IB Mathematics HL

Test - Trigonometric Functions and Equations

SOLUTION KEY

♦ **Part 1** – NO calculator allowed – Questions 1-7

1. Find all **exact** solutions to the equation $2\sin^2 x + 3\cos x = 0$ in the interval $0 \le x \le 2\pi$. [6 marks]

$2(1-\cos^2 x) + 3\cos x = 0$

$2 - 2\cos^2 x + 3\cos x = 0$

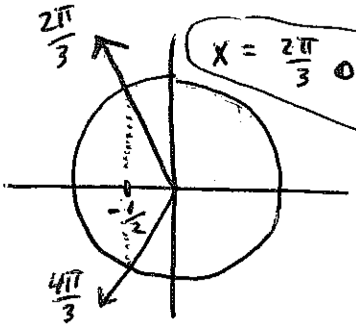
$2\cos^2 x - 3\cos x - 2 = 0$

$(2\cos x + 1)(\cos x - 2) = 0$

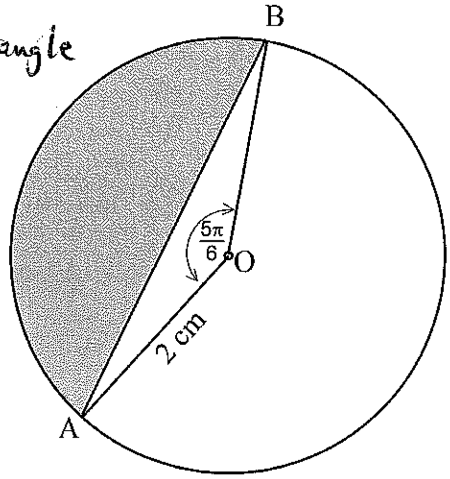
$\cos x = -\frac{1}{2}$ OR $\cos x = 2$

NO SOLUTION

$x = \frac{2\pi}{3}$ OR $\frac{4\pi}{3}$



2. The diagram below shows a circle with centre O and radius OA = 2 cm. The central angle AOB has a measure of $\frac{5\pi}{6}$ radians. Find the **exact** area of the shaded region. [4 marks]



area of shaded region = area of sector – area of triangle

$= \frac{1}{2}\left(\frac{5\pi}{6}\right)2^2 - \frac{1}{2}\cdot 2^2 \sin\frac{5\pi}{6}$

$= \frac{10\pi}{6} - 2\left(\frac{1}{2}\right)$

$= \left(\frac{5\pi}{3} - 1\right) \text{cm}^2$

[OR $\frac{5\pi - 3}{3}$ cm^2]

3. Write down the domain, range, period and amplitude for each function.

(a) $y = 3\cos\left(2x + \frac{\pi}{2}\right)$ [5 marks]

domain: $x \in \mathbb{R}$ range: $-3 \le y \le 3$

period: π amplitude: 3

(b) $y = 2 - \sin\left(\frac{x}{4}\right)$ [5 marks]

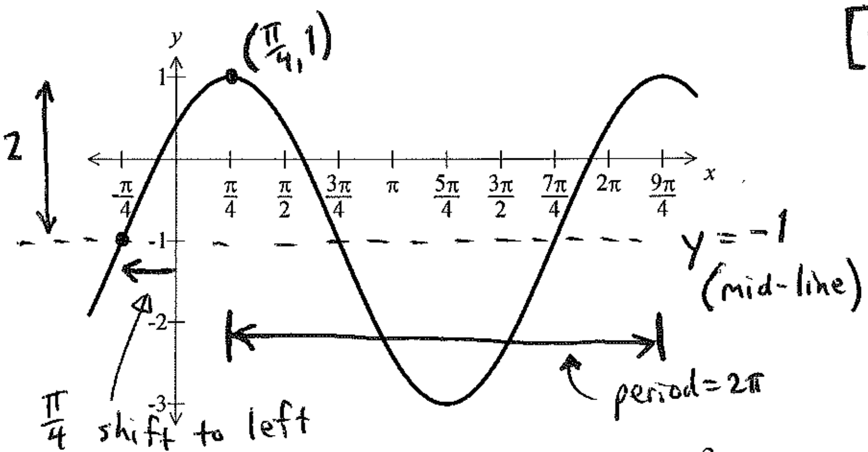
domain: $x \in \mathbb{R}$ range: $1 \le y \le 3$

period: 8π amplitude: 1

4. Write down the domain, range and period of the function $g(x) = \tan\left(\frac{x}{3}\right)$. [5 marks]

domain: $x \neq \frac{3\pi}{2} + k \cdot 3\pi,\ k \in \mathbb{Z}$ range: $y \in \mathbb{R}$ period: 3π

5. The curve shown below has an equation in the form $y = a\sin\left[b(x+c)\right] + d$, where *a, b, c* and *d* are integers. The curve passes through the point $\left(\frac{\pi}{4}, 1\right)$ **Write down** the values of *a, b, c* and *d*. [8 marks]



$a =$ 2

$b =$ 1

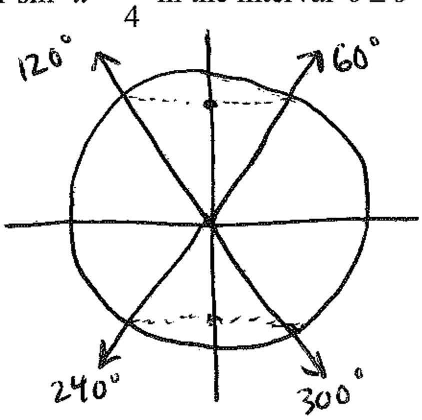
$c =$ $\frac{\pi}{4}$

$d =$ -1

6. Find all **exact** solutions to the equation $\sin^2 x = \frac{3}{4}$ in the interval $0 \le \theta < 360^\circ$. [5 marks]

$\sqrt{\sin^2 x} = \pm\sqrt{\frac{3}{4}}$

$\sin x = \pm\frac{\sqrt{3}}{2}$



$x = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

β is in 1st or 2nd quadrant; but must be 2nd quadrant since $\cos\beta$ is negative

7. If $\cos\beta = -\frac{2}{5}$ and $0 \le \beta \le \pi$, find the **exact** values of the following: [8 marks]

(a) $\sin\beta$ (b) $\tan\beta$ (c) $\csc 2\beta$

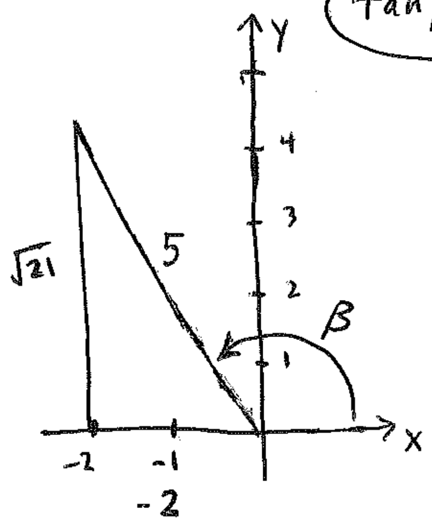
$\sin\beta = \frac{\sqrt{21}}{5}$

$\tan\beta = \frac{\sqrt{21}}{-2}$

$y^2 + 2^2 = 5^2$

$y = \sqrt{25-4}$

$y = \sqrt{21}$



$\csc 2\beta = \frac{1}{\sin 2\beta}$

$= \frac{1}{2\sin\beta\cos\beta}$

$= \frac{1}{2\left(\frac{\sqrt{21}}{5}\right)\left(-\frac{2}{5}\right)}$

$= \frac{1}{-\frac{4\sqrt{21}}{25}} = -\frac{25}{4\sqrt{21}}$

$= -\frac{25}{4\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}} = -\frac{25\sqrt{21}}{84}$

$\csc 2\beta = -\frac{25\sqrt{21}}{84}$

--- end of part 1 ---

♦ **Part 2** – Calculator allowed – Questions 8-11

[5 marks]

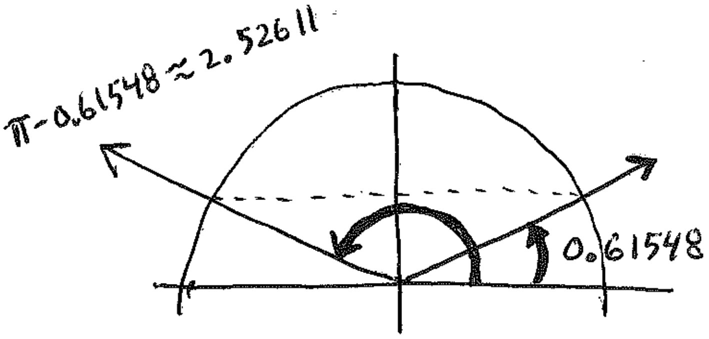
8. Find all of the values of θ in the interval $0 \le \theta \le \pi$ which satisfy the equation $\cos 2\theta = \sin^2\theta$.

$1 - 2\sin^2\theta = \sin^2\theta$

$3\sin^2\theta = 1$

$\sin^2\theta = \frac{1}{3}$

$\sin\theta = \pm\sqrt{\frac{1}{3}}$



$\theta = \sin^{-1}\left(\sqrt{\frac{1}{3}}\right) \approx 0.61548$ (1st quadrant)

OR $\theta \approx 2.52611$ (2nd quadrant

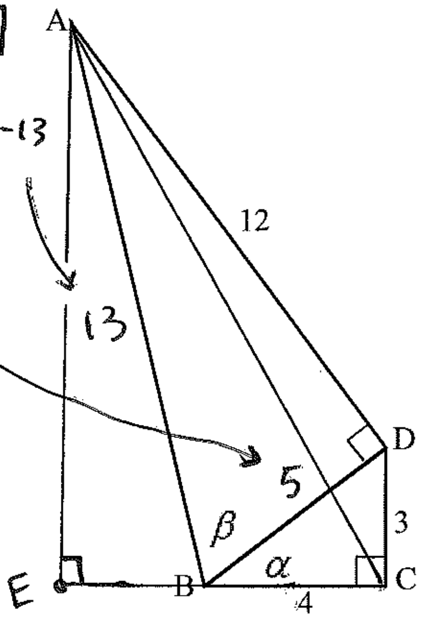
OR $\theta = \sin^{-1}\left(-\sqrt{\frac{1}{3}}\right)$ but here θ will be in 3rd or 4th quadrant which is outside the solution interval of $0 \le \theta \le \pi$

thus, solutions are $\theta \approx 0.615$ OR $\theta \approx 2.53$

(3 significant figures)

9. In the diagram below, triangle ABD and triangle DBC are right triangles, $A\hat{B}C = \alpha + \beta$, DC = 3, BC = 4 and AD = 12.

(a) Show that the exact value of $\cos A\hat{B}C = -\dfrac{16}{65}$ [4 marks]



two Pythagorean triples in diagram: 3-4-5 and 5-12-13

sum identity for cosine

$$\cos A\hat{B}C = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

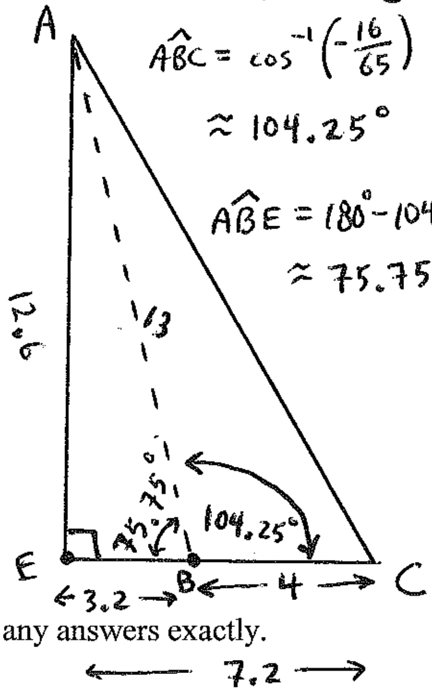
$$= \frac{4}{5}\cdot\frac{5}{13} - \frac{3}{5}\cdot\frac{12}{13}$$

$$= \frac{20}{65} - \frac{36}{65}$$

$$\cos A\hat{B}C = -\frac{16}{65} \quad \text{Q.E.D.}$$

(b) Add the line segment AC to the diagram. Find the length of AC accurate to three significant figures. [3 marks]

[could use cosine rule with ΔABC; or use right ΔAEC]



$$A\hat{B}C = \cos^{-1}\left(-\frac{16}{65}\right) \approx 104.25°$$

$$A\hat{B}E = 180° - 104.25° \approx 75.75°$$

$\sin 75.75° = \dfrac{AE}{13}$ $\quad AE = 13\sin 75.75° = 12.6$

$\cos 75.75° = \dfrac{EB}{13}$ $\quad EB = 13\cos 75.75° = 3.2$

thus, EC = 7.2

$$(AC)^2 = 7.2^2 + 12.6^2$$

$$AC = \sqrt{7.2^2 + 12.6^2}$$

$$AC \approx 14.5$$

10. Given that $\tan 2x = \dfrac{4}{3}$, find all possible values of $\tan x$. Express any answers exactly. [5 marks]

$$\frac{2\tan x}{1-\tan^2 x} = \frac{4}{3} \qquad 6\tan x = 4 - 4\tan^2 x \quad \text{(divide through by 2)}$$

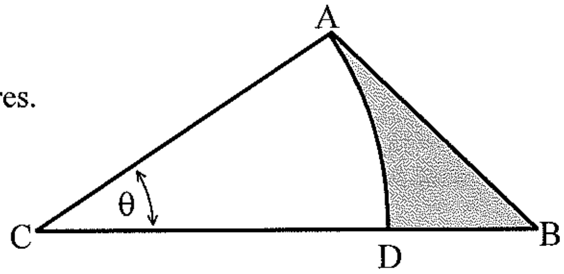
$$2\tan^2 x + 3\tan x - 2 = 0$$

$$(2\tan x - 1)(\tan x + 2) = 0$$

$$\tan x = \frac{1}{2} \quad \text{OR} \quad \tan x = -2$$

11. The triangle ABC, as shown in the diagram, has AC = 8 cm, CB = 12 cm and $A\hat{C}B = \theta$ radians.

The area of triangle ABC = 20 cm^2.



(a) Show that $\theta = 0.430$, correct to three significant figures.
[3 marks]

(b) The point D lies on CB such that AD is an arc of a circle with centre C and radius 8 cm. The region bounded by the arc AD and the line segments DB and AB is shaded in the diagram.

Calculate, to three significant figures:

[2 marks] (i) the length of the arc AD;

[2 marks] (ii) the area of the shaded region.

(a) area of $\Delta = \frac{1}{2}\cdot 12 \cdot 8 \sin\theta = 20$

$\sin\theta = \frac{20}{48}$ $\qquad \theta = \sin^{-1}\left(\frac{20}{48}\right) \approx 0.42978$

$\theta \approx 0.430$ Q.E.D.

(b)(i) arc length $= \theta r$ (θ in radians)

arc AD $= 0.429775(8) \approx 3.4382$ cm

arc AD ≈ 3.44 cm (3 significant figures)

(ii) area shaded region = area of Δ − area of sector

$= 20 - \frac{1}{2}(0.429775)8^2 \approx 6.2472 \text{ cm}^2$

area of shaded region $\approx 6.25 \text{ cm}^2$ (3 significant figures)