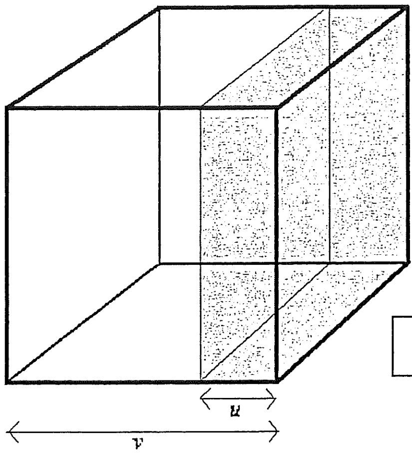

Figure 2

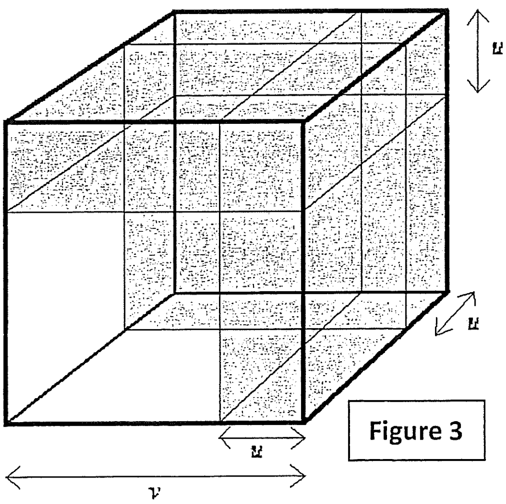

Figure 3

Finally, imagine that there are two additional planes, one parallel to the top face of the cube and one parallel to the back face of the cube, which intersect the cube at a distance, u, from their respective faces, resulting in three "slices" (Anthony).

In figure 4 below, figure 3 had been re-shaded so that shapes with equal volume are the same shade. The next step is to calculate the volume of each section by shade, so that the sum of these expressions is equal to the volume of the cube. Personally, I like this idea of representing an equation visually, as operations involving volume are easy to understand, so it is not difficult to see where the substitutions (which follow) come from.

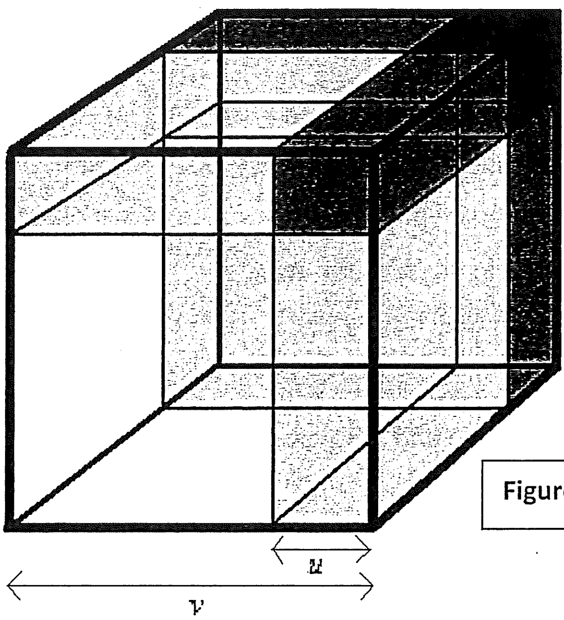

Figure 4

Volume of the large cube (white) $\rightarrow$ $volume = (v-u)^3$

Volume of the three slices (light grey) $\rightarrow$ $volume = 3u(v-u)^2$

Volume of the three bars (medium grey) $\rightarrow$ $volume = 3u^2(v-u)$

Volume of the small cube (dark grey) $\rightarrow$ $volume = u^3$

Volume of the whole cube (1) $\rightarrow$ $volume = v^3$

Volume of the whole cube (2) $\rightarrow$ $volume = (v-u)^3 + 3u(v-u)^2 + 3u^2(v-u) + u^3$

Equating the two expressions for volume $\rightarrow$ $v^3 = (v-u)^3 + 3u(v-u)^2 + 3u^2(v-u) + u^3$

This equation for volume can be rearranged. The purpose of this will soon become evident.

$$\text{Factoring out } 3u \rightarrow v^3 - u^3 = (v-u)^3 + 3u\left[(v-u)^2 + u(v-u)\right]$$

$$\text{Factoring out } v-u \text{ from the square brackets} \rightarrow v^3 - u^3 = (v-u)^3 + 3u\left[(v-u)((v-u)+u)\right]$$

$$\text{Simplifying} \rightarrow v^3 - u^3 = (v-u)^3 + 3u\left[(v-u)(v)\right] \rightarrow v^3 - u^3 = (v-u)^3 + 3uv(v-u)$$

$$\text{Rearranging} \rightarrow (v-u)^3 + 3vu(v-u) = v^3 - u^3$$

At this point, a trained eye will notice that $(v-u)^3 + 3vu(v-u) = v^3 - u^3$ is a cubic equation. More specifically, it is a depressed cubic equation in the form $x^3 + mx = n$, in which $x = v-u$, $m = 3vu$ and $n = v^3 - u^3$. It is "depressed" because it lacks the term $(v-u)^2$. The latter two of these three equations make it possible to solve for the variables u and v in terms of the coefficients m and n. With expressions for u and v, is then becomes possible to solve for x by using the third equation, $x = v-u$. By this method, a formula for the roots of a depressed cubic equation in the form $x^3 + mx = n$ can be determined. The equation for the volume of the cube in figure 4 demonstrates where these substitutions come from.

$$\text{Because } m = 3vu \rightarrow u = \frac{m}{3v}$$

$$\text{And because } n = v^3 - u^3 \rightarrow n = v^3 - \frac{m^3}{27v^3}$$

$$\text{Multiplying both sides by } v^3 \rightarrow nv^3 = v^6 - \frac{m^3v^3}{27v^3} \rightarrow v^6 - nv^3 - \frac{m^3}{27} = 0$$

Multiplying both sides by v^3 achieves two things. Firstly, it eliminates the variable v from the denominator, making the expression much easier to work with, and secondly, it yields a quadratic equation in the form $x^2 + ax + b = 0$, where $x = v^3$ such that $(v^3)^2 - n(v^3) - \frac{m^3}{27} = 0$ (Dunham 145).

Applying the quadratic formula $\rightarrow$ $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\rightarrow$ $v^3=\dfrac{n\pm\sqrt{n^2+\dfrac{4m^3}{27}}}{2}$

Taking the cube root of both sides $\rightarrow$ $v=\sqrt[3]{\dfrac{n\pm\sqrt{n^2+\dfrac{4m^3}{27}}}{2}}$ $\rightarrow$ $\sqrt[3]{\dfrac{n}{2}\pm\dfrac{\sqrt{n^2+\dfrac{4m^3}{27}}}{2}}$

Replacing the second denominator with $\sqrt{4}$ $\rightarrow$ $v=\sqrt[3]{\dfrac{n}{2}\pm\dfrac{\sqrt{n^2+\dfrac{4m^3}{27}}}{\sqrt{4}}}$ $\rightarrow$ $v=\sqrt[3]{\dfrac{n}{2}\pm\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}$

At this stage, the problem can be simplified. Instead of $v=\sqrt[3]{\dfrac{n}{2}\pm\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}$, let $v=\sqrt[3]{\dfrac{n}{2}+\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}$. Although this will eliminate one of the possible values for v, and therefore x, it makes the rest of the solution much simpler. Additionally, the remaining roots which are eliminated here can be uncovered using polynomial division once the first root is known. With an expression for v, the next goal is to derive an expression for u.

Derived from the cube $\rightarrow$ $n=v^3-u^3$

Substitution of v $\rightarrow$ $n=\left(\sqrt[3]{\dfrac{n}{2}+\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}\right)^3-u^3$

Solving for u $\rightarrow$ $u=\sqrt[3]{-\dfrac{n}{2}+\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}$

Expressions for both v and u have been obtained in terms of m and n, the coefficients of a depressed cubic equation. Now, finally, it is possible to solve for x, the roots of such an equation.

Deduced from the cube's volume formula $\rightarrow$ $x=v-u$

Substituting for v and u $\rightarrow$ $x=\sqrt[3]{\dfrac{n}{2}+\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}-\sqrt[3]{-\dfrac{n}{2}+\sqrt{\dfrac{n^2}{4}+\dfrac{m^3}{27}}}$

This is the depressed cubic formula. Due to the fact that it will yield only one of three roots, polynomial division is required to solve for the remaining roots. Cubic equations always have three roots, as the product of these roots must yield an equation in which the highest power of x is three. Additionally, all imaginary roots come in conjugate pairs, means that a cubic equation will always have either three real roots, or one real root and two imaginary roots. If one root is imaginary, its conjugate pair must also be a root, leaving one real root (Hansen). From my experience, the equation is much less tedious to use when one potential solution is removed, and initially solving for one root and then uncovering the eliminated ones using polynomial division is much easier than solving for all the roots using the formula.

Consider the equation $-2x^3+30x=-8$, which can be simplified to $x^3-15x=4$ so that the leading coefficient is 1. Now that the equation is in the form $x^3+mx=n$, its roots can be determined using a combination of the depressed cubic formula and polynomial division.

Applying the depressed cubic formula $\rightarrow$

$$x=\sqrt[3]{\frac{(4)}{2}+\sqrt{\frac{(4)^2}{4}+\frac{(-15)^3}{27}}}-\sqrt[3]{-\frac{(4)}{2}+\sqrt{\frac{(4)^2}{4}+\frac{(-15)^3}{27}}}$$

Simplifying and solving (by hand or with computer algebra) $\rightarrow$

$$x=\sqrt[3]{2+\sqrt{-121}}-\sqrt[3]{-2+\sqrt{-121}} \rightarrow x=4$$

By polynomial long division $\rightarrow$

$$\frac{x^3-15x-4}{x-4}=\left(x+2+\sqrt{3}\right)\left(x+2-\sqrt{3}\right)$$

So the roots of the equation $x^3-15x=4$ are $x=4$, $x=-2+\sqrt{3}$, and $x=-2-\sqrt{3}$. This is represented graphically in figure 5 on the following page.

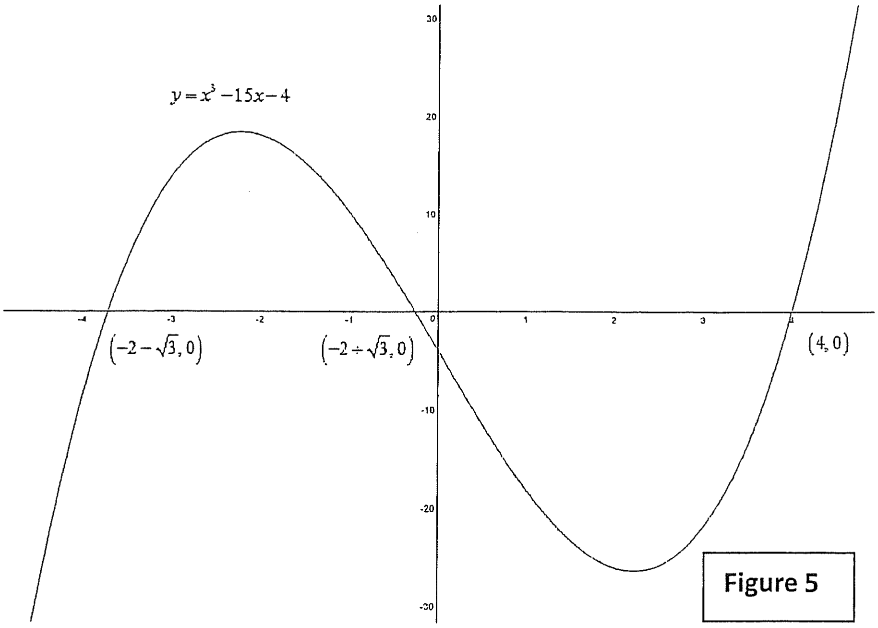

Figure 5

But what about equations which are not in depressed form, that is, cubic equations containing an x^2 term? There is a clever substitution which actually cancels out the squared term, leaving a depressed cubic equation. This new equation can then be plugged in to the depressed cubic formula to yield a general cubic formula (Dunham 148).

Starting with a cubic equation in standard form $\rightarrow$ $y = ax^3 + bx^2 + cx + d$

Setting it equal to zero to solve for the roots $\rightarrow$ $ax^3 + bx^2 + cx + d = 0$

Applying the substitution $\rightarrow$ $x = y - \frac{b}{3a}$ $\rightarrow$ $y = a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d$

So far it seems as if this substitution has only complicated the equation by introducing new, unnecessary, variables, but with some simplification the purpose of the substitution becomes clear.

Expanding and distributing $\rightarrow$

$$ay^3 - \frac{2aby^2}{3a} + \frac{ab^2y}{9a^2} - \frac{aby^2}{3a} + \frac{2ab^2y}{9a^2} - \frac{ab^3}{27a^3} + by^2 - \frac{2b^2y}{3a} + \frac{b^3}{9a^2} + cy - \frac{bc}{3a} + d = 0$$

Simplifying and rearranging $\rightarrow$ $(a)y^3 + \left(\frac{-b^2}{3a} + c\right)y + \left(\frac{2b^3}{27a^2} - \frac{bc}{3a} + d\right) = 0$

Notice that what is left is a depressed cubic equation in terms of a variable y. To see why this substitution is able to reduce the equation to a depressed cubic, examine the following expression, which is comprised of all of the terms containing y^2 from the general cubic equation after the substitution was applied.

$$\frac{3by^2}{3}-\frac{2by^2}{3}-\frac{by^2}{3}=0$$

Clearly, the coefficient of the term containing y^2 is 0. The substitution is so clever because it yields a depressed cubic equation. This is useful because it can now be plugged in to the depressed cubic formula which was previously derived. Recall that the depressed cubic formula works for an equation in the form $x^3+mx=n$, or in this case, $y^3+my=n$ (Knaust).

Reducing the leading coefficient to 1 by dividing through by a $\rightarrow$

$$\left(\frac{a}{a}\right)y^3+\left(\frac{\frac{-b^2}{3a}+c}{a}\right)y+\left(\frac{\frac{2b^3}{27a^2}-\frac{bc}{3a}+d}{a}\right)=0 \rightarrow y^3+\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right)y+\left(\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a}\right)=0$$

Subtracting the constant term from both sides and simplifying $\rightarrow$

$$y^3+\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right)y=-\left(\frac{2b^3}{27a^3}-\frac{bc}{3a^2}+\frac{d}{a}\right) \rightarrow y^3+\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right)y=\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)$$

Finally, a depressed cubic equation in the form $y^3+my=n$ has been obtained, where $\rightarrow$

$$m=\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right) \text{ and } n=\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right).$$

Now, using the depressed cubic formula, it is possible to solve for y. This is desirable because the substitution $x=y-\frac{b}{3a}$, which was initially used to generate a depressed cubic equation from a cubic equation in standard form, can then be used to generate an expression for a root of any cubic equation. With one known root, the other can be determined through polynomial division.

Applying the depressed cubic formula →

$$y=\sqrt[3]{\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}-\sqrt[3]{-\frac{n}{2}+\sqrt{\frac{n^2}{4}+\frac{m^3}{27}}}$$

$$y=\sqrt[3]{\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)}{2}+\sqrt{\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)^2}{4}+\frac{\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right)^3}{27}}}\ldots$$

$$\ldots-\sqrt[3]{-\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)}{2}+\sqrt{\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)^2}{4}+\frac{\left(\frac{-b^2}{3a^2}+\frac{c}{a}\right)^3}{27}}}$$

To simplify this equation, it is necessary to express the denominator of some of the fractions as numbers to the the second or third power.

Rewriting →

$$y=\sqrt[3]{\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)}{2}+\sqrt{\left(\frac{\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}}{2}\right)^2+\left(\frac{\frac{-b^2}{3a^2}+\frac{c}{a}}{3}\right)^3}}\ldots$$

$$\ldots-\sqrt[3]{-\frac{\left(\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}\right)}{2}+\sqrt{\left(\frac{\frac{bc}{3a^2}-\frac{2b^3}{27a^3}-\frac{d}{a}}{2}\right)^2+\left(\frac{\frac{-b^2}{3a^2}+\frac{c}{a}}{3}\right)^3}}$$

Simplifying →

$$y=\sqrt[3]{\left(\frac{bc}{6a^2}-\frac{b^3}{27a^3}-\frac{d}{2a}\right)+\sqrt{\left(\frac{bc}{6a^2}-\frac{b^3}{27a^3}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}\ldots$$

$$\ldots-\sqrt[3]{\left(\frac{b^3}{27a^3}+\frac{d}{2a}-\frac{bc}{6a^2}\right)+\sqrt{\left(\frac{bc}{6a^2}-\frac{b^3}{27a^3}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$

The final step is to substitute this expression for y into the equation $x = y - \frac{b}{3a}$, which was earlier used as a substitution to depress a general cubic equation.

Making this substitution →

$$x = y - \frac{b}{3a}$$

$$x = \sqrt[3]{\left(\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}\right) + \sqrt{\left(\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \ldots$$

$$\ldots - \sqrt[3]{\left(\frac{b^3}{27a^3} + \frac{d}{2a} - \frac{bc}{6a^2}\right) + \sqrt{\left(\frac{bc}{6a^2} - \frac{b^3}{27a^3} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

This is the general cubic formula. To verify this, consider the equation $y = x^3 - 3x^2 + 3x - 1$.

Applying the cubic formula →

$$x = \sqrt[3]{\left(\frac{(-3)(3)}{6(1)^2} - \frac{(-3)^3}{27(1)^3} - \frac{(-1)}{2(1)}\right) + \sqrt{\left(\frac{(-3)(3)}{6(1)^2} - \frac{(-3)^3}{27(1)^3} - \frac{(-1)}{2(1)}\right)^2 + \left(\frac{(3)}{3(1)} - \frac{(-3)^2}{9(1)^2}\right)^3}} \ldots$$

$$\ldots - \sqrt[3]{\left(\frac{(-3)^3}{27(1)^3} + \frac{(-1)}{2(1)} - \frac{(-3)(3)}{6(1)^2}\right) + \sqrt{\left(\frac{(-3)(3)}{6(1)^2} - \frac{(-3)^3}{27(1)^3} - \frac{(-1)}{2(1)}\right)^2 + \left(\frac{(3)}{3(1)} - \frac{(-3)^2}{9(1)^2}\right)^3}} - \frac{(-3)}{3(1)}$$

Computing this leaves $x = 1$. To determine the rest of the roots the original equation must be divided by the known factor, $x - 1$.

Dividing the original polynomial → $\frac{x^3 - 3x^2 + 3x - 1}{x - 1} = (x-1)^2$

So according to the general cubic equation, there is a triple root at $x = 1$. This is confirmed graphically in figure 6 on the following page.

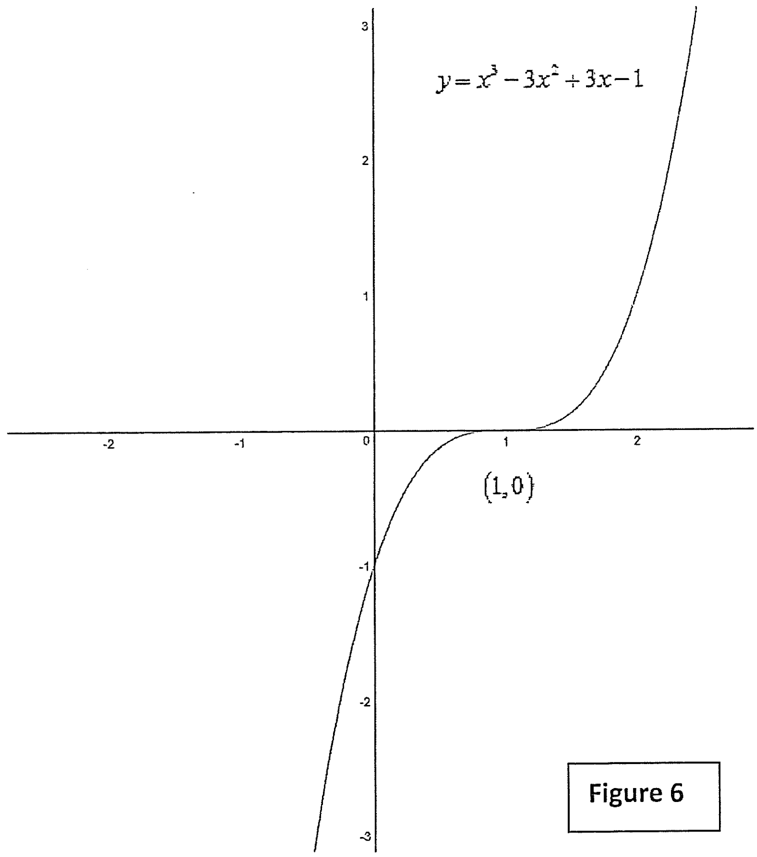

Figure 6

The cubic equation, although tedious to use, is extremely elegant today compared to when it was originally derived. This is because the formula often yields complex solutions, which were poorly, if at all, understood in the 16th century (Merzbach 259). Even the much shorter formula for the depressed cubic often involves the square root of negative numbers. Today, the formula makes much more sense because the complex solutions are much better understood. Furthermore, even negative numbers on their own used to cause many issues with the cubic formula. This is partially due to the geometric approach to such problems. This derivation, used by Cardano, involves a definition of x which means that it has to be positive because it is taken to be the length of an edge of a cube, something which cannot be negative. Due to this, when the formula returned negative roots many mathematicians were sceptical of its credibility.

After using the general cubic formula numerous times to solve for exact roots, I can say that its tediousness varies depending on the problem. When working with unknown constants or non-real roots, the formula, while it does work, is not very efficient. That being said, it is an excellent tool regardless of its intricacy. Its complexity is a likely reason for it not being taught alongside the quadratic formula in high school mathematics. Computers, however, can utilize the formula to give solutions to cubic equations very quickly.

One idea which stuck me while writing this essay is how to reverse the derivation process and make a cube based on a cubic equation, rather than generating an equation based on a cube. As it turns out, this is not very complicated because the length of one edge of the cube is simply equal to a root of its equation. Does this mean that three different cubes can be constructed based on one cubic equation because every cubic equation has three roots? Unfortunately not. This is because of the limitations of geometry in that there can be no negative or complex lengths, even though there can be negative or complex roots. Additionally, even if the roots of a cubic equation are all positive, if there is a triple root there is only 1 cube which can represent that equation. However, given that at least one root of a cubic equation is positive, it is possible to construct a cube to represent it. For example, the equation $x^3+6x=20$ has a root $x-2$, so one of the cubes (in this case the only cube because $x-2$ is a triple root) which can represent this equation has edges of length 2.

Overall, I have learned that geometry and algebra are linked in many ways and that a visual representation of a problem can be crucial for its solution, as well as making it easier to understand. It also occurred to me that it should be fairly simple to construct cubic equations based on known roots. This is useful because I have always been curious as to how one can tell which equations (to any degree) will have integer solutions. After exploring cubic equations, it became clear that it is as simple as choosing roots and then finding the equation which describes them.

Compared to just making algorithmic substitutions, the geometric representation of the cubic equation is much better at explaining why the substitutions work. It must be said that there are many other methods to solve cubic equation or to derive the cubic formula, but Cardano's method is comparatively simple and elegant in the sense that it takes the problem into the well-known realms of geometry and quadratic equations. This is something I particularly like about the solution, as it revolves around the idea that all cubic equations can be re-written as quadratic equations, something which I am very familiar with. Due to this, Cardano's solution makes much more sense to me that methods used by other mathematicians.

This breakthrough in the solving of polynomials led to various methods for solving quartic equations, but after centuries of attempting to find a general solution for quintic equations, it was proven to be impossible by Niels Abel (Dunam 153).

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