11 Valid comparisons and informed decisions: probability distributions

Exercise 11A

1 a $P(\text{French and German}) = P(\text{French}) + P(\text{German}) - P(\text{French or German})$

$$= \frac{10}{15} + \frac{8}{15} - \left(1 - \frac{5}{15}\right) = \frac{8}{15}$$

b $P(\text{exactly one language}) = 1 - P(\text{both}) - P(\text{neither})$

$$= 1 - \frac{8}{15} - \frac{5}{15} = \frac{2}{15}$$

2 Number of ways of picking 4 letters: $\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{362880}{24 \times 120} = 126$

$$P(\text{at least one vowel}) = 1 - P(\text{no vowels}) = 1 - \frac{\binom{6}{4}}{126} = \frac{37}{42}$$

3 Number of ways of picking 2 fruits: $\binom{9}{2} = \frac{9!}{2!(9-2)!} = \frac{362880}{2 \times 5040} = 36$

a $P(\text{2 kiwis}) = \frac{\binom{3}{2}}{36} = \frac{1}{12}$

b $P(\text{two different fruits}) = \frac{6 \times 3}{36} = \frac{1}{2}$

4 Number of ways of picking 3 crayons: $\binom{12}{3} = \frac{12!}{3!(12-3)!} = \frac{479001600}{6 \times 362880} = 220$

a $P(\text{all green}) = \frac{\binom{7}{3}}{220} = \frac{7}{44}$

b $P(\text{not all same colour}) = 1 - P(\text{all same colour}) = 1 - P(\text{all green}) - P(\text{all blue})$

$$= 1 - \frac{7}{44} - \frac{\binom{5}{3}}{220} = \frac{35}{44}$$

5 a $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.5 - 0.2 = 0.6$

b $P(B \cap A') = P(B) - P(A \cap B) = 0.5 - 0.2 = 0.3$

c $P(A' \cap B') = P(A') - P(A' \cap B) = 1 - 0.3 - 0.3 = 0.4$

6 a $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.6 - 0.7 = 0.3$

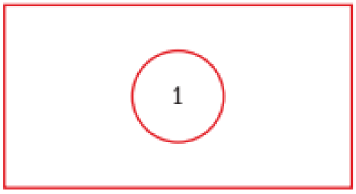
b $P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$

c $P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$
$P(A' \cap B') = P(B') - P(A \cap B') = 0.4 - 0.1 = 0.3$

so $P(A' \cup B') = 0.6 + 0.4 - 0.3 = 0.7$

7 $U = \{1, 2, 3\}$, $A = \{3\}$, $B = \varnothing$

a



b **i** $P(A) = \dfrac{n(A)}{n(U)} = \dfrac{1}{3}$

ii $P(B) = \dfrac{n(B)}{n(U)} = \dfrac{0}{3} = 0$

Exercise 11B

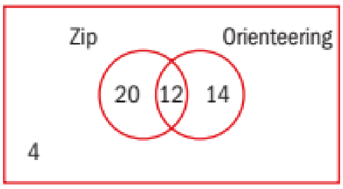
1 **a** Independent

b Independent

c Not independent

d Independent

2 **a**



b **i** P(at least one student went zip lining) $= 1$, as 32 students chose zip lining.

ii $P(\text{orienteering} \mid \text{zip lining}) = \dfrac{P(\text{both})}{P(\text{zip lining})}$,

$P(\text{both}) = P(\text{zip lining}) + P(\text{orienteering}) - P(\text{either}) = \dfrac{32}{50} + \dfrac{26}{50} - \dfrac{46}{50} = \dfrac{6}{25}$,

$P(\text{orienteering} \mid \text{zip lining}) = \dfrac{\frac{6}{25}}{\frac{32}{50}} = \dfrac{3}{8}$

iii $P(\text{orienteering}) = \dfrac{26}{50} = \dfrac{13}{25}$

iv $P(\text{zip lining} \mid \text{orienteering}) = \dfrac{P(\text{both})}{P(\text{orienteering})} = \dfrac{\frac{6}{25}}{\frac{13}{25}} = \dfrac{6}{13}$

$P(\text{orienteering} \mid \text{zip lining}) = \dfrac{\frac{6}{25}}{\frac{32}{50}} = \dfrac{3}{8}$

3 **a** $P(\text{art}) = \dfrac{5+2+7+3}{35} = \dfrac{17}{35}$

b $P(\text{biology}) = \dfrac{4+2+7+6}{35} = \dfrac{19}{35}$

c $P(\text{chemistry}) = \dfrac{1+3+7+6}{35} = \dfrac{17}{35}$

d $P(\text{art} \mid \text{biology}) = \dfrac{2+7}{2+7+4+6} = \dfrac{9}{19}$

e $P(\text{biology} \mid \text{chemistry}) = \dfrac{7+6}{3+7+6+1} = \dfrac{13}{17}$

f $P(\text{chemistry} \mid \text{not art}) = \dfrac{6+1}{35-(5+2+3+7)} = \dfrac{7}{18}$

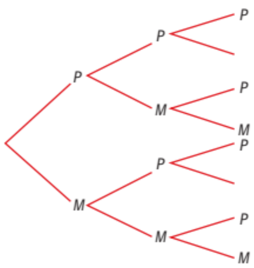
g $P(\text{homework} \mid \text{no homework}) = 0$

4 $P(\text{both blue} \mid \text{same colour}) = \dfrac{P(\text{both blue} \cap \text{same colour})}{P(\text{same colour})}$

$$= \frac{P(\text{both blue})}{P(\text{same colour})} = \frac{P(\text{both blue})}{P(\text{both blue}) + P(\text{both yellow})}$$

$$= \frac{\frac{3}{8}\times\frac{2}{7}}{\frac{3}{8}\times\frac{2}{7}+\frac{5}{8}\times\frac{4}{7}} = \frac{3}{13}$$

5 **a**



b **i** $P(\text{all puzzles}) = \dfrac{8}{13}\times\dfrac{7}{12}\times\dfrac{6}{11} = \dfrac{28}{143}$

ii $P(\text{at least one puzzle}) = 1 - P(\text{no puzzles}) = 1 - \left(\dfrac{5}{13}\times\dfrac{4}{12}\times\dfrac{3}{11}\right) = \dfrac{138}{143}$

6 Let A be the event that he scores on the first shot and B be the event that he scores on the second shot. Now, we have $P(A) = 0.85$, $P(B' \mid A) = 0.1$ and $P(B \mid A') = 0.75$. We want to find

$$\begin{aligned} P((A \cap B') \cup (A' \cap B)) &= P(A \cap B') + P(A' \cap B) \\ &= P(A)P(B' \mid A) + P(A')P(B \mid A') \\ &= 0.85 \times 0.1 + 0.15 \times 0.75 = 0.1975 \end{aligned}$$

7 **a** $P(B') = 1 - P(B) = 1 - (P(A)P(B \mid A) + P(A')P(B \mid A')) = 1 - \left(\dfrac{1}{3}\times\dfrac{3}{5}+\dfrac{2}{3}\times\dfrac{1}{2}\right) = \dfrac{7}{15}$

b $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - P(A)P(B \mid A) = \dfrac{4}{5}$

Exercise 11C

1 P(yellow) = P(both yellow)P(yellow | both yellow) + P(one of each)P(yellow | one of each) $+P$(both green)P(yellow | both green)

$$= \frac{\binom{5}{2}}{\binom{13}{2}} \times \frac{4}{10} + \frac{\binom{5}{1}\binom{8}{1}}{\binom{13}{2}} \times \frac{3}{10} + \frac{\binom{8}{2}}{\binom{13}{2}} \times \frac{2}{10} = \frac{18}{65}$$

$$P(\text{yellow from } A \mid \text{yellow}) = \frac{\binom{5}{2}}{\binom{13}{2}} \times \frac{2}{4} + \frac{\binom{5}{1}\binom{8}{1}}{\binom{13}{2}} \times \frac{1}{3} + \frac{\binom{8}{2}}{\binom{13}{2}} \times 0 = \frac{55}{234}$$

2 **a** $P(\text{male})$ = P(high income)P(male | high) + P(medium income)P(male | medium) $+P$(low income)P(male | low)

$= (0.1 \times 0.5) + (0.65 \times 0.7) + (0.25 \times 0.8) = 0.705$

b P(high income | male)

$= P$(high income and male) $\div P$(male)
$= 0.05 \div 0.705$
$= 0.07$

c P(female | high income) $= 0.5$, from the question.

3 Let D be the event that the transistor is defective and M_i be the event that the transistor is from machine i, now:

$$P(M_2 \mid D') = \frac{P(M_2)P(D' \mid M_2)}{P(M_1)P(D' \mid M_1) + P(M_2)P(D' \mid M_2) + P(M_3)P(D' \mid M_3)}$$
$$= \frac{0.45 \times 0.97}{0.5 \times 0.96 + 0.45 \times 0.97 + 0.05 \times 0.92}$$
$$= 0.4535$$

4 Let A_1 be the event that the teacher took two 320GB laptops, A_2 be the event that the teacher took one 160GB laptop and one 320GB laptop and A_3 be the event that the teacher took two 160GB laptops. Also let B be the event that the student took a 160GB laptop. We wish to find

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + P(A_3)P(B \mid A_3)}$$

$$= \frac{\frac{\binom{8}{2}}{\binom{20}{2}} \times \frac{12}{18}}{\frac{\binom{8}{2}}{\binom{20}{2}} \times \frac{12}{18} + \frac{\binom{12}{1}\binom{8}{1}}{\binom{20}{2}} \times \frac{11}{18} + \frac{\binom{12}{2}}{\binom{20}{2}} \times \frac{10}{18}} = \frac{\frac{14}{95} \times \frac{12}{18}}{\frac{14}{95} \times \frac{12}{18} + \frac{48}{95} \times \frac{11}{18} + \frac{33}{95} \times \frac{10}{18}}$$

$$= \frac{28}{171}$$

5 $P(\text{basket}) = \frac{4}{12} \times 0.009 + \frac{4}{12} \times 0.006 + \frac{4}{12} \times 0.002 = 0.00567 = 0.567\%$

6 **a** $P(\text{Rh+}) = 0.45 \times 0.63 + 0.37 \times 0.84 + 0.14 \times 0.83 + 0.04 \times 0.75 = 0.7405 = 74.05\%$

b $P(\text{AB} \mid \text{Rh-}) = \dfrac{P(\text{AB})P(\text{Rh-} \mid \text{AB})}{P(\text{O})P(\text{Rh-} \mid \text{O}) + P(\text{A})P(\text{Rh-} \mid \text{A}) + P(\text{B})P(\text{Rh-} \mid \text{B}) + P(\text{AB})P(\text{Rh-} \mid \text{AB})}$

$= \dfrac{0.04 \times 0.25}{0.45 \times 0.37 + 0.37 \times 0.16 + 0.14 \times 0.17 + 0.04 \times 0.25}$

$= 0.03853 = 3.853\%$

Exercise 11D

1 a Yes

b No because the sum of the probabilities is not equal to 1

2 a

x	0	1	2	3	4
$P(X = x)$	0	$\frac{1}{2}$	4	$\frac{27}{2}$	32

Not a probability distribution as some of the probabilities are greater than 1

b

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{6}{17}$	$\frac{12}{11}$

Not a probability distribution as one of the probabilities is greater than 1

3 a Need to find k such that $k(10+11+12) = 1$, so $k = \dfrac{1}{10+11+12} = \dfrac{1}{33}$

b The mode is the most likely outcome, which in this case is $x = 12$

c $P(X \text{ is even}) = P(X = 10) + P(X = 12) = \dfrac{10}{33} + \dfrac{12}{33} = \dfrac{2}{3}$

Exercise 11E

1 $E(X) = \Sigma x P(X = x) = 0 \times \dfrac{1}{6} + 1 \times \dfrac{1}{6} + 2 \times \dfrac{1}{3} + 3 \times \dfrac{1}{4} + 4 \times \dfrac{1}{12} = \dfrac{23}{12}$

2 Have to find k such that $0.1 + 0.2 + k + 2k + (k - 0.1) = 1$

so $1 = 4k - 0.2 \Rightarrow 4k = 0.8 \Rightarrow k = 0.2$,

$E(X) = \Sigma x P(X = x) = (4 \times 0.1) + (5 \times 0.2) + (6 \times 0.2) + (7 \times 0.4) + (8 \times 0.1) = 6.2$

3 $P(X = 3) = 1 - P(X \le 2) = 0.15$, $P(X = 2) = P(X \le 2) - P(X \le 1) = 0.35$, let $p_0 = P(X = 0)$ and $p_1 = P(X = 1)$. Then $E(X) = 0 \times p_0 + 1 \times p_1 + 2 \times 0.35 + 3 \times 0.15 = 1.45$ gives us that $p_1 = 0.3$ and therefore $p_0 = 0.2$.

a

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.35	0.15

b The mode is $x = 2$

4 There are $\binom{10}{3} = 120$ ways of picking the three students.

x	0	1	2	3
$P(X = x)$	$\frac{1}{120}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{30}$

5 **a** Need to find p where $p + 2p = 1$, so $p = \frac{1}{3}$

b $E(X) = \Sigma xP(X = x) = 2\left(\frac{1}{3}\right) + 5\left(\frac{1}{3}\right) = \frac{2}{3} + \frac{10}{3} = 4$

Exercise 11F

1 **a** $E(X) = \Sigma xP(X = x) = 1 \times \frac{1}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{2} = \frac{1}{8} + \frac{6}{8} + \frac{3}{2} = \frac{19}{8} = 2.375$

b $E(5X) = 5E(X) = 5 \times 2.375 = 11.875$

c $E(X^2) = 1^2 \times \frac{1}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{2} = \frac{1}{8} + \frac{12}{8} + \frac{9}{2} = \frac{49}{8} = 6.125$

d $Var(X) = E(X^2) - (E(X))^2 = 6.125 - 2.375^2 = 0.484$ (3.s.f.)

e Standard deviation of $X = \sqrt{Var(X)}$

$= \sqrt{0.484...} = 0.696$ (3s.f.)

2 **a** $P(X \geq 1) = 2P(X \leq 2) \Rightarrow a + a + 2a + b = 2(a + a + a) \Rightarrow b = a$, so combining with $5a + b = 1$ gives us that $6a = 1 \Rightarrow a = b = \frac{1}{6}$

b $E(X) = 0 \times \frac{1}{6} + 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{2}{6} + 4 \times \frac{1}{6} = \frac{13}{6}$

$E(X^2) = 0^2 \times \frac{1}{6} + 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{2}{6} + 4^2 \times \frac{1}{6} = \frac{13}{2}$

c $Var(X) = E(X^2) - (E(X))^2 = \frac{13}{2} - \left(\frac{13}{6}\right)^2 = \frac{65}{36}$

3 Let X_1 be the value of the bottom card and let X_2 be the value of the top card

a $P(S = 4) = P(X_1 = A \cap X_2 = 3) + P(X_1 = 3 \cap X_2 = A) = \frac{1}{10} \times \frac{1}{9} + \frac{1}{10} \times \frac{1}{9} = \frac{2}{90} = \frac{1}{45}$

$P(S = 8) = P(X_1 = A \cap X_2 = 7) + P(X_1 = 2 \cap X_2 = 6) + P(X_1 = 3 \cap X_2 = 5) +$
$+P(X_1 = 5 \cap X_2 = 3) + P(X_1 = 6 \cap X_2 = 2) + P(X_1 = 7 \cap X_2 = A) =$
$= 6 \times \left(\frac{1}{10} \times \frac{1}{9}\right) = \frac{6}{90} = \frac{1}{15}$

$$P(S=11)=P(X_1=A\cap X_2=10)+P(X_1=2\cap X_2=9)+P(X_1=3\cap X_2=8)+$$
$$+P(X_1=4\cap X_2=7)+P(X_1=5\cap X_2=6)+P(X_1=6\cap X_2=7)+P(X_1=7\cap X_2=4)+$$
$$+P(X_1=8\cap X_2=3)+P(X_1=9\cap X_2=2)+P(X_1=10\cap X_2=A)=$$
$$=10\times\left(\frac{1}{10}\times\frac{1}{9}\right)=\frac{10}{90}=\frac{1}{9}$$

b

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$P(S=s)$	$\frac{2}{90}$	$\frac{2}{90}$	$\frac{4}{90}$	$\frac{4}{90}$	$\frac{6}{90}$	$\frac{6}{90}$	$\frac{8}{90}$	$\frac{8}{90}$	$\frac{10}{90}$	$\frac{8}{90}$	$\frac{8}{90}$	$\frac{6}{90}$	$\frac{6}{90}$	$\frac{4}{90}$	$\frac{4}{90}$	$\frac{2}{90}$	$\frac{2}{90}$

c $E(S)=11$ (because of symmetry)

$E\left(S^2\right)=\Sigma s^2P(S=s)-E(S)^2=\frac{407}{3}-121=\frac{44}{3}$

4 a Need to find k such that $k\left((8-4)^2+(8-5)^2+(8-6)^2+(8-7)^2\right)+k\left(1^2+2^2+3^2\right)=1$, so

$30k+14k=1\Rightarrow k=\frac{1}{30+14}=\frac{1}{44}$

b $P(T=4)=\frac{1}{44}(8-4)^2=\frac{4}{11}$

$P(T\le 4)=\frac{1}{44}\left(1^2+2^2+3^2\right)+\frac{4}{11}=\frac{15}{22}$

$P(T=4\mid T\le 4)=\frac{P(T=4\cap T\le 4)}{P(T\le 4)}=\frac{\frac{4}{11}}{\frac{15}{22}}=\frac{8}{15}$

c $E(T)=1\times\frac{1^2}{44}+2\times\frac{2^2}{44}+3\times\frac{3^2}{44}+4\times\frac{(8-4)^2}{44}+5\times\frac{(8-5)^2}{44}+6\times\frac{(8-6)^2}{44}+7\times\frac{(8-7)^2}{44}$

$=4$

$Var(T)=E\left(T^2\right)-(E(T))^2$

$=1^2\times\frac{1^2}{44}+2^2\times\frac{2^2}{44}+3^2\times\frac{3^2}{44}+4^2\times\frac{(8-4)^2}{44}+5^2\times\frac{(8-5)^2}{44}+6^2\times\frac{(8-6)^2}{44}+7^2\times\frac{(8-7)^2}{44}-16$

$=\frac{193}{11}-16=\frac{17}{11}$

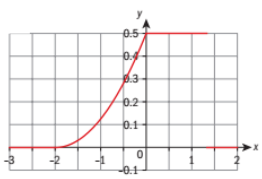
d The mode is the most likely value which is $t=4$

Exercise 11G

1 a Need to find k such that $1=\int_{-2}^{0}k(x+2)^2\,\mathrm{d}x+\int_{0}^{\frac{4}{3}}4k\,\mathrm{d}x$,

$1=\int_{-2}^{0}k(x+2)^2\,\mathrm{d}x+\int_{0}^{\frac{4}{3}}4k\,\mathrm{d}x=\left[\frac{k}{3}(x+2)^3\right]_{-2}^{0}+\left[4kx\right]_{0}^{\frac{4}{3}}=\frac{8k}{3}+\frac{16k}{3}=8k\Rightarrow k=\frac{1}{8}$

b

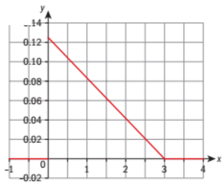


c **i** $P(X \le -1) = \int_{-2}^{-1} \frac{(x+2)^2}{8} \mathrm{d}x = \left[\frac{(x+2)^3}{24}\right]_{-2}^{-1} = \frac{1}{24}$

ii $P(0 \le X \le 1) = \int_0^1 \frac{1}{2} \mathrm{d}x = \left[\frac{x}{2}\right]_0^1 = \frac{1}{2}$

iii $P(-1 \le X \le 1) = P(-1 \le X \le 0) + \frac{1}{2} = \int_{-1}^{0} \frac{(x+2)^2}{8} \mathrm{d}x + \frac{1}{2} = \left[\frac{(x+2)^3}{24}\right]_{-1}^{0} + \frac{1}{2} = \frac{7}{24} + \frac{1}{2} = \frac{19}{24}$

2 **a**



b Need to find k such that $1 = \int_0^3 k(3-x)\,\mathrm{d}x$, $1 = \int_0^3 k(3-x)\,\mathrm{d}x = \left[-k\left(\frac{x^2}{2} - 3x\right)\right]_0^3 = \frac{9k}{2} \Rightarrow k = \frac{2}{9}$

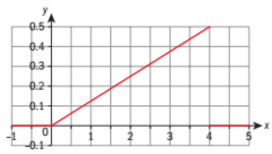
c $P(1.2 \le X \le 2.3) = \int_{1.2}^{2.3} \frac{2(3-x)}{9} \mathrm{d}x = \left[\frac{6x - x^2}{9}\right]_{1.2}^{2.3} = \frac{11}{36}$

3 **a** Need to find c such that $1 = \int_0^1 2x^c\,\mathrm{d}x$, $1 = \int_0^1 2x^c\,\mathrm{d}x = \left[\frac{2x^{c+1}}{c+1}\right]_0^1 = \frac{2}{c+1} \Rightarrow c = 1$

b $P(X < 0.5) = \int_0^{0.5} 2x\,\mathrm{d}x = \left[x^2\right]_0^{0.5} = \frac{1}{4}$

Exercise 11H

1



a This function is a probability density function because it is non-negative for all possible values and the integral is equal to 1. $\int_0^4 \frac{x}{8} \mathrm{d}x = \left[\frac{x^2}{16}\right]_0^4 = 1$

b $P(1 < x < 20) = \int_1^4 \frac{x}{8} \mathrm{d}x + \int_4^{20} 0\,\mathrm{d}x = \left[\frac{x^2}{16}\right]_1^4 + 0 = \frac{15}{16}$

c Mean $= \int_0^4 \frac{x^2}{8} \mathrm{d}x = \left[\frac{x^3}{24}\right]_0^4 = \frac{8}{3}$

Mode: As $f(x)$ is strictly increasing, the mode occurs at the point $x = 4$

Median: Need to find M such that $0.5=\int_0^M \frac{x}{8}\,dx$, so

$$0.5=\int_0^M \frac{x}{8}\,dx=\left[\frac{x^2}{16}\right]_0^M=\frac{M^2}{16}\Rightarrow M=\pm 2\sqrt{2}\Rightarrow M=2\sqrt{2}$$

$$\text{Standard deviation}=\sqrt{\int_0^4 \frac{x^3}{8}\,dx-\left(\frac{8}{3}\right)^2}=\sqrt{\left[\frac{x^4}{32}\right]_0^4-\left(\frac{8}{3}\right)^2}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$$

2 Need to find a such that $1=\int_0^2 ax(2-x)\,dx$, so $1=\int_0^2 ax(2-x)\,dx=\left[a\left(x^2-\frac{x^3}{3}\right)\right]_0^2=\frac{4a}{3}\Rightarrow a=\frac{3}{4}$

a $E(X)=\int_0^2 \frac{3}{4}x^2(2-x)\,dx=\left[\frac{8x^3-3x^4}{16}\right]_0^2=1$

b $Var(X)=\int_0^2 \frac{3}{4}x^3(2-x)\,dx-1=\left[\frac{3(5x^4-2x^5)}{40}\right]_0^2-1=\frac{6}{5}-1=\frac{1}{5}$

c Need to find M such that $0.5=\int_0^M \frac{3}{4}x(2-x)\,dx$, so

$$0.5=\int_0^M \frac{3}{4}x(2-x)\,dx=\left[\frac{3}{4}\left(x^2-\frac{x^3}{3}\right)\right]_0^M=\frac{3M^2}{4}-\frac{M^3}{4}\Rightarrow M=1,1\pm\sqrt{3}\Rightarrow M=1$$

d As $f(x)$ is symmetric, the mode occurs at the middle value $x=1$

3 a $P\left(\frac{\pi}{12}<X<\frac{\pi}{6}\right)=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} 2\cos(2x)\,dx=\left[\sin(2x)\right]_{\frac{\pi}{12}}^{\frac{\pi}{6}}=\frac{1}{2}\left(\sqrt{3}-1\right)$

b Median: Need to find M such that $0.5=\int_0^M 2\cos(2x)\,dx$, so

$$0.5=\int_0^M 2\cos(2x)\,dx=\left[\sin(2x)\right]_0^M=\sin(2M)\Rightarrow M=\frac{\sin^{-1}(0.5)}{2}=\frac{\pi}{12}$$

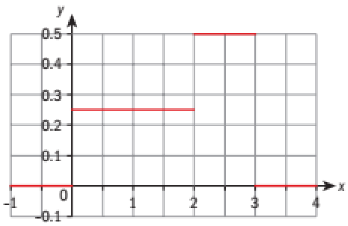
$$\text{Mean}=\int_0^{\frac{\pi}{4}} 2x\cos(2x)\,dx=\left[\frac{1}{2}\cos(2x)+x\sin(2x)\right]_0^{\frac{\pi}{4}}=\frac{1}{4}(\pi-2)$$

Mode: As $f(x)$ is decreasing over the whole interval (from knowledge about cos) we know that the mode is at $x=0$

4 a Need to find k such that $1=\int_0^2 k\,dx+\int_2^3 2k\,dx$, so

$$1=\int_0^2 k\,dx+\int_2^3 2k\,dx=\left[kx\right]_0^2+\left[2kx\right]_2^3=2k+2k=4k\Rightarrow k=\frac{1}{4}$$

b Median is $x=1$.



c $E(X) = \int_0^2 \frac{x}{4}\,dx + \int_2^3 \frac{x}{2}\,dx = \left[\frac{x^2}{8}\right]_0^2 + \left[\frac{x^2}{4}\right]_2^3 = \frac{1}{2} + \frac{5}{4} = \frac{7}{4}$

$Var(X) = \int_0^2 \frac{x^2}{4}\,dx + \int_2^3 \frac{x^2}{2}\,dx - \left(\frac{7}{4}\right)^2 = \left[\frac{x^3}{12}\right]_0^2 + \left[\frac{x^3}{6}\right]_2^3 - \frac{49}{16} = \frac{2}{3} + \frac{19}{6} - \frac{49}{16} = \frac{37}{48}$

5 a Need to find b in terms of a such that $1 = \int_0^3 ax^2 + b\,dx$, so

$1 = \int_0^3 ax^2 + b\,dx = \left[\frac{ax^3}{3} + bx\right]_0^3 = 9a + 3b \Rightarrow b = \frac{1}{3} - 3a$

b $0.5 = \int_0^1 ax^2 - 3a + \frac{1}{3}\,dx = \left[\frac{1}{3}\left(x - 9ax + ax^3\right)\right]_0^1 = \frac{1}{3} - \frac{8a}{3} \Rightarrow a = -\frac{1}{16}$

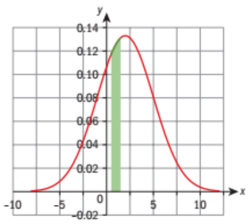
c $E(X) = \int_0^3 \frac{25x}{48} - \frac{x^3}{16}\,dx = \left[\frac{1}{48}\left(\frac{25x^2}{2} - \frac{3x^4}{4}\right)\right]_0^3 = \frac{69}{64}$

$Var(X) = \int_0^3 \frac{25x^2}{48} - \frac{x^4}{16}\,dx - \left(\frac{69}{64}\right)^2 = \left[\frac{1}{48}\left(\frac{25x^3}{3} - \frac{3x^5}{5}\right)\right]_0^3 - \frac{4761}{4096} = \frac{33}{20} - \frac{4761}{4096} = \frac{9987}{20480}$

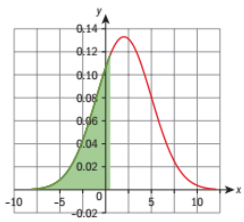
Exercise 11J

1 a $P(0.5 < X < 1.5) = P\left(\frac{0.5-2}{3} < X < \frac{1.5-2}{3}\right) = \Phi(-0.1667) - \Phi(-0.5) = 0.4338 - 0.3085$

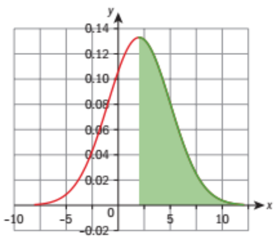
$= 0.1253$



b $P(X < 0.5) = P\left(X < \frac{0.5-2}{3}\right) = \Phi(-0.5) = 0.3085$



c $P(X \geq 2) = P\left(X \geq \frac{2-2}{3}\right) = 1 - \Phi(0) = 0.5$



2 **a** $P(X < 25) = P\left(X < \frac{25-30}{8}\right) = \Phi(-0.625) = 0.2660$

b $P(17 \le X < 35) = P\left(\frac{17-30}{8} \le X < \frac{35-30}{8}\right) = \Phi(0.625) - \Phi(-1.625) = 0.7340 - 0.0521$

$= 0.6819$

c $P(X \ge 12) = P\left(X \ge \frac{12-30}{8}\right) = 1 - \Phi(-2.25) = 1 - 0.5 = 1 - 0.0122 = 0.9878$

3 **a** Mean: 5, Standard deviation: 3

b $P(X < 4) = P\left(Z < \frac{4-5}{3}\right) = \Phi(-0.3333) = 0.3695$

c $0.011 = P(X < a) = P\left(Z < \frac{a-5}{3}\right) = \Phi\left(\frac{a-5}{3}\right) \Rightarrow \Phi^{-1}(0.011) = -2.2904 = \frac{a-5}{3} \Rightarrow a = -1.8712$

$0.871 = P(X \ge b) = P\left(Z \ge \frac{b-5}{3}\right) = 1 - \Phi\left(\frac{b-5}{3}\right) \Rightarrow \frac{b-5}{3} = \Phi^{-1}(1-0.871) = -1.1311$

$\Rightarrow a = -1.8712$

4 **a** Mean $= \frac{220 \times 11 + 240 \times 21 + 260 \times 38 + 280 \times 17 + 300 \times 13}{11 + 21 + 38 + 17 + 13} = \frac{26000}{100} = 260$

Standard deviation

$= \sqrt{\frac{220^2 \times 11 + 240^2 \times 21 + 260^2 \times 38 + 280^2 \times 17 + 300^2 \times 13}{11 + 21 + 38 + 17 + 13} - 260^2} = \sqrt{\frac{6813600}{100} - 67600}$

$= 25.152$

b $\Phi^{-1}(0.05) = -1.6449$ and $\Phi^{-1}(0.95) = 1.6449$, so need to find a and b such that

$\frac{a-260}{25.152} = -1.6449$ and $\frac{b-260}{25.152} = 1.6449$. Therefore $a = 218.6$ g and $b = 301.4$ g

5 $X \sim N(150, 0.5)$

a $P(X < 149) = P\left(Z < \frac{149-150}{0.5}\right) = \Phi(-2) = 0.0228$

b $P(X > 151.5) = P\left(Z > \frac{151.5-150}{0.5}\right) = 1 - \Phi(3) = 0.00135$

c $P(149 < X < 151) = P\left(\frac{149-150}{0.5} < Z < \frac{151-150}{0.5}\right) = \Phi(2) - \Phi(-2) = 0.9773 - 0.2275 = 0.955$

6 $T \sim N(13.2, 1.5)$

a $P(T < 12.1) = P\left(Z < \frac{12.1-13.2}{1.5}\right) = \Phi(-0.7333) = 0.232$

$P(T > 14.9) = P\left(Z > \frac{14.9-13.2}{1.5}\right) = \Phi(1.1333) = 0.871$

b If $P(T < t) = 0.444$ then $\frac{t-13.2}{1.5} = \Phi^{-1}(0.444) = -0.1408 \Rightarrow t = 12.989$

Exercise 11K

1 $0.321 = P(X \le 4) = P\left(Z < \frac{4-8}{\sigma}\right) = \Phi\left(\frac{4-8}{\sigma}\right) \Rightarrow \Phi^{-1}(0.321) = -0.4649 = \frac{4-8}{\sigma}$

$\Rightarrow \sigma = 8.604 \Rightarrow \sigma^2 = 74.03$

2 $0.345 = P(X \le 1) = P\left(Z < \frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.345) = -0.3989 = \frac{1-\mu}{\sigma}$ and

$0.943 = P(X \le 3) = P\left(Z < \frac{3-\mu}{\sigma}\right) = \Phi\left(\frac{3-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.943) = 1.5805 = \frac{3-\mu}{\sigma}$ solving simultaneously gives $\mu = 1.403$ and $\sigma = 1.010$

3 $0.013 = P(X \le 1) = P\left(Z < \frac{1-1.02}{\sigma}\right) = \Phi\left(\frac{1-1.02}{\sigma}\right) \Rightarrow \Phi^{-1}(0.013) = -2.2262 = \frac{1-1.02}{\sigma}$

$\Rightarrow \sigma = 0.00898$

4 **a** $0.203 = P(X > 46.8) = P\left(Z > \frac{46.8-\mu}{\sigma}\right) = 1-\Phi\left(\frac{46.8-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.203) = 0.8310$

$= \frac{46.8-\mu}{\sigma}$ and $0.315 = P(X \le 42.6) = P\left(Z < \frac{42.6-\mu}{\sigma}\right) = \Phi\left(\frac{42.6-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.315)$

$= -0.4817 = \frac{42.6-\mu}{\sigma}$ solving simultaneously gives $\mu = 44.141$ and $\sigma = 3.200$

b $P\left(|X-\mu| < \frac{\sigma}{2}\right) = P\left(\mu - \frac{\sigma}{2} < X < \mu + \frac{\sigma}{2}\right) = P\left(\frac{-1}{2} < Z < \frac{1}{2}\right) = \Phi\left(\frac{1}{2}\right) - \Phi\left(\frac{-1}{2}\right)$

$= 0.6915 - 0.3085 = 0.383$

5 **a** $P(200 < X < 350) = P\left(\frac{200-320}{20} < Z < \frac{350-320}{20}\right) = \Phi\left(\frac{350-320}{20}\right) - \Phi\left(\frac{200-320}{20}\right)$

$= 0.9332 - 0 = 0.9332$

b $0.1 = P(X > m) = P\left(Z > \frac{m-320}{20}\right) = 1-\Phi\left(\frac{m-320}{20}\right) \Rightarrow \Phi^{-1}(1-0.1) = 1.2816 = \frac{m-320}{20}$

$\Rightarrow m = 345.63\text{g}$

c $P(X > 350) = P\left(Z > \frac{350-320}{20}\right) = 1-\Phi\left(\frac{350-320}{20}\right) = 0.0668$, the expected number sold that weighed more than 350g is $500 \times 0.0668 = 33.4$

d $0.15 = P(Y > 400) = P\left(Z > \frac{400-\mu}{\sigma}\right) = 1-\Phi\left(\frac{400-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.15) = 1.0364 = \frac{400-\mu}{\sigma}$ and

$0.1 = P(X \le 370) = P\left(Z < \frac{370-\mu}{\sigma}\right) = \Phi\left(\frac{370-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.1) = -1.2816 = \frac{370-\mu}{\sigma}$ solving simultaneously gives $\mu = 386.59$ g and $\sigma = 12.942$ g

6 We wish to find the maximal value of σ such that

$0.01 < P(X < 1) = P\left(Z < \frac{1-1.03}{\sigma}\right) = \Phi\left(\frac{1-1.03}{\sigma}\right) \Rightarrow \Phi^{-1}(0.01) = -2.3264 < \frac{1-1.03}{\sigma} \Rightarrow \sigma < 0.0129$

Chapter review

1 $U=\{Mon,Tue,Wed,Thu,Fri,Sat,Sun\}^2$, $A=\{(Mon,Mon)\}$,
$B=\{(Mon,Mon),(Tue,Tue),(Wed,Wed),(Thu,Thu),(Fri,Fri),(Sat,Sat),(Sun,Sun)\}$,
$C=\{(Mon,Tue),(Tue,Wed),(Wed,Thu),(Thu,Fri),(Fri,Sat),(Sat,Sun),(Sun,Mon),$
$(Mon,Sun),(Sun,Sat),(Sat,Fri),(Fri,Thu),(Thu,Wed),(Wed,Tue),(Tue,Mon)\}$

a $P(A)=\frac{n(A)}{n(U)}=\frac{1}{49}$

b $P(B)=\frac{n(B)}{n(U)}=\frac{7}{49}=\frac{1}{7}$

c This question should be clearer, what is defined as the same week? $P(C)=\frac{n(C)}{n(U)}=\frac{14}{49}=\frac{2}{7}$

2 $U=\{0,1,2,3,4,5,6,7,8,9\}$, $A=\{3,6,7,9\}$, $P(A)=\frac{n(A)}{n(U)}=\frac{4}{10}=\frac{2}{5}$

3 Want to find n such that $P(\text{six at least once})>\frac{9}{10}\Rightarrow P(\text{no sixes})=\frac{1}{10}$, so

$$\frac{1}{10}\le\binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n=1\times1\times\left(\frac{5}{6}\right)^n\Rightarrow n\ge\frac{\log(10)}{\log\left(\frac{6}{5}\right)}\Rightarrow n=12$$

4 **a** $P(\text{first red})=\frac{7}{10}$

b $P(\text{second red})=P(\{RR,WR\})=\frac{7}{10}\times\frac{6}{9}+\frac{3}{10}\times\frac{7}{9}=\frac{7}{10}$

c $P(\text{exactly one red})=P(\{RW,WR\})=\frac{7}{10}\times\frac{3}{9}+\frac{3}{10}\times\frac{7}{9}=\frac{7}{15}$

d $P(\text{at least one red})=P(\{RW,WR,RR\})=\frac{7}{10}\times\frac{3}{9}+\frac{3}{10}\times\frac{7}{9}+\frac{7}{10}\times\frac{6}{9}=\frac{14}{15}$

5 **a** $P(\text{infection})=0.2\times0.1+0.8\times0.75=0.62$

b $P(\text{vaccinated}\mid\text{infection})=\frac{P(\text{vaccinated}\cap\text{infection})}{P(\text{infection})}=\frac{0.1}{0.62}=0.161$

c $P(\text{not vaccinated}\mid\text{infection})=1-P(\text{vaccinated}\mid\text{infection})=1-0.161=0.839$

6 $P(\text{2 wins})=\binom{8}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0=\frac{1}{256}$,

$P(\text{1 win})=\binom{8}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1+\binom{8}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3+\binom{8}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5+\binom{8}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^7=\frac{1}{2}$,

$P(\text{loss})=1-\frac{1}{256}-\frac{1}{2}=\frac{127}{256}$. By encoding a win as a 1 and a loss as a -1, the expected number of wins and losses is $2\times\frac{1}{256}+1\times\frac{1}{2}+(-1)\times\frac{127}{256}=\frac{3}{256}$

7 **a** $P(\text{no defective})=\binom{5}{0}\left(\frac{4}{20}\right)^0\left(\frac{16}{20}\right)^5=0.328$

b $P(\text{at least 3 defective}) = P(\text{3 defective}) + P(\text{4 defective}) + P(\text{5 defective}) = \sum_{i=3}^{5}\binom{5}{i}\left(\frac{4}{20}\right)^{i}\left(\frac{16}{20}\right)^{5-i}$

$= 0.0579$

8 a $P(H > 1500) = P\left(Z > \frac{1500-1300}{125}\right) = 1 - \Phi\left(\frac{1500-1300}{125}\right) = 0.0548$

b $P(H < 1050) = P\left(Z < \frac{1050-1300}{125}\right) = \Phi\left(\frac{1050-1300}{125}\right) = 0.0228 = 2.28\%$

c $(P(1200 < H < 1400))^2 = \left(P\left(\frac{1200-1300}{125} < H < \frac{1400-1300}{125}\right)\right)^2$

$= \left(\Phi\left(\frac{1400-1300}{125}\right) - \Phi\left(\frac{1200-1300}{125}\right)\right)^2 = (0.7881 - 0.2119)^2 = 0.332$

9 $0.04 = P(X > 2.01) = P\left(Z > \frac{2.01-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{2.01-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(1-0.04) = 1.7507 = \frac{2.01-\mu}{\sigma}$ and

$0.07 = P(X \le 1.99) = P\left(Z < \frac{1.99-\mu}{\sigma}\right) = \Phi\left(\frac{1.99-\mu}{\sigma}\right) \Rightarrow \Phi^{-1}(0.07) = -1.4758 = \frac{1.99-\mu}{\sigma}$ solving

simultaneously gives $\mu = 2.00$ and $\sigma = 0.00620$

10 a $P(\text{faulty}) = 0.01 \times 0.35 + 0.03 \times 0.20 + 0.025 \times 0.24 + 0.02 \times 0.21 = 0.0197$

b $P(\text{conveyor } D \mid \text{faulty}) = \frac{P(\text{conveyor } D \cap \text{faulty})}{P(\text{faulty})} = \frac{0.02 \times 0.21}{0.0197} = 0.213$

11 To show that $k = \pi$, have to show that $\int_0^1 \pi x \sin(\pi x)\,dx = 1$,

$\int_0^1 \pi x \sin(\pi x)\,dx = \left[\frac{\sin(\pi x)}{\pi} - x\cos(\pi x)\right]_0^1 = 1 - 0 = 1$.

Mean $= \int_0^1 \pi x^2 \sin(\pi x)\,dx = \left[\frac{(2-\pi^2x^2)\cos(\pi x) + 2\pi x\sin(\pi x)}{\pi^2}\right]_0^1 = 1 - \frac{4}{\pi^2}$

Variance

$= \int_0^1 \pi x^3 sin(\pi x)\,dx - \left(1 - \frac{4}{\pi^2}\right)^2$

$= \left[\frac{\pi x(6-\pi^2x^2)\cos(\pi x) + 3(\pi^2x^2-2)\sin(\pi x)}{\pi^3}\right]_0^1 - \left(1-\frac{4}{\pi^2}\right)^2 = \left(1-\frac{6}{\pi^2}\right) - \left(1-\frac{4}{\pi^2}\right)^2$

$= \frac{2(\pi^2-8)}{\pi^4}$

Exam-style questions

12 a If they were mutually exclusive, then $P(A \cap B) = 0$, (1 mark)

but since they are independent, we have $P(A \cap B) = P(A)P(B) = 0.3 \times 0.8 \neq 0$.

Therefore, we have a contradiction, and so A and B are not mutually exclusive.

(1 mark)

b i $P(A \cap B) = P(A)P(B) = 0.3 \times 0.8 = 0.24$ (2 marks)

ii $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.8 - 0.24 = 0.86$ (2 marks)

iii $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{P(A) - P(A \cap B)}{P(B')} = \dfrac{0.3 - 0.24}{0.2} = \dfrac{0.06}{0.2} = 0.3$ (2 marks)

iv $P(A' \cap B) = P(B) - P(A \cap B) = 0.8 - 0.24 = 0.56$ (2 marks)

13 a $\dfrac{k}{2} + k + k^2 + 2k^2 + \dfrac{k}{2} = 1$ (1 mark)

$3k^2 + 2k - 1 = 0$ (1 mark)

$(3k - 1)(k + 1) = 0$ (1 mark)

$\Rightarrow k = \dfrac{1}{3}$ (1 mark)

b $E(X) = \sum xP(X = x) = 0 \times \dfrac{k}{2} + 0.5 \times k + 1 \times k^2 + 1.5 \times 2k^2 + 2 \times \dfrac{k}{2}$ (1 mark)

$E(X) = \dfrac{k}{2} + k^2 + 3k^2 + k$

$= 4k^2 + \dfrac{3k}{2}$ (1 mark)

$= 4\left(\dfrac{1}{3}\right)^2 + \dfrac{3}{2}\left(\dfrac{1}{3}\right)$

$= \dfrac{4}{9} + \dfrac{1}{2}$ (1 mark)

$= \dfrac{17}{18}$

c $P(X \ge 1.25) = 2k^2 + \dfrac{k}{2}$ (1 mark)

$= 2\left(\dfrac{1}{3}\right)^2 + \dfrac{1}{6}$ (1 mark)

$= \dfrac{2}{9} + \dfrac{1}{6}$

$= \dfrac{7}{18}$ (1 mark)

d $\text{Var}(X) = E(X^2) - [E(X)]^2$ (1 mark)

$= \left[0^2 \times \dfrac{1}{6} + \left(\dfrac{1}{2}\right)^2 \times \dfrac{1}{3} + 1^2 \times \dfrac{1}{9} + \left(\dfrac{3}{2}\right)^2 \times \dfrac{2}{9} + 2^2 \times \dfrac{1}{6}\right] - \left(\dfrac{17}{18}\right)^2$ (1 mark)

$= 0.469$ (1 mark)

14 a $X \sim B(24, 0.04)$

$P(X=2)=\binom{24}{2}(0.04)^2(0.96)^{22}$ (1 mark)

$=0.180$ (1 mark)

b $P(X\leq 4)=0.998$ (2 marks)

c $P(X\geq 2)=0.249$ (2 marks)

d $\text{Var}(X)=np(1-p)$ (1 mark)

$=24\times 0.04\times 0.96$

$=0.922$ (1 mark)

15 a $X\sim N(36,3.12^2)$

$P(X>40)=P\left(Z>\frac{40-36}{3.12}\right)$ (1 mark)

$=0.1$ (1 mark)

b $P(34<X<38)=P\left(\frac{34-36}{3.12}<Z<\frac{38-36}{3.12}\right)$ (1 mark)

$=P(-0.641<Z<0.641)$ (1 mark)

$=P(Z<0.641)-P(Z<-0.641)$ (1 mark)

$=0.739-0.261$

$=0.478$ (1 mark)

c $P\left(Z>\frac{M-36}{3.12}\right)=0.015$ (1 mark)

$P\left(Z<\frac{M-36}{3.12}\right)=0.985$

$\frac{M-36}{3.12}=2.170$ (1 mark)

$\Rightarrow M=42.77$

$\Rightarrow M=42$ minutes, 46 seconds (1 mark)

d $P(X<30)=P\left(Z<\frac{30-36}{3.12}\right)$ (1 mark)

$=P(Z<-1.923)$

$=0.027$ (1 mark)

$195\times 0.027=5.3$ (1 mark)

Therefore, the expected number of days is 5. (1 mark)

16 a Let X be the discrete random variable 'mass of a can of baked beans'.

Then $X\sim N(415,12^2)$

$P(X > m) = 0.65 \Rightarrow P(X \le m) = 1 - 0.65 = 0.35$ (1 mark)

Using inverse normal distribution on GDC $\Rightarrow m = 410.4$ (2 marks)

b You require $P(X > 422.5 \mid X > 420)$. (1 mark)

$$P(X > 422.5 \mid X > 420) = \frac{P(X > 422.5)}{P(X > 420)}$$

$$= \frac{1 - P(X \le 422.5)}{1 - P(X \le 420)}$$ (1 mark)

$$= \frac{0.266}{0.338}$$ (1 mark)

$= 0.787$ (1 mark)

c Using GDC

$P(X < 413.5) = 0.450$ (2 marks)

Let Y be the random variable 'Number of cans of beans having a mass less than 413.5 g'.

In Ashok's experiment, Y is Binomially distributed across 144 trials with probability of 'success' (i.e. mass less than 413.5 g) being 0.450.

So $Y \sim B(144, 0.450)$ (1 mark)

$P(Y \ge 75) = 0.0524$ (1 mark)

17 a $k\int_0^1 (10x^2 - x^3)\ \mathrm{d}x = 1$ (1 mark)

$$k\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1$$ (1 mark)

$$k\left(\frac{10}{3} - \frac{1}{4}\right) = 1$$ (1 mark)

$$\frac{37k}{12} = 1$$ (1 mark)

$$k = \frac{12}{37}$$

b $E(X) = \int_0^1 xf(x)\ \mathrm{d}x = \frac{12}{37}\int_0^1 (10x^3 - x^4)\ \mathrm{d}x$ (1 mark)

$$= \frac{12}{37}\left[\frac{10x^4}{4} - \frac{x^5}{5}\right]_0^1$$ (1 mark)

$$= \frac{12}{37}\left(\frac{5}{2} - \frac{1}{5}\right)$$ (1 mark)

$$= \frac{276}{370}\left(= \frac{138}{185}\right)(= 0.746)$$ (1 mark)

c $E(X^2) = \int_0^1 x^2 f(x)\ \mathrm{d}x = \frac{12}{37}\int_0^1 (10x^4 - x^5)\ \mathrm{d}x$ (1 mark)

$= \frac{12}{37}\left[2x^5 - \frac{x^6}{6}\right]_0^1$ (1 mark)

$= \frac{12}{37}\left(2 - \frac{1}{6}\right)$

$= \frac{22}{37}$ (1 mark)

$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$ (1 mark)

$= \frac{22}{37} - \left(\frac{138}{185}\right)^2$

$= 0.0382$ (1 mark)

d $\frac{12}{37}\int_0^m (10x^2 - x^3)\ \mathrm{d}x = \frac{1}{2}$ (1 mark)

$\frac{12}{37}\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^m = \frac{1}{2}$ (1 mark)

$\frac{12}{37}\left(\frac{10m^3}{3} - \frac{m^4}{4}\right) = \frac{1}{2}$ (1 mark)

$\frac{10m^3}{3} - \frac{m^4}{4} = \frac{37}{24}$

$80m^3 - 6m^4 = 37$

GDC $\Rightarrow m = 0.789$ (2 marks)

18 a $P\left(Z < \frac{110-\mu}{\sigma}\right) = 0.10 \Rightarrow \frac{110-\mu}{\sigma} = -1.282$ (2 marks)

$P\left(Z > \frac{130-\mu}{\sigma}\right) = 0.45 \Rightarrow \frac{130-\mu}{\sigma} = 0.126$ (2 marks)

Attempt to solve simultaneously: (1 mark)

$\mu = 128$ (1 mark)

$\sigma = 14.2$ (1 mark)

b $P(|X-\mu|) < 0.22$

$0.5 - \frac{0.22}{2} = 0.39$ (1 mark)

$P(X < a) = 0.39 \Rightarrow a = 124.2$ (1 mark)

$P(X > b) = 0.39 \Rightarrow b = 132.2$ (1 mark)

So $124.2 < X < 132.2$ (1 mark)

19 a $k\int_0^{\frac{\pi}{2}} \cos x \text{ d}x = 1$ (1 mark)

$k\left[\sin x\right]_0^{\frac{\pi}{2}} = 1$ (1 mark)

$k\left(\sin\frac{\pi}{2} - \sin 0\right) = 1$ (1 mark)

$k(1-0) = 1$ (1 mark)

$k = 1$

b $\int_0^{\frac{\pi}{2}} x\cos x \text{ d}x = \left[x\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \text{ d}x$ (2 marks)

$\int_0^{\frac{\pi}{2}} x\cos x \text{ d}x = \left[x\sin x\right]_0^{\frac{\pi}{2}} + \left[\cos x\right]_0^{\frac{\pi}{2}}$ (1 mark)

$= \left(\frac{\pi}{2} - 0\right) + (0-1)$ (1 mark)

$= \frac{\pi}{2} - 1$ (1 mark)

$\int_0^{\frac{\pi}{2}} x^2\cos x \text{ d}x = \left[x^2\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x \text{ d}x$ (2 marks)

$= \left[x^2\sin x\right]_0^{\frac{\pi}{2}} - 2\left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}}$ (1 mark)

$= \left[x^2\sin x - 2\sin x + 2x\cos x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{4} - 2$ (1 mark)

$\text{Var}(X) = \int_0^{\frac{\pi}{2}} x^2\cos x \text{ d}x - \left(\int_0^{\frac{\pi}{2}} x\cos x \text{ d}x\right)^2$ (1 mark)

$= \frac{\pi^2}{4} - 2 - \left(\frac{\pi}{2} - 1\right)^2$ (1 mark)

$= \frac{\pi^2}{4} - 2 - \left(\frac{\pi^2}{4} - \pi + 1\right)$ (1 mark)

$= \pi - 3$