Complex Numbers – three different forms

Notes: In IB mathematics a complex number z can be written in three different forms.

$z = a + ib$ **Cartesian** form

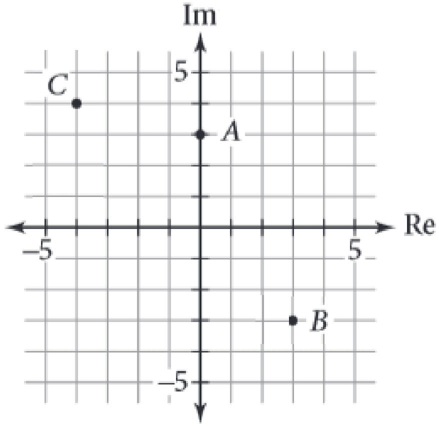
$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$ **modulus-argument** form* (also known as polar form, or trigonometric form)

$z = re^{i\theta}$ **Euler's** form* (also known as exponential form)

* the argument θ can have multiple values, but best to give the principal value such that $-\pi < \theta \leq \pi$

♦ do **not** use a calculator ♦

Exercises



1. For each complex number represented by the letters A, B & C plotted in the complex plane, write it in each of the three forms: Cartesian form, modulus-argument form and Euler's form.

2. Write each of the following complex numbers in modulus-argument form.

(a) $w = -\sqrt{3} + i$ (b) $w = 2 + 2i\sqrt{3}$ (c) $w = -\frac{1}{2} - \frac{i}{2}$

3. Write each of the following complex numbers in Cartesian form.

(a) $z = 5e^{\frac{\pi}{2}i}$ (b) $z = 8e^{-\frac{5\pi}{6}i}$ (c) $z = 2e^{\frac{2\pi}{3}i}$

4. For each of the two expressions below, first write it as a complex number in the form $a + ib$ and then write it in the form $r\,\text{cis}\,\theta$.

(a) $\frac{4}{1+i}$ (b) $\frac{2i}{1-i}$

5. Consider the complex numbers $z_1 = 1 + i$ and $z_2 = \frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$.

(a) Show that, in modulus-argument form, $z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{6}\right)$.

(b) For two complex numbers in modulus-argument form, $w_1 = r_1\,\text{cis}\,\theta_1$ and $w_2 = r_2\,\text{cis}\,\theta_2$, it can be shown that $\frac{w_1}{w_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$. Use this to express $\frac{z_1}{z_2}$ in modulus-argument form.

(c) Express $\frac{z_1}{z_2}$ in Cartesian form.

(d) Use the results from (b) and (c) to write down the exact values of $\sin\frac{5\pi}{12}$ and $\cos\frac{5\pi}{12}$.

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Answers

1. **A**: 3i ; $3\text{cis}\dfrac{\pi}{2}$; $3e^{\frac{\pi}{2}\text{i}}$ **B**: $3-3\text{i}$; $3\sqrt{2}\text{cis}\left(-\dfrac{\pi}{4}\right)$; $3\sqrt{2}e^{-\frac{\pi}{4}\text{i}}$ **C**: $-4+4\text{i}$; $4\sqrt{2}\text{cis}\left(\dfrac{3\pi}{4}\right)$; $4\sqrt{2}e^{\frac{3\pi}{4}\text{i}}$

2. (a) $w=2\,\text{cis}\dfrac{5\pi}{6}$ (b) $w=4\text{cis}\dfrac{\pi}{3}$ (c) $w=\dfrac{\sqrt{2}}{2}\text{cis}\left(-\dfrac{3\pi}{4}\right)$

3. (a) $z=5\text{i}$ (b) $z=-4\sqrt{3}-4\text{i}$ (c) $z=-1+\text{i}\sqrt{3}$

4. (a) $2-2\text{i}$; $2\text{cis}\left(-\dfrac{\pi}{4}\right)$ (b) $-1+\text{i}$; $\sqrt{2}\text{cis}\left(\dfrac{3\pi}{4}\right)$

5. (b) $\text{cis}\dfrac{5\pi}{12}$ (c) $\dfrac{z_1}{z_2}=\dfrac{\sqrt{6}}{4}-\dfrac{\sqrt{2}}{4}+\text{i}\left(\dfrac{\sqrt{6}}{4}+\dfrac{\sqrt{2}}{4}\right)$ (d) $\sin\dfrac{5\pi}{12}=\dfrac{\sqrt{6}}{4}+\dfrac{\sqrt{2}}{4}$, $\cos\dfrac{5\pi}{12}=\dfrac{\sqrt{6}}{4}-\dfrac{\sqrt{2}}{4}$