9 Modelling 3D space: vectors

Skills check

1 **a** $c = \sqrt{21^2 + 20^2} = 29$

b $a = \sqrt{13^2 - 7^2} = 10.95$

2 $d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2} = \sqrt{(-5-3)^2 + (4+2)^2} = 10$

3 We calculate the slopes and get that $m_a = -\frac{2}{3} = m_c$ so a and c are parallel

$m_b = \frac{3}{2}$, $m_a \times m_b = m_a \times m_d = -1$ so b and d are parallel, and both perpendicular to a. e is not parallel or perpendicular to any of the other lines.

4 We use method of elimination

$$\begin{cases} 16x - 12y = 4 \\ -15x + 12y = -6 \end{cases}$$

We subtract both and get that $x = -2$

and so $4(-2) - 3y = 1$

$y = \frac{1+8}{-3} = -3$

Exercise 9A

1 correct vectors drawn

2 **a** $\boldsymbol{a} + \boldsymbol{b}$ **b** $-\boldsymbol{a} - \boldsymbol{b}$ **c** $\boldsymbol{b} - \boldsymbol{a}$ **d** $-2\boldsymbol{a} - \boldsymbol{b}$

e $-2\boldsymbol{a} - \frac{3}{2}\boldsymbol{b}$ **f** $2\boldsymbol{b} + \frac{3}{2}\boldsymbol{a}$ **g** $-\frac{3\boldsymbol{a}}{2} + \frac{\boldsymbol{b}}{2}$ **h** $\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

3 **a** AG = AB + BC +CG = **a** + **b** + **c**

b CE = CB + BA + AE = **−b** − **a** + **c**

c DF = DA + AB + BF = −b + a + c

d MN = MB + BC + CG + GN = $-\frac{1}{2}\boldsymbol{c} + \boldsymbol{b} + \boldsymbol{c} - \frac{1}{2}\boldsymbol{a} = \boldsymbol{b} + \frac{1}{2}\boldsymbol{c} - \frac{1}{2}\boldsymbol{a}$

4 We have that $DC = \frac{5}{3}y$ where y is some length. Then the ratio gives us that

$\frac{3}{5}PC = y$

5 **a** $HC = HF + FE + EC$

where $FE = AB$

$HC - HQ = AB$

$HC - \lambda HC = AB$

$HC = (1-k)AB$

Hence they are parallel.

This means we can form a right angled triangle HQA and Pythagoras' theorem gives us

$HC = \left(1+\sqrt{2}\right)AB$

b $MN = ME + ED + DN$

$MN = \frac{1}{2}FE + ED + \frac{1}{2}DC$

$MN = AD + \frac{1}{2}OD + \frac{1}{2}DC$

Hence they are parallel.

Again we have a right angled triangle which gives us that

$MN = \left(1+\frac{\sqrt{2}}{2}\right)AB$

6 $KL = KB + BL$

$NM = ND + DM$

We know that they are the midpoints, so

$KL = \frac{AB}{2} + \frac{BC}{2}$

and

$NM = \frac{AD}{2} + \frac{DC}{2}$

Then we form a parallelogram.

Exercise 9B

1 We need to show that

$BD = \lambda PQ$

so we use the triangle rule for both diagonals to get

$BC + CD = BD$

and

$PC + CQ = PQ$

where

$PC = \frac{1}{2}BC$

and

$CQ = \frac{1}{2}CD$

Then

$BD = 2(PC + CQ) = 2PQ$

hence they are parallel. Additionally,

$PQ = \frac{1}{2}BD$

2 If PQ is perpendicular to AC, then PQ is parallel to BD, as the diagonals are perpendicular. Then

$PQ = QC + CP$

$AC = AD + DC$

$BD = DC + CB$

$BD = 2QC + 2CP = 2PQ \Rightarrow BD$ and PQ are parallel, hence PQ is orthogonal to AC

$AD = 2CP$, $DC = 2QC$

Then

$AC = 2CP + 2QC = 2(CP + QC) = 2PQ$

as requested

3 **a** $AG = \boldsymbol{a} + \boldsymbol{c} + \boldsymbol{b}$

b $CE = \boldsymbol{c} - \boldsymbol{b} - \boldsymbol{a}$

c $DF = \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$

d $MN = \frac{1}{2}\boldsymbol{c} + \boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

4 **i** $\lambda(\mu a) = \lambda\mu a = (\lambda\mu)a = (\mu\lambda)a = \mu(\lambda a)$

ii We just the associativity of scalar multiplication

$(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \lambda\boldsymbol{a}$

iii We just the distributivity of scalar multiplication

$(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \lambda\boldsymbol{a}$

iv 1 is the identity so the operation from the left and the right returns the same vector

$1 \cdot \boldsymbol{a} = \boldsymbol{a} \cdot 1 = \boldsymbol{a}$

v multiplying by zero will always return zero, as each component of the vector is multiplied by zero.

$0 \cdot \boldsymbol{a} = 0$

Exercise 9C

1 **a** $\boldsymbol{a}+\boldsymbol{b}=(2-3)\boldsymbol{i}+(-5+4)\boldsymbol{j}=-\boldsymbol{i}-\boldsymbol{j}$

b $\boldsymbol{a}-\boldsymbol{b}=(2+3)\boldsymbol{i}+(-5-4)\boldsymbol{j}=5\boldsymbol{i}-9\boldsymbol{j}$

c $5\boldsymbol{a}-6\boldsymbol{b}=5(2\boldsymbol{i}-5\boldsymbol{j})-6(-3\boldsymbol{i}+4\boldsymbol{j})=10\boldsymbol{i}-25\boldsymbol{j}+18\boldsymbol{i}-24\boldsymbol{j}=28\boldsymbol{i}-49\boldsymbol{j}$

d $7\boldsymbol{b}-4\boldsymbol{a}=7(-3\boldsymbol{i}+4\boldsymbol{j})-4(2\boldsymbol{i}-5\boldsymbol{j})=-21\boldsymbol{i}+28\boldsymbol{j}-8\boldsymbol{i}+20\boldsymbol{j}=-29\boldsymbol{i}+48\boldsymbol{j}$

e $\frac{3}{5}\boldsymbol{a}+\frac{3}{4}\boldsymbol{b}=\frac{3}{5}(2\boldsymbol{i}-5\boldsymbol{j})+\frac{3}{4}(-3\boldsymbol{i}+4\boldsymbol{j})=\frac{6}{5}\boldsymbol{i}-\frac{15}{5}\boldsymbol{j}-\frac{9}{4}\boldsymbol{i}+\frac{12}{4}\boldsymbol{j}=-\frac{21}{20}\boldsymbol{i}$

2 **a** Let $\alpha(3\boldsymbol{i}+2\boldsymbol{j})+\beta(\boldsymbol{i}+5\boldsymbol{j})=5\boldsymbol{i}-\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i}+2\alpha\boldsymbol{j}+\beta\boldsymbol{i}+5\beta\boldsymbol{j}=5\boldsymbol{i}-\boldsymbol{j}$

This gives two equations

$3\alpha+\beta=5$

$2\alpha+5\beta=-1$

Then

$\beta=5-3\alpha$

we substitute to get α

$2\alpha+5(5-3\alpha)=-1$

$-13\alpha=-26$

$\alpha=2$

and we substitute again to get β

$\beta=5-3(2)=-1$

$5\mathbf{i}-\mathbf{j}=2\mathbf{p}-\mathbf{q}$

b Let

$\alpha(3\boldsymbol{i}+2\boldsymbol{j})+\beta(\boldsymbol{i}+5\boldsymbol{j})=10\boldsymbol{i}+9\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i}+2\alpha\boldsymbol{j}+\beta\boldsymbol{i}+5\beta\boldsymbol{j}=10\boldsymbol{i}-9\boldsymbol{j}$

This gives two equations

$3\alpha+\beta=10$

$2\alpha+5\beta=9$

Then

$\beta=10-3\alpha$

we substitute to get α.

$2\alpha + 5(10 - 3\alpha) = 9$

$-13\alpha = -41$

$\alpha = \frac{41}{13}$

and we substitute again to get β

$\beta = 10 - 3\left(\frac{41}{13}\right) = \frac{7}{13}$

$\frac{41}{13}\boldsymbol{p} + \frac{7}{13}\boldsymbol{q}$

c Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -9\boldsymbol{i} + 7\boldsymbol{j}$

here α and β are constants. Then

$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -9\boldsymbol{i} + 7\boldsymbol{j}$

This gives two equations

$3\alpha + \beta = -9$

$2\alpha + 5\beta = 7$

Then

$\beta = -9 - 3\alpha$

we substitute to get α

$2\alpha + 5(-9 - 3\alpha) = 7$

$-13\alpha = 52$

$\alpha = -4$

and we substitute again to get β

$\beta = -9 - 3(-4) = 3$

$-4\mathbf{p} + 3\mathbf{q}$

d Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = \boldsymbol{i}$

where α and β are constants. Then

$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = 10\boldsymbol{i} - 9\boldsymbol{j}$

This gives two equations

$3\alpha + \beta = 1$

$2\alpha + 5\beta = 0$

Then

$\beta = 1 - 3\alpha$

we substitute to get α

$2\alpha + 5(1 - 3\alpha) = 0$

$-13\alpha = -5$

$\alpha = \frac{5}{13}$

and we substitute again to get β

$\beta = 1 - 3\left(\frac{5}{13}\right) = \frac{-2}{13}$

$\frac{5}{13}\boldsymbol{p} - \frac{2}{13}\boldsymbol{q}$

e Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -\boldsymbol{j}$

where α and β are constants. Then

$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -\boldsymbol{j}$

This gives two equations

$3\alpha + \beta = 0$

$2\alpha + 5\beta = -1$

Then

$\beta = -3\alpha$

we substitute to get α

$2\alpha + 5(-3\alpha) = -1$

$-13\alpha = -1$

$\alpha = \frac{1}{13}$

and we substitute again to get β

$\beta = -3\left(\frac{1}{13}\right) = \frac{-3}{13}$

$\frac{1}{13}\boldsymbol{p} - \frac{3}{13}\boldsymbol{q}$

f Let

$\alpha(3\boldsymbol{i} + 2\boldsymbol{j}) + \beta(\boldsymbol{i} + 5\boldsymbol{j}) = -\frac{1}{2}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}$

where α and β are constants. Then

$$3\alpha\boldsymbol{i} + 2\alpha\boldsymbol{j} + \beta\boldsymbol{i} + 5\beta\boldsymbol{j} = -\frac{1}{2}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}$$

This gives two equations

$$3\alpha + \beta = -\frac{1}{2}$$

$$2\alpha + 5\beta = \frac{2}{3}$$

Then

$$\beta = -\frac{1}{2} - 3\alpha$$

we substitute to get α

$$2\alpha + 5\left(-\frac{1}{2} - 3\alpha\right) = \frac{2}{3}$$

$$-13\alpha = \frac{19}{6}$$

$$\alpha = \frac{-19}{78}$$

and we substitute again to get β

$$\beta = -\frac{1}{2} - 3\left(\frac{-19}{78}\right) = \frac{3}{13}$$

$$-\frac{19}{78}\boldsymbol{p} + \frac{3}{13}\boldsymbol{q}$$

3 Note that

$QR \equiv PS$

so we calculate

$QR = -i - 3j + 4i + j = 3i - 2j$

and so

$PS = (3 - x)i + (3 - y)j$

where $P = xi + yj$, corresponding to (x, y) coordinates of P. Then we equate both expressions and get

$3 - x = 3$

$3 - y = -2$

so $x = 0$ and $y = 5$. Then $P = (0,5)$

4 In the notation bellow, any vector with a single letter is measured from the origin (e.g. OA=A)

$$OA = OB + CD = \begin{pmatrix}3\\2\\0\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+\begin{pmatrix}-1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$OE = OA + CG = \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+\begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}0\\0\\-1\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$)F = OB + CG = \begin{pmatrix}3\\2\\0\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+\begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

$$OH = OD + CG = \begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}+\begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}i\\j\\k\end{pmatrix}=\begin{pmatrix}-1\\1\\0\end{pmatrix}\begin{pmatrix}i\\j\\k\end{pmatrix}$$

Then $A = (2,0,1)$, $E = (0,0,-1)$, $F = (1,2,-2)$, $H = (-1,1,0)$

5 **i** Commutative: Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$, and $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$ for real x,y,m,n . Then for all $\boldsymbol{a},\boldsymbol{b}$

$$\boldsymbol{a}+\boldsymbol{b} = (x+m)\boldsymbol{i} + (y+n)\boldsymbol{j} = (m+x)\boldsymbol{i} + (n+y)\boldsymbol{j} = \boldsymbol{b}+\boldsymbol{a}$$

where we have used the commutativity of addition of real numbers.

ii Associative: Let $\boldsymbol{c} = l\boldsymbol{i} + p\boldsymbol{j}$ for real l,p, then for all $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$

$$(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = (m+x)\boldsymbol{i} + (n+y)\boldsymbol{j} + l\boldsymbol{i} + p\boldsymbol{j} = (m+x+l)\boldsymbol{i} + (n+y+p)\boldsymbol{j}$$

$$= m\boldsymbol{i} + n\boldsymbol{j} + (x+l)\boldsymbol{i} + (y+p)\boldsymbol{j} = \boldsymbol{a} + (\boldsymbol{b}+\boldsymbol{c})$$

where we have used the commutativity and associativity of addition of real numbers.

iii Identity: for $0 = 0\boldsymbol{i} + 0\boldsymbol{j}$ and for all $\boldsymbol{a}$

$$0 + \boldsymbol{a} = (0+x)\boldsymbol{i} + (0+y)\boldsymbol{j} = (x+0)\boldsymbol{i} + (y+0)\boldsymbol{j} = x\boldsymbol{i} + y\boldsymbol{j} = \boldsymbol{a}$$

where we have used the identity and commutativity of addition of real numbers.

iv Let $-\boldsymbol{a} = -x\boldsymbol{i} - y\boldsymbol{j}$. Then for all $\boldsymbol{a},-\boldsymbol{a}$

$$\boldsymbol{a} + (-\boldsymbol{a}) = (x-x)\boldsymbol{i} + (y-y)\boldsymbol{j} = (-x+x)\boldsymbol{i} + (-y+y)\boldsymbol{j} = -\boldsymbol{a} + \boldsymbol{a} = 0\boldsymbol{i} + 0\boldsymbol{j} = 0$$

where we have used the identity and commutativity of addition of real numbers.

6 **i** For any real λ, μ and for all $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ for real x,y we have

$$(\lambda\mu)\boldsymbol{a} = (\lambda\mu)(x\boldsymbol{i} + y\boldsymbol{j}) = \lambda\mu x\boldsymbol{i} + \lambda\mu y\boldsymbol{j} = \lambda(\mu x\boldsymbol{i}) + \lambda(\mu y\boldsymbol{j}) = \lambda(\mu\boldsymbol{a})$$

and

$$\lambda(\mu x\boldsymbol{i}) + \lambda(\mu y\boldsymbol{j}) = \mu(\lambda x\boldsymbol{i}) + \mu(\lambda y\boldsymbol{j}) = \mu(\lambda\boldsymbol{a})$$

where we have used the commutativity of the multiplication of real numbers.

ii Let $b = m\boldsymbol{i} + n\boldsymbol{j}$ for any real n,m

$$\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda((x+m)\boldsymbol{i} + (y+n)\boldsymbol{j}) = \lambda x\boldsymbol{i} + \lambda m\boldsymbol{i} + \lambda y\boldsymbol{j} + \lambda n\boldsymbol{j} = \lambda x\boldsymbol{i} + \lambda y\boldsymbol{j} + \lambda m\boldsymbol{i} + \lambda n\boldsymbol{j} = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$$

where we have used the commutativity and associativity of the multiplication of real numbers.

iii $(\lambda+\mu)\boldsymbol{a} = (\lambda+\mu)(x\boldsymbol{i}+y\boldsymbol{j}) = \lambda x\boldsymbol{i} + \mu x\boldsymbol{i} + \lambda y\boldsymbol{j} + \mu y\boldsymbol{j} = \lambda x\boldsymbol{i} + \lambda y\boldsymbol{j} + \mu x\boldsymbol{i} + \mu y\boldsymbol{j} = \lambda\boldsymbol{a} + \mu\boldsymbol{b}$

where we have used the commutativity and associativity of the multiplication of real numbers.

iv $1\boldsymbol{a} = 1(x\boldsymbol{i}+y\boldsymbol{j}) = (1\times x)\boldsymbol{i} + (1\times y)\boldsymbol{j} = x\boldsymbol{i}+y\boldsymbol{j} = \boldsymbol{a}$

where we have used the identity of multiplication of real numbers

v $0\boldsymbol{a} = 0(x\boldsymbol{i}+y\boldsymbol{j}) = (0\times x)\boldsymbol{i} + (0\times y)\boldsymbol{j} = 0\boldsymbol{i}+0\boldsymbol{j} = 0$

and

$\lambda(0\boldsymbol{i}+0\boldsymbol{j}) = (\lambda\times 0)\boldsymbol{i} + (\lambda\times 0)\boldsymbol{j} = 0\boldsymbol{i}+0\boldsymbol{j} = 0$

Exercise 9D

1 **a** $\hat{\boldsymbol{a}} = \frac{7\boldsymbol{i}+24\boldsymbol{j}}{\sqrt{7^2+24^2}} = \frac{7\boldsymbol{i}+24\boldsymbol{j}}{25} = \frac{7}{25}\boldsymbol{i} + \frac{24}{25}\boldsymbol{j}$

b $\hat{\boldsymbol{b}} = \frac{-3\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{3^2+2^2}} = \frac{-3\boldsymbol{i}+2\boldsymbol{j}}{\sqrt{13}} = \frac{-3}{\sqrt{13}}\boldsymbol{i} + \frac{2}{\sqrt{13}}\boldsymbol{j}$

c $\hat{\boldsymbol{c}} = \frac{4\boldsymbol{i}-5\boldsymbol{j}+20\boldsymbol{k}}{\sqrt{4^2+5^2+20^2}} = \frac{4\boldsymbol{i}-5\boldsymbol{j}+20\boldsymbol{k}}{21} = \frac{4}{21}\boldsymbol{i} - \frac{5}{21}\boldsymbol{j} + \frac{20}{21}\boldsymbol{k}$

d $\hat{\boldsymbol{d}} = \frac{-\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}}{\sqrt{1^2+3^2+4^2}} = \frac{-\boldsymbol{i}+3\boldsymbol{j}+4\boldsymbol{k}}{\sqrt{26}} = -\frac{1}{\sqrt{26}}\boldsymbol{i} + \frac{3}{\sqrt{26}}\boldsymbol{j} + \frac{4}{\sqrt{26}}\boldsymbol{k}$

2 **a** $\hat{\boldsymbol{a}} = \frac{20\boldsymbol{i}-21\boldsymbol{j}}{\sqrt{20^2+21^2}} = \frac{20\boldsymbol{i}-21\boldsymbol{j}}{29} = \frac{20}{29}\boldsymbol{i} - \frac{21}{29}\boldsymbol{j}$

All vectors parallel to $\hat{\boldsymbol{a}}$ are of the form $\lambda\hat{\boldsymbol{a}}$ for real λ

b $\hat{\boldsymbol{b}} = \frac{\boldsymbol{i}-3\boldsymbol{j}}{\sqrt{1^2+3^2}} = \frac{\boldsymbol{i}-3\boldsymbol{j}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\boldsymbol{i} - \frac{3}{\sqrt{10}}\boldsymbol{j}$

All vectors parallel to $\hat{\boldsymbol{b}}$ are of the form $\lambda\hat{\boldsymbol{b}}$ for real λ

c $\hat{\boldsymbol{c}} = \frac{5\boldsymbol{i}+6\boldsymbol{j}-30\boldsymbol{k}}{\sqrt{4^2+5^2+20^2}} = \frac{5\boldsymbol{i}+6\boldsymbol{j}-30\boldsymbol{k}}{\sqrt{61}} = \frac{5}{\sqrt{61}}\boldsymbol{i} + \frac{6}{\sqrt{61}}\boldsymbol{j} - \frac{30}{\sqrt{61}}\boldsymbol{k}$

All vectors parallel to $\hat{\boldsymbol{c}}$ are of the form $\lambda\hat{\boldsymbol{c}}$ for real λ

d $\hat{\boldsymbol{d}} = \frac{2\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{2^2+1^2+5^2}} = \frac{2\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}}{\sqrt{30}} = \frac{2}{\sqrt{30}}\boldsymbol{i} + \frac{1}{\sqrt{30}}\boldsymbol{j} - \frac{5}{\sqrt{30}}\boldsymbol{k}$

All vectors parallel to $\hat{\boldsymbol{d}}$ are of the form $\lambda\hat{\boldsymbol{d}}$ for real λ

3 We write the equation explicitly $\mathbf{b} = -\mathbf{i} + 5\mathbf{j} + (\lambda - 5)\mathbf{k}$

$2|\boldsymbol{a}| = |\boldsymbol{b}|$

$4|a|^2 = |b|^2$

$4(3^2 + 2^2 + \lambda^2) = 1^2 + 5^2 + (\lambda - 5)^2$

$3\lambda^2 + 10\lambda + 1 = 0$

$\lambda = \frac{-10 \pm \sqrt{100-12}}{6} = \frac{-5 \pm \sqrt{22}}{3}$

4 **a** $\hat{\boldsymbol{a}} = \frac{5\boldsymbol{i} - \boldsymbol{j}}{\sqrt{5^2 + 1^2}} = \frac{5\boldsymbol{i} - \boldsymbol{j}}{\sqrt{26}} = \frac{5}{\sqrt{26}}\boldsymbol{i} - \frac{1}{\sqrt{26}}\boldsymbol{j}$

Then the required vector is $m\hat{\boldsymbol{a}} = \frac{5 \times 6}{\sqrt{26}}\boldsymbol{i} - \frac{6}{\sqrt{26}}\boldsymbol{j} = \frac{30}{\sqrt{26}}\boldsymbol{i} - \frac{6}{\sqrt{26}}\boldsymbol{j}$

b $\hat{\boldsymbol{b}} = \frac{-4\boldsymbol{i} + 5\boldsymbol{j} + 20\boldsymbol{k}}{\sqrt{4^2 + 5^2 + 20^2}} = \frac{-4\boldsymbol{i} + 5\boldsymbol{j} + 20\boldsymbol{k}}{21} = \frac{-4}{21}\boldsymbol{i} + \frac{5}{21}\boldsymbol{j} + \frac{20}{21}\boldsymbol{k}$

Then the required vector is $m\hat{\boldsymbol{b}} = \frac{-4 \times 63}{21}\boldsymbol{i} + \frac{5 \times 63}{21}\boldsymbol{j} + \frac{20 \times 63}{21}\boldsymbol{k} = -12\boldsymbol{i} + 15\boldsymbol{j} + 60\boldsymbol{k}$

5 **a** This is the same cuboid as in exercise 9C, 4. A space diagonal could be

$$\boldsymbol{AG} = \left[\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} = -2\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$$

$\Rightarrow |\boldsymbol{AG}| = \sqrt{17}$

b Recall that $A = (2,0,1)$, $E = (0,0,-1)$, $F = (1,2,-2)$, $H = (-1,1,0)$

Then

$AD = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

$AE = -2\boldsymbol{i} + 0\boldsymbol{j} - 2\boldsymbol{k}$

$AB = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

$V = A_{base} \times h = |AD||AE||AB| = \sqrt{1^2 + 1^2 + 1^2}\sqrt{2^2 + 2^2}\sqrt{1^2 + 2^2 + 1^2} = 12$

Exercise 9E

1 **a** $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \sqrt{3} \times 4\cos 30° = 6$

b $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 12 \times 8\cos 115° = -40.6$

c $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 3 \times 5\cos\frac{\pi}{7} = 13.5$

d $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 5\sqrt{2} \times 17\cos\frac{3\pi}{4} = -85$

2 **a** $\boldsymbol{a} \cdot \boldsymbol{b} = 3 \cdot 6 + (-4) \cdot 5 = -2$

b $|\boldsymbol{a}| = \sqrt{3^2 + 4^2} = 5$

$|\boldsymbol{b}| = \sqrt{6^2 + 5^2} = \sqrt{61}$

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 5\sqrt{61}\cos\theta = -2$

$\cos\theta = \dfrac{-2}{5\sqrt{61}}$

$\theta = 1.62\text{rad} = 93°$

3 a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \cdot (-2) + 4 \cdot 3 + (-3) \cdot 1 = 7$

b $|\boldsymbol{a}| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$

$|\boldsymbol{b}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = \sqrt{26 \times 14}\cos\theta = 7$

$\cos\theta = \dfrac{7}{2\sqrt{91}}$

$\theta = 1.19\text{rad} = 68.5°$

4 $(\boldsymbol{a} - 2\boldsymbol{b}) \cdot (2\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{b} - 2\boldsymbol{b} \cdot 2\boldsymbol{a} - 2\boldsymbol{b} \cdot \boldsymbol{b} = 0$

$2 \times 2 \times 2 + \boldsymbol{a} \cdot \boldsymbol{b} - 4(\boldsymbol{a} \cdot \boldsymbol{b}) - 2\sqrt{3}\sqrt{3} = 0$

$\boldsymbol{a} \cdot \boldsymbol{b} = \dfrac{2}{3} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = 2\sqrt{3}\cos\theta$

$\theta = \cos^{-1}\dfrac{1}{3\sqrt{3}}$

$\theta = 1.38\text{rad} = 78.9°$

5 Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ and $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$ for any real x, y, m, n. Then

i $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = |\boldsymbol{b}||\boldsymbol{a}|\cos\theta = \boldsymbol{b} \cdot \boldsymbol{a}$

ii $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos 0 = |\boldsymbol{a}|^2$

We prove it for the two dimensional case.

iii Let $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$, $\boldsymbol{b} = m\boldsymbol{i} + n\boldsymbol{j}$, and $\boldsymbol{c} = s\boldsymbol{i} + t\boldsymbol{j}$ for any real x, y, m, n. Then

$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = x \times (m + s) + y \times (n + t) = xm + xs + yn + yt = xm + yn + xs + yt = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

We have used multiplicative properties for real numbers, therefore this can be extended to any dimension of vector, as the associativity and distributivity of scalar multiplication holds.

iv Let $\lambda \in \mathbb{R} \Rightarrow \lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = \lambda(|\boldsymbol{a}||\boldsymbol{b}|\cos\theta) = \lambda|\boldsymbol{a}||\boldsymbol{b}|\cos\theta = (\lambda|\boldsymbol{a}|)|\boldsymbol{b}|\cos\theta = |\boldsymbol{a}|(\lambda|\boldsymbol{b}|)\cos\theta$

Hence

$\lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = (\lambda\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (\lambda\boldsymbol{b})$.

6 i $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{a} + 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{b} = |\boldsymbol{a}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

ii $(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{a} - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{b} = |\boldsymbol{a}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

Each of these cases correspond to the cosine rule for a triangle with sides $\boldsymbol{a}, \boldsymbol{b},$ and $\boldsymbol{a}\pm\boldsymbol{b}$.

7 We use both definitions to write the scalar product between $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e.

$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$

Hence both definitions are equivalent.

8 We form the systems of equations

and

$(a-2b)\cdot(3a+b) = a\cdot 3a + a\cdot b - 2b\cdot 3a - 2b\cdot b = 0$

This simplifies to

$2(\boldsymbol{a}\cdot\boldsymbol{a}) - \boldsymbol{b}\cdot\boldsymbol{b} - \boldsymbol{a}\cdot\boldsymbol{b} = 0$

and

$3(\boldsymbol{a}\cdot\boldsymbol{a}) - 2\boldsymbol{b}\cdot\boldsymbol{b} - 5(\boldsymbol{a}\cdot\boldsymbol{b}) = 0$

We will express $|\boldsymbol{a}|^2$ and $|\boldsymbol{b}|^2$ in terms of the scalar product between a and b. This means we solve the system of equations for the norms of a and b

$\boldsymbol{b}\cdot\boldsymbol{b} = 2(\boldsymbol{a}\cdot\boldsymbol{a}) - \boldsymbol{a}\cdot\boldsymbol{b}$

$3(\boldsymbol{a}\cdot\boldsymbol{a}) - 4(\boldsymbol{a}\cdot\boldsymbol{a}) + 2(\boldsymbol{a}\cdot\boldsymbol{b}) - 5(\boldsymbol{a}\cdot\boldsymbol{b}) = 0$

$(\boldsymbol{a}\cdot\boldsymbol{a}) = -3(\boldsymbol{a}\cdot\boldsymbol{b})$

Note that the dot product is negative. This will be important as it allows us to take square roots of negative numbers multiplied by the dot product. Then we substitute into form for the norm of b

$\boldsymbol{b}\cdot\boldsymbol{b} = -6(\boldsymbol{a}\cdot\boldsymbol{b}) - \boldsymbol{a}\cdot\boldsymbol{b} = -7(\boldsymbol{a}\cdot\boldsymbol{b})$

Then we write

$\boldsymbol{a}\cdot\boldsymbol{b} = \sqrt{3(-\boldsymbol{a}\cdot\boldsymbol{b})}\sqrt{7(-\boldsymbol{a}\cdot\boldsymbol{b})}\cos\theta$

or equivalently

$1 = \sqrt{21}\cos\theta$

so $\cos\theta = 1/\sqrt{21}$, giving $\theta = 77.4°$

Exercise 9F

1 a $\boldsymbol{d} = (2-0)\boldsymbol{i} + (3-0)\boldsymbol{j} = 2\boldsymbol{i} + 3\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

b $\boldsymbol{d} = (-1-2)\boldsymbol{i} + (3-1)\boldsymbol{j} = -3\boldsymbol{i} + 2\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} + k\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

c $\boldsymbol{d} = (3+2)\boldsymbol{i} + (-6+5)\boldsymbol{j} = 5\boldsymbol{i} - \boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + k\begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

d $\boldsymbol{d} = \left(-\frac{1}{2} - \frac{2}{3}\right)\boldsymbol{i} + \left(\frac{3}{4} + 1\right)\boldsymbol{j} = -\frac{7}{6}\boldsymbol{i} + \frac{7}{4}\boldsymbol{j}$

Then the vector equation of the line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \end{pmatrix} + k\begin{pmatrix} -\frac{7}{6} \\ \frac{7}{4} \end{pmatrix}$$

2 a $\boldsymbol{p} = \boldsymbol{a} - \lambda\boldsymbol{d} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} x = 2 + \lambda \\ y = -7 + \lambda \end{cases} \Rightarrow \begin{cases} x - 2 = \lambda \\ y + 7 = \lambda \end{cases}$$

$$\Rightarrow x - 2 = y + 7 \Rightarrow y = x - 9$$

b $\boldsymbol{n} \cdot (\boldsymbol{p} \cdot \boldsymbol{a}) = 0 \Rightarrow \boldsymbol{n} \cdot \boldsymbol{p} = \boldsymbol{n} \cdot \boldsymbol{a}$

$$\Rightarrow \begin{pmatrix} 2 \\ -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}\begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$\Rightarrow 2x - 3y = 4 + 21$$

$$2x - 3y = 25$$

$$y = \frac{2x - 25}{3}$$

3 a We obtain the direction vector of L,

$$\frac{x-3}{2} = \frac{y+1}{-3}$$

Then

$$-3x + 9 = 2y + 2 = \lambda$$

Then

$$-3x + 9 = \lambda \Rightarrow x = 3 - \frac{1}{3}\lambda$$

and

$2y+2=\lambda \Rightarrow y=-1+\frac{1}{2}\lambda$

Hence the direction vector is $\boldsymbol{d}=-\frac{1}{3}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j}$

Then the vector equation parallel to L and passing through T is

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -3 \\ 8 \end{pmatrix}+\lambda\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{2} \end{pmatrix}$$

b The perpendicular line must have a normal vector for its direction vector

$\boldsymbol{d}=-\frac{1}{3}\boldsymbol{i}+\frac{1}{2}\boldsymbol{j} \Rightarrow \boldsymbol{n}=\frac{1}{2}\boldsymbol{i}+\frac{1}{3}\boldsymbol{j}$

Then the vector equation perpendicular to L passing through T is

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -3 \\ 8 \end{pmatrix}+\lambda\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$$

c We find the intersection between L and $\boldsymbol{r}$ as

$$\boldsymbol{r}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

We write this in Cartesian notation

$x=1+2\lambda$

$y=-1+4\lambda$

Then

$\frac{x-1}{2}=\frac{y+1}{4} \Rightarrow 4x-4=2y+2 \Rightarrow y=2x-3$

We write L in Cartesian form and get

$\frac{x-3}{2}=\frac{y+1}{3} \Rightarrow 6x-9=2y+2 \Rightarrow y=3x-\frac{11}{2}$

To find the intersection between the two, we equate both lines and get

$2x-3=3x-\frac{11}{2} \Rightarrow x=\frac{5}{2}$

and

$y=2\left(\frac{5}{2}\right)-3=2$

So we must find a line passing through T and $\left(\frac{5}{2},2\right)$, so be obtain the direction vector as

$\boldsymbol{d}=\left(-3-\frac{5}{2}\right)\boldsymbol{i}+(8-2)\boldsymbol{j}=-\frac{11}{2}\boldsymbol{i}+6\boldsymbol{j}$

Then the equation of the line passing through T and the intersection between the two lines is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{11}{2} \\ 6 \end{pmatrix}$$

4 a For them to be parallel, their direction vectors have to be proportional to each other.

Note that

$$\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(-\boldsymbol{i} + a\boldsymbol{j}) \Rightarrow \boldsymbol{d}_1 = -\boldsymbol{i} + a\boldsymbol{j}$$

$$\boldsymbol{p} = (1 + 2\mu)\boldsymbol{i} + 5(\mu - 2)\boldsymbol{j} \Rightarrow \boldsymbol{d}_2 = 2\boldsymbol{i} + 5\boldsymbol{j}$$

For them to be parallel, we must have

$$\boldsymbol{d}_1 = \gamma \boldsymbol{d}_2$$

for real γ. Then the normalised

$$-\boldsymbol{i} + a\boldsymbol{j} = \gamma(2\boldsymbol{i} + 5\boldsymbol{j})$$

Note that $2\gamma = -1$ gives $\gamma = \frac{-1}{2}$, so

$$a = 5 \times \frac{-1}{2} = -\frac{5}{2}$$

b For them to be perpendicular, their scalar product must be zero, so

$$\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0 \Rightarrow -1 \times 2 + a \times 5 = 0 \Rightarrow 5a = 2 \Rightarrow a = \frac{2}{5}$$

Exercise 9G

1 a $\boldsymbol{d} = (4 - 1)\boldsymbol{i} + (2 - 3)\boldsymbol{j} + (1 + 2)\boldsymbol{k} = 3\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

Then we write the vector equation simply as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

b $\boldsymbol{d} = (5 - 3)\boldsymbol{i} + (7 - 0)\boldsymbol{j} + (-2 + 5)\boldsymbol{k} = 2\boldsymbol{i} + 7\boldsymbol{j} + 3\boldsymbol{k}$

Then we write the vector equation simply as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$$

2 a We write the form for $\boldsymbol{r}$ as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

We substitute with P, and get

$0 = 3 - \lambda$

$2 = -1 + \lambda$

$5 = 2 + 3\lambda$

From the first equations, $\lambda = 3$, and then substituting with that value of λ in the last one gives

$2 + 3(3) = 11 \neq 5$

Hence there is a contradiction, and so P does not lie on the line.

b A parallel line has the same direction vector, and now the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

c We substitute T into the equation of the line to get the system of equations

$-2 = 3 - \lambda$

$4 = -1 + \lambda$

$a = 2 + 3\lambda$

Then $\lambda = 5$, and is consistent in the first two equations. Then

$a = 2 + 3(5) = 17$

3 a If the lines are parallel, their direction vectors are proportional to each other. We obtain them by rewriting in the equations for the lines in vector form

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+3}{5} = \lambda$$

so

$x = 2\lambda + 1$

$y = 3\lambda + 2$

$z = 5\lambda - 3$

so

$\boldsymbol{d}_1 = 2\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$

and

$$L_2 : x + 2 = \frac{y-1}{-2} = \frac{z-2}{4} = \mu$$

so

$x = \mu - 2$

$y = -2\mu + 1$

$z = 4\mu + 2$

so

$\boldsymbol{d}_2 = \boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$

The lines are not parallel as there is no real γ for which

$\boldsymbol{d}_1 = \gamma \boldsymbol{d}_2$

b The lines are skew if they are not parallel or perpendicular to each other. We check the scalar product between their direction vectors:

$d_1 \cdot d_2 = 2 \cdot 1 + 3 \cdot (-2) + 5 \cdot 4 = 16 \neq 0$

Hence the lines are skew.

4 **a** We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$3 + 5\lambda = 7 - \mu$

$-2 + 4\lambda = 4 + 2\mu$

$1 + 3\lambda = -1 - 3\mu$

From the first equation we get that

$\mu = -3 - 5\lambda + 7 = -5\lambda + 4$

and so

$-2 + 4\lambda = 4 + 2(-5\lambda + 4) \Rightarrow 14\lambda = 14 \Rightarrow \lambda = 1$

so

$\mu = -5(1) + 4 = -1$

We check that these values satisfy the third equation

$1 + 3(1) \neq -1 - 3(-1)$

so the lines do not intersect.

b We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$\lambda = -7 + 3\mu$

$-1 + 2\lambda = \mu$

$3 - \lambda = 7 - 2\mu$

We substitute the first equation into the second equation and get

$-1 + 2(-7 + 3\mu) = \mu$

$-1 - 14 + 6\mu = \mu$

$5\mu = 15$

$\mu = 3$

and so

$\lambda = -7 + 3(3) = 2$

which is consistent with the third equation.

c We rewrite the lines in parametric form

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$2\lambda = 1 + 3\mu$

$2 + 5\lambda = -1 + 2\mu$

$4\lambda = -3 + \mu$

The first equation gives us that

$$\lambda = \frac{1 + 3\mu}{2}$$

We substitute this into the third equation and get

$$4\left(\frac{1 + 3\mu}{2}\right) + 3 = \mu$$

$2 + 6\mu + 3 = \mu$

$5\mu = -5$

$\mu = -1$

and so

$$\lambda = \frac{1+3(-1)}{2} = -1$$

which is consistent with the second equation.

5 First we find the point of intersection between L_1 and L_2

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

At the intersection, both lines will take on the same values, so we construct the system of equations

$$-1 + 2\lambda = -3 + \mu$$

$$5 + 3\lambda = 6 - 2\mu$$

$$-4 + 4\lambda = 4 - \mu$$

The first equation gives us that

$$\mu = -1 + 2\lambda + 3 = 2\lambda + 2$$

We substitute this into the third equation and get

$$-4 + 4\lambda = 4 - 2\lambda - 2$$

$$6\lambda = 6$$

$$\lambda = 1$$

and so

$$\mu = 2(1) + 2 = 4$$

This is not consistent with the second equation, so the lines are not concurrent.

Exercise 9H

1 **a** We use the provided formula

$$\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} 3\times3-(-5)\times2 \\ (-5)\times1-2\times3 \\ 2\times(-2)-3\times1 \end{pmatrix} = \begin{pmatrix} 1+10 \\ -5-6 \\ -4-3 \end{pmatrix} = \begin{pmatrix} 9 \\ -11 \\ -7 \end{pmatrix}$$

b $$\begin{pmatrix} 1\times(-2)-0\times0 \\ 0\times3-1\times(-2) \\ 1\times0-1\times3 \end{pmatrix} = \begin{pmatrix} -2-0 \\ 0+2 \\ 0-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$$

c $$\begin{pmatrix} -4\times2-(-1)\times1 \\ -1\times2-3\times2 \\ 3\times1-(-4)\times2 \end{pmatrix} = \begin{pmatrix} -8+1 \\ -2-6 \\ 3+8 \end{pmatrix} = \begin{pmatrix} -7 \\ -8 \\ 11 \end{pmatrix}$$

d $$\begin{pmatrix} -\frac{3}{4}\times(-2)-1\times\left(-\frac{2}{3}\right) \\ 1\times 1-\left(\frac{1}{2}\right)\times(-2) \\ \left(\frac{1}{2}\right)\times\left(-\frac{2}{3}\right)-\left(-\frac{3}{4}\right)\times 1 \end{pmatrix} = \begin{pmatrix} \frac{6}{4}+\frac{2}{3} \\ 1+1 \\ -\frac{2}{6}+\frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{13}{6} \\ 2 \\ \frac{5}{12} \end{pmatrix}$$

2 $A = |\boldsymbol{a}\times\boldsymbol{b}|$

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix} 3\times(-4)-(-6)\times(-1) \\ (-6)\times 3-2\times(-4) \\ 1\times(-1)-(3)\times 3 \end{pmatrix} = \begin{pmatrix} -12-6 \\ -18+8 \\ -1-9 \end{pmatrix} = \begin{pmatrix} -18 \\ -10 \\ -10 \end{pmatrix}$$

Then

$$|\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{18^2+10^2+10^2} = 22.9$$

3 **a** $\boldsymbol{AB} = (-2-1)\boldsymbol{i}+(0-4)\boldsymbol{j}+(3-2)\boldsymbol{k} = -3\boldsymbol{i}-4\boldsymbol{j}+\boldsymbol{k}$

$\boldsymbol{AC} = (-1-1)\boldsymbol{i}+(2-4)\boldsymbol{j}+(4-2)\boldsymbol{k} = -2\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$

b $A = \frac{1}{2}|\boldsymbol{AB}\times\boldsymbol{AC}|$

$$\boldsymbol{AB}\times\boldsymbol{AC} = \begin{pmatrix} -4\times 2-1\times -2 \\ 1\times(-2)-(-3)\times 2 \\ -3\times(-2)-(-4)(-2) \end{pmatrix} = \begin{pmatrix} -8+1 \\ -2+6 \\ 6-8 \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \\ -2 \end{pmatrix}$$

Then

$$|\boldsymbol{AB}\times\boldsymbol{AC}| = \sqrt{7^2+4^2+2^2} = \sqrt{49+16+4} = 8.31$$

Then the area is

$$A = \frac{1}{2}(8.31) = 8.15$$

4 **i** Let the vectors $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ be well defined. Then

$$\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta = -|\boldsymbol{a}||\boldsymbol{b}|\sin(-\theta) = -|\boldsymbol{b}||\boldsymbol{a}|\sin(-\theta) = -(\boldsymbol{b}\times\boldsymbol{a})$$

where we have used the commutativity of real numbers and properties of sines. Note that if the angle from a to b is θ, then the angle from b to a is $-\theta$

ii We calculate

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \begin{pmatrix} a_2(b_1c_2-b_2c_1)+a_3(b_1c_3+a_3c_1) \\ a_3(a_2b_3-a_3b_2)-a_1(b_1c_2-b_2c_1) \\ -a_1(b_1c_3+b_3c_1)-a_2(a_2b_3-a_3b_2) \end{pmatrix} = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$$

iii $$\lambda\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix} = \begin{pmatrix} \lambda(a_2b_3-a_3b_2) \\ \lambda(a_3b_1-a_1b_3) \\ \lambda(a_1b_2-a_2b_1) \end{pmatrix} = \begin{pmatrix} a_2(\lambda b_3)-a_3(\lambda b_2) \\ a_3(\lambda b_1)-a_1(\lambda b_3) \\ a_1(\lambda b_2)-a_2(\lambda b_1) \end{pmatrix} = \begin{pmatrix} (\lambda a_2)b_3-(\lambda a_3)b_2 \\ (\lambda a_3)b_1-(\lambda a_1)b_3 \\ (\lambda a_1)b_2-(\lambda a_2)b_1 \end{pmatrix}$$

Hence

$\lambda(a \times b) = (\lambda a) \times b = a \times (\lambda b)$, $\lambda \in \mathbb{R}$

iv We can expand out the cross products explicitly as

$$(\boldsymbol{a}+\boldsymbol{b})\times = \begin{pmatrix} (a_2+b_2)c_3-(a_3+b_3)c_2 \\ (a_3+b_3)c_1-(a_1+b_1)c_3 \\ (a_1+b_1)c_2-(a_2+b_2)c_1 \end{pmatrix} = \begin{pmatrix} a_2c_3-a_3c_2 \\ a_3c_1-a_1c_3 \\ a_1c_2-a_2c_1 \end{pmatrix} + \begin{pmatrix} b_2c_3-b_3c_2 \\ b_3c_1-b_1c_3 \\ b_1c_2-b_2c_1 \end{pmatrix} = (\boldsymbol{a}\times\boldsymbol{c})+(\boldsymbol{b}\times\boldsymbol{c})$$

5 We write out the vectors

$$\boldsymbol{AB} = (2-1)\boldsymbol{i}+(-1-1)\boldsymbol{j}+(0-1)\boldsymbol{k} = \boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}$$

$$\boldsymbol{AC} = (2-1)\boldsymbol{i}+(4-1)\boldsymbol{j}+(2-1)\boldsymbol{k} = \boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$$

$$\boldsymbol{AD} = (-2-1)\boldsymbol{i}+(2-1)\boldsymbol{j}+(2-1)\boldsymbol{k} = -3\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$$

Then

$$(\boldsymbol{AB}\times\boldsymbol{AC})\cdot\boldsymbol{AD} = \begin{pmatrix} (-2)\times 1-(-1)\times 3 \\ -1\times 1-1\times 1 \\ 1\times 3-(-2)\times 1 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -1-1 \\ 3+2 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$= 1\times(-3)+(-2)\times(1)+5\times(1) = 0$$

Hence the three points are coplanar.

6 **a** We find D such that AB = DC. Then

$$AB = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

and

$$DC = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 4-d_1 \\ 5-d_2 \\ -1-d_3 \end{pmatrix}$$

Then we have the equations

$$4-d_1 = 1$$

$$5-d_2 = -3$$

$$-1-d_3 = 2$$

Then

$$D = (3, 8, -3)$$

b Note that the vectors DC, DA and DH enclose the parallelepiped, so

$$\boldsymbol{DC} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

$$DA = \begin{pmatrix} 1-3 \\ 2-8 \\ 1+3 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

$$DH = \begin{pmatrix} 4-3 \\ 3-8 \\ 6+3 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} i \\ j \\ k \end{pmatrix}$$

c The volume of the parallelepiped is given by

$$V = |(\boldsymbol{DC} \times \boldsymbol{DA}) \cdot \boldsymbol{DH}| = |(0)(1) + (-8)(-5) + (-12)(9)| = 68$$

7 Assuming D is the apex, we obtain

$$\boldsymbol{BC} = \boldsymbol{BD} + \boldsymbol{DC} = (1-2)\boldsymbol{i} + (-2+1)\boldsymbol{j} + (4+5)\boldsymbol{k} = -\boldsymbol{i} - \boldsymbol{j} + 9\boldsymbol{k}$$

$$V = \frac{1}{3} Base \cdot h = \frac{1}{3} \cdot \frac{1}{2} |\boldsymbol{BA} \times \boldsymbol{BD}| |\boldsymbol{BC}|$$

$$\boldsymbol{BA} \times \boldsymbol{BD} = \begin{pmatrix} -3 \times 4 - 2 \times (-2) \\ 2 \times 1 - (-2) \times 4 \\ -2 \times (-2) - (-3) \times 1 \end{pmatrix} = \begin{pmatrix} -12+4 \\ 2+8 \\ 4+3 \end{pmatrix} = \begin{pmatrix} -8 \\ 10 \\ 7 \end{pmatrix}$$

Then

$$V = \frac{1}{6}\left(\sqrt{8^2 + 10^2 + 7^2}\sqrt{1^2 + 1^2 + 9^2}\right) = 22.2$$

8 $(\boldsymbol{a} \cdot \boldsymbol{b})^2 + |(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b})| = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \cos^2\theta + |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 \sin^2\theta = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 (\cos^2\theta + \sin^2\theta) = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2$ �

9 We calculate

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 + a_3c_1) \\ a_3(a_2b_3 - a_3b_2) - a_1(b_1c_2 - b_2c_1) \\ -a_1(b_1c_3 + b_3c_1) - a_2(a_2b_3 - a_3b_2) \end{pmatrix} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})$$

Exercise 9I

1 a $\boldsymbol{p} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

b $\boldsymbol{p} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

c $\boldsymbol{p} = \boldsymbol{a} + \lambda\boldsymbol{u} + \mu\boldsymbol{v} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

2 a $\boldsymbol{AB} = (-1-0)\boldsymbol{i} + (2-1)\boldsymbol{j} + (0-3)\boldsymbol{k} = -\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

$\boldsymbol{AC} = (3-0)\boldsymbol{i} + (-2-1)\boldsymbol{j} + (4-3)\boldsymbol{k} = 3\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

Then we can write the vector equation of the plane as

$$p = \begin{pmatrix}0\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\-3\\1\end{pmatrix}$$

b $x = -\lambda + 3\mu$

$y = 1 + \lambda - 3\mu$

$z = 3 - 3\lambda + \mu$

c We eliminate the parameters in **b**

$\lambda = 3\mu - x$

which we substitute into the equation for y and z to get

$y = 1 + 3\mu - x - 3\mu \Rightarrow y + x = 1$

$z = 3 - 3(3\mu - x) + \mu$

We cannot express the equation in terms of x, y , and z, so the Cartesian equation is

$y + x = 1$

3 **a** The normal vector is

$$\begin{pmatrix}2\\1\\4\end{pmatrix} \times \begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}9\\-2\\5\end{pmatrix}$$

$$\begin{pmatrix}9\\-2\\5\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}9\\-2\\5\end{pmatrix} \cdot \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$9x - 2y + 5z = -4 - 5$

$9x - 2y + 5z = -9$

b
$$\begin{pmatrix}0\\-3\\2\end{pmatrix} \times \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}-2\\2\\3\end{pmatrix}$$

$$\begin{pmatrix}-2\\2\\3\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-2\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\3\end{pmatrix}$$

$-2x + 2y + 3z = -2 - 4 + 9$

$-2x + 2y + 3z = 3$

c
$$\begin{pmatrix}-2\\0\\3\end{pmatrix} \times \begin{pmatrix}2\\1\\5\end{pmatrix} = \begin{pmatrix}-3\\16\\-2\end{pmatrix}$$

$$\begin{pmatrix}-3\\16\\-2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-3\\16\\-2\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\2\end{pmatrix}$$

$$-3x+16y-2z=69$$

4 **a** We substitute the point into the equation of the plane

$$3(5)-4(4)+2(-2)=-5\neq 5$$

Hence the point is not on the plane

b The normal vector is

$$\boldsymbol{d}=3\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$$

Then we are searching for a plane with the same normal vector but a different point.

$$\begin{pmatrix}3\\-4\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-4\\2\end{pmatrix}\cdot\begin{pmatrix}5\\4\\-2\end{pmatrix}$$

$$3x-4y+2z=15-16-4$$

$$3x-4y+2z=-5$$

5 We equate solve the equations of a plane as a system of equations

$$x+y-z=1$$

$$2x-3y-9z=10$$

$$x+2y-3z=-4$$

We subtract the third from the first and get

$$-y+2z=5$$

$$y=2z-5$$

We subtract two times the first from the second, and get

$$-5y-7z=8$$

Then substituting our value for y we get that

$$-5(2z-5)-7z=8$$

$$z=1$$

Then

$$y=2(1)-5=-3$$

and so

$$x+(-3)-1=1$$

$$x=5$$

6 a We express y and z in terms of x

$$y = 3 + 2z - x$$

$$z = 1 + 3y - 2x$$

Then

$$y = 3 + 2(x - 2) - x$$

$$y = 3 + 2x - 4 - x$$

Which we can then substitute into the equation for z as

$$z = 1 + 3(3 + 2z - x) - 2x$$

this simplifies into

$$z = x - 2$$

and so

$$y = x - 1$$

We let $x = \lambda$, and so

$$x = \lambda$$

$$y = \lambda - 1$$

$$z = \lambda - 2$$

We eliminate λ to find the Cartesian equation, as

$$x = y + 1 = z - 2$$

b We can set the new equation to be generated by

$$A = \lambda \boldsymbol{i} + (\lambda - 1)\boldsymbol{j} + (\lambda - 2)\boldsymbol{k}$$

and

$$T = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$$

so we can write it as

$$p = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} x - 4y + 3z = 21$$

7 If two planes are parallel, their normal vectors are parallel, then

$$n_1 \times n_2 = |n_1||n_2|\sin\theta = |n_1||n_2| \times 0 = 0$$

If the vector product of the normal vectors is zero, we have

$$n_1 \times n_2 = |n_1||n_2|\sin\theta = 0 \Rightarrow \theta = 0$$

hence they are parallel

Exercise 9J

1 a $x-5=\lambda$

$\frac{y+1}{2}=\lambda$

$\frac{1-z}{3}=\lambda$

or equivalently

$x=\lambda+5$

$y=2\lambda-1$

$z=1-3\lambda$

We substitute in the equation of the plane

$2(\lambda+5)-4(2\lambda-1)+1-3\lambda=-3$

$2\lambda+10-8\lambda+4+1-3\lambda=-3$

$-9\lambda=-18$

$\lambda=2$

There is a unique solution, so the line and the plane intersect at a point. This point is

$x=2+5=7$

$y=2(2)-1=3$

$z=1-3(2)=-5$

So they intersect at $(7,3,-5)$.

b $1-2x=\lambda$

$\frac{y-3}{4}=\lambda$

$\frac{2z+2}{3}=\lambda$

or equivalently

$x=\frac{1-\lambda}{2}$

$y=4\lambda+3$

$z=\frac{3\lambda-2}{2}$

We substitute in the equation of the plane

$5\left(\frac{1-\lambda}{2}\right)+(4\lambda+3)-4\left(\frac{3\lambda-2}{2}\right)=3$

$\lambda = \frac{13}{9}$

There is a unique solution, so the line and the plane intersect at a point. This point is determined by

$x = -\frac{2}{9}$

$y = \frac{79}{9}$

$z = \frac{7}{6}$

c $\frac{x-5}{4} = \lambda$

$\frac{y+2}{-2} = \lambda$

$\frac{z-3}{3} = \lambda$

or equivalently

$x = 4\lambda + 5$

$y = -2\lambda - 2$

$z = 3\lambda + 3$

We substitute in the equation of the plane

$2(4\lambda + 5) + (-2\lambda - 2) - 2(3\lambda + 3) = 3$

This has no solutions, so there is no intersection.

$y = \frac{79}{9}$

$z = \frac{7}{6}$

d $\frac{1-x}{2} = \lambda$

$\frac{y+2}{3} = \lambda$

$1 - 3z = \lambda$

or equivalently

$x = 1 - 2\lambda$

$y = 3\lambda - 2$

$z = \frac{1-\lambda}{3}$

We substitute in the equation of the plane

$$2(1-2\lambda)+(3\lambda-2)-3\left(\frac{1-\lambda}{3}\right)=-1$$

This has infinite solutions, so the line is contained in the plane.

2 The normal of the plane and the direction vector of the line must be orthogonal, so their dot product must be zero. We obtain the parametric equation of the line as

$$\frac{x}{m}=\lambda \Rightarrow x=\lambda m$$

$$\frac{y-1}{2}=\lambda \Rightarrow y=2\lambda+1$$

$$\frac{z+2}{4}=\lambda \Rightarrow z=4\lambda-2$$

Then

$$\boldsymbol{d}=\begin{pmatrix} m \\ 2 \\ 4 \end{pmatrix}$$

and the normal of the plane is

$$\boldsymbol{n}=\begin{pmatrix} 2 \\ m \\ -3 \end{pmatrix}$$

Then

$$\boldsymbol{d}\cdot\boldsymbol{n}=\begin{pmatrix} m \\ 2 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ m \\ -3 \end{pmatrix}=2m+2m-12=0$$

gives

$m=3$

3 This is precisely what we have calculated above, as

$$\boldsymbol{d}\cdot\boldsymbol{n}=|\boldsymbol{d}||\boldsymbol{n}|\cos\theta$$

and $\theta=\pi/2$, so $\boldsymbol{d}\cdot\boldsymbol{n}=0$.

Exercise 9K

1 **a** Let $x=\lambda$

$$3\lambda+y-2z=-1$$

$$\lambda-4y+2x=3$$

Then

$$y=\frac{4\lambda-2}{3}$$

and

$z = \frac{1+13\lambda}{6}$

which determine the equation of the line

b $n_1 = (3,1,-2)$

$n_2 = (1,-4,2)$

$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|} = \frac{|3(1)+(1)(-4)+(-2)(2)|}{\sqrt{3^2+1^2+2^2}\sqrt{1^2+4^2+2^2}} = \frac{5}{7\sqrt{6}}$

Then

$\theta = \cos^{-1}\frac{5}{7\sqrt{6}} = 1.275$

2 Any system of equations formed has no solution, so these lines do not intersect.

3 **a** We have that

$x = \lambda + 2$

$y = -3\lambda + 1$

$z = 2\lambda + 2$

Substitute in the equation of the plane as

$3(\lambda+2)+2(-3\lambda+1)-(2\lambda+2)=1$

$\lambda = 1$

Then the point of intersection is $P = (3,-2,4)$

b The direction vector of the line is

$d = i - 3j + 2k$

The normal vector of the plane is

$n = 3i + 2j - k$

Then

$\sin\theta = \frac{|d \cdot n|}{|d||n|} = \frac{|(1)(3)+(-3)(2)+(2)(-1)|}{\sqrt{1^2+3^2+2^2}\sqrt{3^2+2^2+1^2}} = \frac{5}{14}$

$\theta = 0.365$

4 We look at the angle between the normal vectors

$n_1 = (a,0,a)$

and

$n_2 = (b,-b,0)$

Note that

$n_1 \cdot n_2 = a \cdot b$

$|n_1| = \sqrt{2}|a|$

$|n_2| = \sqrt{2}|b|$

$ab = 2|a||b|\cos\theta$

so

$\cos\theta = \pm\frac{1}{2}$

It is the angle between their normal vectors if it is acute and it is the supplementary angle if it is obtuse, hence for both the positive and the negative case, the angle will be $\frac{\pi}{3}$

Exercise 9L

1 **a** For a, we have that the direction vector will be

$$\boldsymbol{d}_a = (0 + 3000)\boldsymbol{i} + (0 - 5000)\boldsymbol{j}$$

and since the speed is 4m/s we have to normalise and multiply by this so that the magnitude holds. Then

$$\boldsymbol{d}_a = \frac{4 \times 3000\boldsymbol{i} - 4 \times 5000\boldsymbol{j}}{\sqrt{3000^2 + 5000^2}} = \frac{12}{\sqrt{34}}\boldsymbol{i} - \frac{20}{\sqrt{34}}\boldsymbol{j}$$

and so with the point $(-3000, 5000)$, the equation of the position becomes

$$\boldsymbol{a} = (-3000\boldsymbol{i} + 5000\boldsymbol{j}) + t\left(\frac{12}{\sqrt{34}}\boldsymbol{i} - \frac{20}{\sqrt{34}}\boldsymbol{j}\right)$$

similarly for l we have

$$\boldsymbol{d}_l = (0 - 7000)\boldsymbol{i} + (0 - 9000)\boldsymbol{j}$$

and since the speed is 4 ms^{-1} we have to normalise and multiply by this so that the magnitude holds. Then

$$\boldsymbol{d}_a = \frac{6 \times (-7000)\boldsymbol{i} - 6 \times (-9000)\boldsymbol{j}}{\sqrt{7000^2 + 9000^2}} = -\frac{42}{\sqrt{130}}\boldsymbol{i} - \frac{54}{\sqrt{130}}\boldsymbol{j}$$

and so with the point $(7000, 9000)$, the equation of the position becomes

$$\boldsymbol{l} = (7000\boldsymbol{i} + 9000\boldsymbol{j}) + t\left(-\frac{42}{\sqrt{130}}\boldsymbol{i} - \frac{54}{\sqrt{130}}\boldsymbol{j}\right)$$

b We check when each boat gets to the point (0,0). For a

$$x = -3000 + \frac{12}{\sqrt{34}}t$$

$$y = 5000 - \frac{20}{\sqrt{34}}t$$

Then at (0,0)

$$-3000+\frac{12}{\sqrt{34}}t=5000-\frac{20}{\sqrt{34}}t$$

$$\frac{32}{\sqrt{34}}t=2000$$

$$t=\frac{125\sqrt{34}}{2}\approx 364.4s$$

For I we have

$$x=7000-\frac{42}{\sqrt{130}}t$$

$$y=9000-\frac{54}{\sqrt{130}}t$$

Then at (0,0)

$$7000-\frac{42}{\sqrt{130}}t=9000-\frac{54}{\sqrt{130}}t$$

$$\frac{12}{\sqrt{130}}t=2000$$

$$t=500\frac{\sqrt{130}}{3}\approx 1900.3s$$

Boat a will arrive first.

2 **a** The initial position is given at time $t=0$ so

$$\boldsymbol{p}(0)=23\boldsymbol{i}+8\boldsymbol{j}+43\boldsymbol{k}$$

b The speed is given by the magnitude of the direction vector

$$|d|=\sqrt{2^2+1^2+4^2}=\sqrt{21}\approx 4.58\,\text{ms}^{-1}$$

c Intersection between the line given and the plane. The components of p are

$$x=23+2t$$

$$y=8-t$$

$$z=43+4t$$

We substitute into the equation of the plane to get

$$12(23+2t)-3(8-t)-5(43+4t)=-2$$

$$276+24t-24+3t-215-20t=-2$$

$$39=13t$$

$$3=t$$

d Total distance = $3\sqrt{21}\approx 13.75$ m

3 **a** Assuming distance is in km and time in hours

Speed of p_1

$v_1 = \sqrt{8^2 + 9^2 + 0.25^2} \approx 12.04\ \text{kmh}^{-1}$

Speed of p_2

$v_2 = \sqrt{7^2 + 11^2 + 0.2^2} = 13.04\text{kmh}^{-1}$

b Assume that there is an intersection. We write out the components of p_1 and p_2

$x_1 = 147 - 8t$

$y_1 = -156 + 9t$

$z_1 = 5 + 0.25t$

$x_2 = -118 + 7\mu$

$y_2 = 189 - 11\mu$

$z_2 = 7 + 0.2\mu$

We equate the components to get a value of t

$147 - 8t = -118 + 7\mu$

$-156 + 9t = 189 - 11\mu$

$5 + 0.25t = 7 + 0.2\mu$

This gives $\mu = 15$ and $t = 20$ which is consistent in all three equations. Hence the paths intersect. The point of intersection is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 147 - 8(20) \\ -156 + 9(20) \\ 5 + 0.25(20) \end{pmatrix} = \begin{pmatrix} -13 \\ 24 \\ 25 \end{pmatrix}$$

c The times at which they reach this point are different, and unique. Hence they will not collide.

Chapter review

1 a $\boldsymbol{a} + \boldsymbol{b} = AB$

Hence the midpoint will have half of that length, so

$$\boldsymbol{m} = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$$

b $AD = \frac{10}{3}DC$, so AD and DC are the parallel sides of the trapezium.

c Midpoints are (6, 1), (4.5, 3.5), (2, -1), (8.5, 5.5), which give two pairs of parallel lines with equal length and thus form a rhombus.

2 a We calculate the Cartesian form

$x = 2 + \lambda - 3\mu$

$y = 2\lambda + \mu$

$z = \mu - 1$

We subtract the second from twice the first

$2x - y = 4 + 2\lambda - 2\lambda - 6\mu - \mu$

Then

$2x - y = 4 - 7\mu$

We add 7 times the third equation as

$2x - y + 7z = 7\mu + 4 - 7 - 7\mu$

$2x - y + 7z = -3$

b We substitute with each of the points, leading to the equations

$2(2) - (0) + 7(a) = -3$

$2(b) - 4 + 7(-1) = -3$

$2(-1) - d + 7(0) = -3$

Then we solve them and get

$$a = \frac{-3-4}{7} = -1$$

$$b = \frac{-3+4+7}{2} = 4$$

$$d = \frac{-3+2}{-1} = 1$$

c We write (taking all vectors from the origin)

$C = B - A + D = A = B + D - 2A$

$$C = \begin{pmatrix} 5 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$$

d We substitute with the point E and get

$2(1) - (-2) + 7(1) = 11 \neq 3$

so the point E does not lie in the plane

e We use the formula for the volume of the pyramid. We calculate

$$AC = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

$$AE = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$$

Then

$$V = \frac{1}{6}\left|(AC \times AB) \cdot AE\right|$$

$$V = \frac{1}{6}\left|(-8)(-1) + (4)(-2) + (-22)(2)\right| = 8$$

3 a The direction vector of the line will be the normal to the plane

$$d = (2, -2, 1)$$

Then the equation of the line is

$$p = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

b The point of intersection is obtained by substituting

$$x = 2 + 2\lambda$$

$$y = -2 - 2\lambda$$

$$z = 1 + \lambda$$

into the equation of the plane

$$2(2 + 2\lambda) - 2(-2 - 2\lambda) + (1 + \lambda) = 0$$

$$4 + 4\lambda + 4 + 4\lambda + 1 + \lambda = 0$$

$$\lambda = -1$$

Then the point of intersection is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then the distance between this point and the plane is

$$|OA| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

c A point on the plane is $B = (0, 0, 0)$ and we define the vector

$$BP = (x_0)i - 2y_0 j + z_0 k$$

The normal of the plane is n

$$n = 2i - 2j + k$$

Then the distance we need is

$$d = \frac{|AB \cdot n|}{|n|} = \frac{|2x_0 - 2y_0 + z_0|}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{|2x_0 - 2y_0 + z_0|}{3}$$

4 a Note that

$a \cdot b = pr + 4 + rp$

and

$a \cdot b = \left(p^2 + 4 + r^2\right)\cos\theta$

since the components form an arithmetic sequence with common difference d, we have the relationship

$p + d = 2$

$2 + d = r$

We use this to rewrite the formula for the dot product in terms of d and get

$$\frac{(2-d)(2-d)+4+(2-d)(2-d)}{(2-d)^2+4+(2+d)^2} = \frac{12-2d^2}{12+2d^2} = \frac{6-d^2}{6+d^2}$$

as required.

b When the angle is $60°$, the cosine is $\frac{1}{2}$ so

$\frac{6-d^2}{6+d^2} = \frac{1}{2}$

$12 - 2d^2 = 6 + d^2$

$3d^2 = 6$

$d^2 = 2$

Then $d = \pm\sqrt{2}$

5 If these planes are perpendicular, then their normal vectors are always perpendicular, so we check

$n_1 \cdot n_2 = \sin\alpha \cdot \cos\alpha + \cos\alpha \sin\alpha - 1$

These planes are not perpendicular

6 If they are perpendicular, their dot product will be equal to zero. We use the fact that their magnitude is 1 to calculate

$(2\boldsymbol{u} - 3\boldsymbol{v}) \cdot (5\boldsymbol{u} + 2\boldsymbol{v}) = 10\boldsymbol{u} \cdot \boldsymbol{u} + 4\boldsymbol{u} \cdot \boldsymbol{v} - 15\boldsymbol{v} \cdot \boldsymbol{u} - 6\boldsymbol{v} \cdot \boldsymbol{v}$

$-11\,\boldsymbol{u} \cdot \boldsymbol{v} + 4 = -11\cos\theta + 4 = 0$

$\cos\theta = \frac{4}{11}$

$\theta = 69°$

7 a $x = 3\lambda + 4$

$y = 1 - \lambda$

$z = 2\lambda + 5$

$$4(3\lambda+4)-3(1-\lambda)+(2\lambda+5)=1$$

$$15\lambda+16-3+3\lambda+2\lambda+5=1$$

$$\lambda=\frac{-17}{20}$$

Then the point P is at $\lambda=-\frac{17}{20}$ and so the point is

$$x=3\left(-\frac{17}{20}\right)+4=\frac{131}{20}$$

$$y=1-\left(-\frac{17}{20}\right)=\frac{37}{20}$$

$$z=2\left(-\frac{17}{20}\right)+5=\frac{33}{10}$$

b Angle between the line and the plane

$$\boldsymbol{d}=(3,-1,2)$$

$$\boldsymbol{n}=(4,-3,1)$$

$$\sin\theta=\frac{|\boldsymbol{d}\cdot\boldsymbol{n}|}{|\boldsymbol{d}||\boldsymbol{n}|}=\frac{|(3)(4)+(-1)(-3)+(2)(1)|}{\sqrt{3^2+1^2+2^2}\sqrt{4^2+3^2+1^2}}\Rightarrow\theta=84.8°$$

8 a $\boldsymbol{a}\cdot\boldsymbol{b}=2^x(2^x)+(4^x)(0.5^x)+(5)(-4)=0$

$$4^x+2^x-20=0$$

This is true for $x=2$.

b The equation of the plane is given by

$$\boldsymbol{p}=\begin{pmatrix}1\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}4\\16\\5\end{pmatrix}+\mu\begin{pmatrix}4\\0.25\\-4\end{pmatrix}$$

9 a We have the following relations

$$\boldsymbol{a}\cdot\hat{\boldsymbol{k}}=|\boldsymbol{a}||\hat{\boldsymbol{k}}|\cos\gamma$$

$$\boldsymbol{a}\cdot\hat{\boldsymbol{j}}=|\boldsymbol{a}||\hat{\boldsymbol{j}}|\cos\beta$$

$$\boldsymbol{a}\cdot\hat{\boldsymbol{i}}=|\boldsymbol{a}||\hat{\boldsymbol{i}}|\cos\alpha$$

Note that the norm of the unit vectors is one, and

$$|\boldsymbol{a}|=\sqrt{(\boldsymbol{a}\cdot\hat{\boldsymbol{k}})^2+(\boldsymbol{a}\cdot\hat{\boldsymbol{j}})^2+(\boldsymbol{a}\cdot\hat{\boldsymbol{i}})^2}$$

We substitute with the relations obtained and get

$$|\boldsymbol{a}|^2=|\boldsymbol{a}|^2\cos^2\gamma+|\boldsymbol{a}|^2\cos^2\beta+|\boldsymbol{a}|^2\cos^2\alpha$$

$1 = \cos^2\gamma + \cos^2\beta + \cos^2\alpha$

b The norm of a is

$|\boldsymbol{a}| = \sqrt{3^2 + 6^2 + 2^2} = 7$

We substitute into the relations obtained in a to get

$3 = 7\cos\alpha \Rightarrow \alpha = 64.6°$

$-6 = 7\cos\beta \Rightarrow \beta = 149°$

$2 = 7\cos\gamma \Rightarrow \gamma = 73.4°$

c When the plane passes through zero, the normal vector will correspond precisely to the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. As we saw in **a**, these can be written as the cosines of the angles. Hence

$\boldsymbol{n} = \cos\alpha\boldsymbol{i} + \cos\beta\boldsymbol{j} + \cos\gamma\boldsymbol{k}$

Then the equation of the plane can be written as

$x\cos\alpha + y\cos\beta + z\cos\gamma = 0$

10 a We calculate the vectors

$$AP = \begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\4\end{pmatrix}$$

and

$$AQ = \begin{pmatrix}0\\1\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$$

These will be the two vectors on the plane equation. Additionally we take a point, choosing for simplicity A = (2,0,0). Then the plane equation in vector form is

$$p = \begin{pmatrix}2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

To write it in Cartesian form, we write out the system of equations

$x = 2 - \lambda - 2\mu$

$y = 2\lambda + \mu$

$z = 4\lambda + 4\mu$

We subtract the third one from twice the second one, to get

$z - 2y = 2\mu$

so

$$\mu = \frac{z - 2y}{2}$$

and we add the second one to twice the first one, to get

$y + 2x = 4 + 2\lambda - 2\lambda + \mu - 4\mu$

or equivalently

$y + 2x = 4 - 3\mu$

Then we substitute with our value for μ to get

$$y + 2x = 4 - 3\left(\frac{z - 2y}{2}\right)$$

This simplifies to

$4x - 4y + 3z = 8$

b using the equation of the plane written in a. BG gives the direction vector of the line.

$$BG = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

Then the equation of the line is written as

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

c Angle between plane

$4x - 4y + 3z = 8$

and line

$$\boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$$

We have that

$$\sin\theta = \frac{|\boldsymbol{d} \cdot \boldsymbol{n}|}{|\boldsymbol{d}||\boldsymbol{n}|} = \frac{|(-2)(4) + (-2)(-4) + (4)(3)|}{\sqrt{2^2 + 2^2 + 4^2}\sqrt{4^2 + 4^2 + 3^2}} = \frac{12}{2\sqrt{246}} = \frac{6}{\sqrt{246}}$$

Then

$\theta = 22.5°$

Exam-style questions

11 a $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ (1 mark)

$\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ (1 mark)

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}3\\2\\2\end{pmatrix}$$ (1 mark)

$$=\begin{pmatrix}2\\-7\\4\end{pmatrix}$$ (1 mark)

b $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{2^2+(-7)^2+4^2}$ (2 marks)

$=\frac{\sqrt{69}}{2}$ (1 mark)

c $$\boldsymbol{r}.\begin{pmatrix}2\\-7\\4\end{pmatrix}=\begin{pmatrix}3\\0\\0\end{pmatrix}.\begin{pmatrix}2\\-7\\4\end{pmatrix}$$ (2 marks)

$$\boldsymbol{r}.\begin{pmatrix}2\\-7\\4\end{pmatrix}=6$$ (1 mark)

$2x-7y+4z=6$

d $$\begin{pmatrix}3\\-5\\1\end{pmatrix}\times\begin{pmatrix}2\\-7\\4\end{pmatrix}=\begin{pmatrix}-13\\-10\\-11\end{pmatrix}$$ (2 marks)

$$\boldsymbol{n}=\begin{pmatrix}13\\10\\11\end{pmatrix}$$

$y=0\Rightarrow x=-\frac{1}{5},z=\frac{8}{5}$ (or equivalent) (2 marks)

$$\boldsymbol{r}=\begin{pmatrix}-\frac{1}{5}\\0\\\frac{8}{5}\end{pmatrix}+\lambda\begin{pmatrix}13\\10\\11\end{pmatrix}$$ (or equivalent) (1 mark)

12 $$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-2\\2\\3\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-3\\2\\2\end{pmatrix}$$ (1 mark)

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}0\\4\\2\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\4\\1\end{pmatrix}$$ (1 mark)

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}3\\1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$$ (1 mark)

Volume $= \frac{1}{6}\left|\overrightarrow{AB}.\overrightarrow{AC}\times\overrightarrow{AD}\right| = \frac{1}{6}\left|\begin{pmatrix}-3\\2\\2\end{pmatrix}.\begin{pmatrix}7\\4\\-9\end{pmatrix}\right|$ (2 marks)

$= \frac{1}{6}\left|(-21+8-18)\right|$

$= \frac{31}{6}$ units^2. (1 mark)

13 $\overrightarrow{AP} = \boldsymbol{p}-\boldsymbol{a}$ (1 mark)

$\overrightarrow{BP} = \boldsymbol{p}-\boldsymbol{b}$ (1 mark)

$\overrightarrow{AP}.\overrightarrow{BP} = (\boldsymbol{p}-\boldsymbol{a}).(\boldsymbol{p}-\boldsymbol{b})$ (1 mark)

$= (\boldsymbol{p}-\boldsymbol{a}).(\boldsymbol{p}+\boldsymbol{a})$ (1 mark)

$= \boldsymbol{p.p}-\boldsymbol{a.p}+\boldsymbol{a.p}-\boldsymbol{a.a}$ (1 mark)

$= \boldsymbol{p.p}-\boldsymbol{a.a}$ (1 mark)

$= |\boldsymbol{p}|^2-|\boldsymbol{a}|^2$ (1 mark)

$= 0$ since $|\boldsymbol{p}| = |\boldsymbol{a}|$ (1 mark)

Therefore $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$ and $\angle APB = 90^\circ$

14 a Equation of line perpendicular to Π and passing through P is $\boldsymbol{r} = \begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}4\\-3\\1\end{pmatrix}$

(2 marks)

Attempting to solve P and Π simultaneously: (1 mark)

$4(1+4\lambda)-3(-3\lambda)+(2+\lambda) = 19$

$4+16\lambda+9\lambda+2+\lambda = 19$

$26\lambda+6 = 19$

$\lambda = \frac{1}{2}$ (1 mark)

Therefore $\overrightarrow{OQ} = \begin{pmatrix}1\\0\\2\end{pmatrix}+2\times\frac{1}{2}\times\begin{pmatrix}4\\-3\\1\end{pmatrix}$ (1 mark)

$= \begin{pmatrix}5\\-3\\3\end{pmatrix}$ (1 mark)

b Distance between $P(1,0,2)$ and $Q(5,-3,3)$ is given by

$\sqrt{(5-1)^2+(-3-0)^2+(3-2)^2}$ (2 marks)

$=\sqrt{16+9+1}$

$=\sqrt{26}$ (1 mark)

15 a $4(1+6\lambda)+3(5-2\lambda)-(-3+2\lambda)=14$ (1 mark)

$4+24\lambda+15-6\lambda+3-2\lambda=14$

$22+16\lambda=14$

$\lambda=-\frac{1}{2}$ (1 mark)

$\boldsymbol{r}=\begin{pmatrix}1\\5\\-3\end{pmatrix}-\frac{1}{2}\begin{pmatrix}6\\-2\\2\end{pmatrix}=\begin{pmatrix}-2\\6\\-4\end{pmatrix}$ (2 marks)

So $P(-2,6,4)$.

b $\begin{pmatrix}-2\\6\\-4\end{pmatrix}$ lies on the plane and $\boldsymbol{n}=\begin{pmatrix}4\\3\\-1\end{pmatrix}$ (2 marks)

So distance $=\dfrac{\begin{pmatrix}-2\\6\\-4\end{pmatrix}.\begin{pmatrix}4\\3\\-1\end{pmatrix}}{\sqrt{4^2+3^2+(-1)^2}}$ (1 mark)

$=\dfrac{-8+18+4}{\sqrt{26}}$

$=\dfrac{14}{\sqrt{26}}\left(=\dfrac{14\sqrt{26}}{26}=\dfrac{7\sqrt{26}}{13}\right)$ (1 mark)

16 a $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\0\\6\end{pmatrix}-\begin{pmatrix}8\\2\\0\end{pmatrix}=\begin{pmatrix}-6\\-2\\6\end{pmatrix}$ (1 mark)

$\boldsymbol{r}=\begin{pmatrix}8\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}-6\\-2\\6\end{pmatrix}$ (2 marks)

b $\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}=\begin{pmatrix}12\\3\\0\end{pmatrix}-\begin{pmatrix}4\\4\\4\end{pmatrix}=\begin{pmatrix}8\\-1\\-4\end{pmatrix}$ (1 mark)

$\boldsymbol{r}=\begin{pmatrix}4\\4\\4\end{pmatrix}+\mu\begin{pmatrix}8\\-1\\4\end{pmatrix}$ (2 marks)

c Direction vectors are $\begin{pmatrix}-6\\-2\\6\end{pmatrix}$ and $\begin{pmatrix}8\\-1\\4\end{pmatrix}$

$\begin{pmatrix}-6\\-2\\6\end{pmatrix}\times\begin{pmatrix}8\\-1\\4\end{pmatrix}=\begin{pmatrix}-2\\72\\22\end{pmatrix}$ (2 marks)

$(8,2,0)$ lies on AB and $(4,4,4)$ lies on CD

$\overrightarrow{AC}=\begin{pmatrix}-4\\2\\4\end{pmatrix}$ (1 mark)

Projection of $\overrightarrow{AC}$ to the vector $\begin{pmatrix}-2\\72\\22\end{pmatrix}$ is $\dfrac{\left|\begin{pmatrix}-4\\2\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\72\\22\end{pmatrix}\right|}{\sqrt{(-2)^2+72^2+22^2}}$ (2 marks)

$=\dfrac{8+144+88}{\sqrt{(-2)^2+72^2+22^2}}$ (1 mark)

$=\dfrac{240}{\sqrt{5672}}$ (1 mark)

$\left(=\dfrac{240\sqrt{5672}}{5672}=\dfrac{480\sqrt{1418}}{5672}=\dfrac{60\sqrt{1418}}{709}(=3.19)\right)$

17 Choosing $\lambda=1$ (say), gives $\boldsymbol{r}=\begin{pmatrix}10\\-4\\4\end{pmatrix}+\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}11\\-2\\5\end{pmatrix}$ (1 mark)

Therefore $A(5,8,0), B(10,-4,4)$ and $C(11,-2,5)$ lie on Π (2 marks)

$\overrightarrow{AB}=\begin{pmatrix}5\\-12\\4\end{pmatrix}$ and $\overrightarrow{AC}=\begin{pmatrix}6\\-10\\5\end{pmatrix}$ (2 marks)

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}5\\-12\\4\end{pmatrix}\times\begin{pmatrix}6\\-10\\5\end{pmatrix}=\begin{pmatrix}-20\\-1\\22\end{pmatrix}$ (2 marks)

So equation of plane is $\boldsymbol{r}\cdot\begin{pmatrix}-20\\-1\\22\end{pmatrix}=\begin{pmatrix}5\\8\\0\end{pmatrix}\cdot\begin{pmatrix}-20\\-1\\22\end{pmatrix}$ (2 marks)

$\boldsymbol{r}\cdot\begin{pmatrix}-20\\-1\\22\end{pmatrix}=-108$ (1 mark)

$-20x-y+22z=-108$ (1 mark)

$$\frac{20}{108}x + \frac{1}{108}y - \frac{22}{108}z = 1$$ (1 mark)

$$\left(\frac{5}{27}x + \frac{1}{108}y - \frac{11}{54}z = 1\right)$$

18 Direction vector of line is $\begin{pmatrix} 2 \\ 5 \\ p \end{pmatrix}$ (1 mark)

Direction normal to plane is $\begin{pmatrix} 5 \\ p \\ p \end{pmatrix}$ (1 mark)

If the angle between the line and the plane is θ , then

$$\sin\theta = \frac{\begin{pmatrix} 2 \\ 5 \\ p \end{pmatrix} . \begin{pmatrix} 5 \\ p \\ p \end{pmatrix}}{\sqrt{2^2 + 5^2 + p^2}\sqrt{5^2 + p^2 + p^2}}$$ (3 marks)

$$= \frac{10 + 5p + p^2}{\sqrt{2^2 + 5^2 + p^2}\sqrt{5^2 + p^2 + p^2}}$$ (1 mark)

θ is maximum when $\sin\theta$ is maximum. (1 mark)

By GDC, maximum occurs when $p = 6.797$ (1 mark)

So maximum value of $\sin\theta$ is 0.96 (1 mark)

$\Rightarrow \theta_{MAX} = 73.7^\circ$ (1 mark)

19 $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, so L_1 and L_2 are not parallel. (2 marks)

Consider ***i*** and ***j*** components: (1 mark)

$1 + 3\lambda = 2 + \mu$ and $-\lambda = 1 - \mu$ (1 mark)

Solving simultaneously: (1 mark)

$\lambda = 1$, $\mu = 2$ (1 mark)

Substitute into ***k*** component: (1 mark)

$2 + \lambda = 1 + \mu$, $2 + 1 = 1 + 2$ (so equations are consistent). (1 mark)

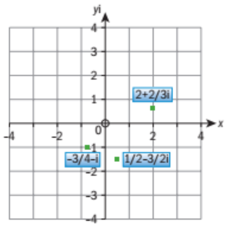
Therefore L_1 and L_2 intersect at the point where $\lambda = 1$ and $\mu = 2$, so are not skew.

(1 mark)

10 Equivalent systems of representation: more complex numbers

Skills check

1



2 $\text{Re}(z_1) = 2, \text{Im}(z_1) = \frac{2}{3}$,

$\text{Re}(z_2) = -\frac{3}{4}, \text{Im}(z_2) = -1$,

$\text{Re}(z_3) = \frac{1}{2}, \text{Im}(z_3) = -\frac{3}{2}$.

3 **a** $1 - 13i$ **b** $-\frac{17}{4} - \frac{7}{4}i$

4 **a** $z^* = 2 + 3i, -z = -2 + 3i$,

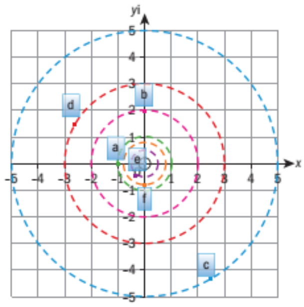
$\frac{1}{z} = \frac{2}{13} + \frac{2}{13}i, |z| = \sqrt{13}$

b $z^* = \frac{4}{5} - \frac{3}{5}i, -z = -\frac{4}{5} - \frac{3}{5}i$,

$\frac{1}{z} = \frac{4}{5} - \frac{3}{5}i, |z| = 1$

Exercise 10A

1



2 a $r = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}$

$\theta = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$2 + 2i = 2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$

b $r = \frac{3}{2}$

$\theta = \frac{\pi}{2}$

$\frac{3}{2}i = \frac{3}{2}\text{cis}\left(\frac{\pi}{2}\right)$

c $r = \sqrt{(-4)^2 + (-3)^2} = 5$

$\theta = \pi + \arctan\left(\frac{3}{4}\right) = 3.78$

$\therefore -4 - 3i = 5\text{cis}(3.78)$

d $r = \sqrt{21^2 + (-20)^2} = 29$

$\theta = 2\pi\arctan\left(\frac{-20}{21}\right) = 5.52$

$21 - 20i = 29\text{cis}(5.52)$

e $r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2$

$\theta = \pi - \arctan\left(\sqrt{3}\right) = \frac{2\pi}{3}$

$-1 + \sqrt{3}i = 2\text{cis}\left(\frac{2\pi}{3}\right)$

f $-\frac{4}{3}i = \frac{4}{3}\text{cis}\left(\frac{3\pi}{2}\right)$

g $r = \sqrt{\left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = \frac{5\sqrt{2}}{12}$

$\theta = 2\pi + \arctan\left(\frac{-3}{4}\right) = 5.64$

$\therefore \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{4}i = \frac{5\sqrt{2}}{12}\text{cis}(5.64)$

4 a $-z = \frac{7}{12}\text{cis}\left(\frac{\pi}{9} + \pi\right) = \frac{7}{12}\text{cis}\left(\frac{10\pi}{9}\right)$

b $z^* = \frac{7}{12}\text{cis}\left(-\frac{\pi}{9}\right)$

c $-z^* = \frac{7}{12}\text{cis}\left(\pi - \frac{\pi}{9}\right) = \frac{7}{12}\text{cis}\left(\frac{8\pi}{9}\right)$

Exercise 10B

1 a $z_1 z_2 = 8e^{i\left(\frac{\pi}{3} + \frac{\pi}{4}\right)} = 8e^{i\frac{7\pi}{12}}$

b $z_3 z_4 = 30\text{cis}(90^o + 45^o) = 30\text{cis}(135^o)$

c $z_5 z_6 = \frac{5}{9}e^{i\left(\frac{11\pi}{7} + \frac{23\pi}{14}\right)} = \frac{5}{9}e^{i\left(\frac{45}{14}\pi\right)} = \frac{5}{9}e^{i\left(\frac{17}{14}\pi\right)}$ □

d $z_7 z_8 = \text{cis}(220^o + 275^o) = \text{cis}(495^o) = \text{cis}(135^o)$

2 a $z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

b $z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \text{cis}\frac{2\pi}{3} = e^{i\frac{2\pi}{3}}$

c $z_1 z_2 = e^{i\frac{3\pi}{4}}e^{i\frac{2\pi}{3}} = e^{i\left(\frac{3\pi}{4} + \frac{2\pi}{3}\right)} = e^{i\left(\frac{17\pi}{12}\right)} = \frac{-\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} + \sqrt{2}}{4}i$

d $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right) + i\left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right)$

$\cos\frac{17\pi}{12} = \frac{\sqrt{2} - \sqrt{6}}{4}; \sin\frac{17\pi}{12} = -\frac{\sqrt{2} + \sqrt{6}}{4}$

e $\tan\frac{17\pi}{12} = \frac{\sin\frac{17\pi}{12}}{\cos\frac{17\pi}{12}} = \frac{\frac{\sqrt{2} + \sqrt{6}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}} = \frac{\left(\sqrt{6} + \sqrt{2}\right)^2}{6 - 2} = \frac{8 + 2\sqrt{12}}{4} = \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}$

3 a $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2e^{-i\frac{\pi}{6}}$

$z\left(\sqrt{3}-i\right)=2re^{i\left(\theta-\frac{\pi}{6}\right)}$

$2r<3 \Rightarrow r<\frac{3}{2}$

$\theta-\frac{\pi}{6}=0 \Rightarrow \theta=\frac{\pi}{6}$

$\theta-\frac{\pi}{6}=\pi \Rightarrow \theta=\frac{7\pi}{6}$

$\therefore$ Real if $\theta=\frac{\pi}{6}$ or $\theta=\frac{7\pi}{6}$ (up to multiples of π)

and less than 3 if $r<\frac{3}{2}$

b $z(-1+i)=\sqrt{2}z\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\sqrt{2}z\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\sqrt{2}ze^{i\frac{5\pi}{4}}$

$=\sqrt{2}re^{i\left(\theta+\frac{5\pi}{4}\right)}$

$\theta+\frac{5\pi}{4}=\pi \Rightarrow \theta=-\frac{\pi}{4}$ or $\frac{7\pi}{4}$

$\theta+\frac{5\pi}{4}=2\pi \Rightarrow \theta=\frac{3\pi}{4}$

$\therefore$ Imaginary if $\theta=\frac{3\pi}{4}$ or $\theta=-\frac{\pi}{4}$ (up to multiples of π)

Modulus greater than 4 if $\left|\sqrt{2}r\right|>4 \Rightarrow r>2\sqrt{2}$

4 $\left(\sin\frac{\pi}{12}+i\cos\frac{\pi}{12}\right)\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)\left(\sin\frac{\pi}{4}+i\cos\frac{\pi}{4}\right)$

$i^3\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right)\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=-i\,\text{cis}\left(-\frac{\pi}{12}\right)\bullet\text{cis}\left(-\frac{\pi}{6}\right)\bullet\text{cis}\left(-\frac{\pi}{4}\right)$

$=-ie^{-i\left(\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right)}=-ie^{-i\frac{\pi}{2}}=(-i)^2=-1$

Exercise 10C

1 a $\frac{z_1}{z_2}=\frac{3\text{cis}\frac{\pi}{4}}{4\text{cis}\frac{5\pi}{3}}=\frac{3}{4}\text{cis}\left(\frac{\pi}{4}-\frac{5\pi}{3}\right)=\frac{3}{4}\text{cis}\left(-\frac{17\pi}{12}\right)=\frac{3}{4}\text{cis}\left(\frac{7\pi}{12}\right)$

b $\frac{z_3{}^*}{-z_2}=-\frac{5\text{cis}\left(\pi-\frac{7\pi}{6}\right)}{4\text{cis}\left(\pi+\frac{5\pi}{3}\right)}=\frac{5\,\text{cis}\left(-\frac{\pi}{6}\right)}{4\,\text{cis}\left(\frac{8\pi}{3}\right)}=\frac{5}{4}\text{cis}\left(-\frac{\pi}{6}-\frac{2\pi}{3}\right)=\frac{5}{4}\text{cis}\left(\frac{-5\pi}{6}\right)=\frac{5}{4}\text{cis}\left(\frac{7\pi}{6}\right)$

c $\frac{z_1}{z_2z_3}=\frac{3\text{cis}\frac{\pi}{4}}{\left(4\text{cis}\frac{5\pi}{3}\right)\left(5\text{cis}\frac{7\pi}{6}\right)}=\frac{3}{20}\text{cis}\left(\frac{\pi}{4}-\frac{5\pi}{3}-\frac{7\pi}{6}\right)=\frac{3}{20}\text{cis}\left(-\frac{31\pi}{12}\right)=\frac{3}{20}\text{cis}\left(\frac{17\pi}{12}\right)$

d $-\left(\frac{z_3}{z_1z_2}\right)^*=-\left(\frac{5}{12}\text{cis}\left(\frac{7\pi}{6}-\frac{\pi}{4}-\frac{5\pi}{3}\right)\right)^*=-\left(\frac{5}{12}\left(\text{cis}\left(-\frac{3\pi}{4}\right)\right)^*\right)=\frac{5}{12}\text{cis}\left(2\pi+\frac{3\pi}{4}\right)=\frac{5}{12}\text{cis}\left(\frac{3\pi}{4}\right)$

2 $1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\sqrt{2}e^{i\frac{\pi}{4}}$

$1-\sqrt{3}i=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=2e^{-i\frac{\pi}{3}}$

a $\frac{z_1}{z_2}=\frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{-i\frac{\pi}{3}}}=\frac{\sqrt{2}}{2}e^{i\frac{7\pi}{12}}=\frac{\sqrt{2}}{2}e^{i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}$

b $-\frac{z_2{}^*}{z_1}=\sqrt{2}\,e^{i\left(-\frac{2\pi}{3}-\frac{\pi}{4}\right)}=\sqrt{2}\,e^{i\left(-\frac{11\pi}{12}\right)}$ or $\sqrt{2}\,e^{i\left(\frac{13\pi}{12}\right)}$

c $\frac{1}{z_1z_2}=\frac{1}{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)\left(2e^{-i\frac{\pi}{3}}\right)}=\frac{1}{2\sqrt{2}e^{i\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}}=\frac{\sqrt{2}}{4}e^{i\frac{\pi}{12}}$

d $-\frac{z_1{}^*}{(z_1z_2)^*}=-\left(\frac{z_1}{z_1z_2}\right)^*=-\frac{1}{z_2{}^*}=\frac{1}{2}\text{cis}\frac{2\pi}{3}=\frac{1}{2}e^{\frac{i\pi}{3}}$

3 **a** $\frac{3}{2+2i}=\frac{3}{2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}=\frac{3}{2\sqrt{2}\text{cis}\frac{\pi}{4}}=\frac{3\sqrt{2}}{4}\text{cis}\left(-\frac{\pi}{4}\right)=\frac{3\sqrt{2}}{4}\text{cis}\left(\frac{7\pi}{4}\right)$

b $\frac{4-4i}{-1+\sqrt{3}i}=\frac{4\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)}{2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}=\frac{2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)}{\text{cis}\left(\frac{2\pi}{3}\right)}=2\sqrt{2}\text{cis}\left(-\frac{11\pi}{12}\right)=2\sqrt{2}\,\text{cis}\left(\frac{13\pi}{12}\right)$

c $\frac{\sqrt{15}-\sqrt{5}i}{\sqrt{2}+\sqrt{6}i}=\frac{2\sqrt{5}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)}{2\sqrt{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}=\frac{\sqrt{10}}{2}\frac{\text{cis}\left(-\frac{\pi}{6}\right)}{\text{cis}\left(\frac{\pi}{3}\right)}=\frac{\sqrt{10}}{2}\text{cis}\left(-\frac{\pi}{2}\right)\frac{\sqrt{10}}{2}\text{cis}\left(\frac{3\pi}{2}\right)$

4 **a** $z_1=5\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=\frac{5}{2}+\frac{5\sqrt{3}}{2}i$

b $z_2=3+3i=3\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=3\sqrt{2}\text{cis}\frac{\pi}{4}$

c $\frac{z_1}{z_2}=\frac{5\text{cis}\frac{\pi}{3}}{3\sqrt{2}\text{cis}\frac{\pi}{4}}=\frac{5\sqrt{2}}{6}\text{cis}\frac{\pi}{12}$

$$\frac{z_1}{z_2}=\frac{\frac{5}{2}+\frac{5\sqrt{3}}{2}i}{3+3i}=\frac{5}{6}\cdot\frac{1+\sqrt{3}i}{1+i}=\frac{5}{6}\cdot\frac{\left(1+\sqrt{3}i\right)(1-i)}{(1+i)(1-i)}$$
$$=\frac{5}{6}\cdot\frac{1+\sqrt{3}+i\left(\sqrt{3}-1\right)}{2}=\frac{5\left(1+\sqrt{3}\right)}{12}+i\frac{5\left(\sqrt{3}-1\right)}{12}$$

d $\frac{5\sqrt{2}}{6}\cos\frac{\pi}{12}=\frac{5}{12}\left(1+\sqrt{3}\right)$

$$\cos\frac{\pi}{12} = \frac{6}{5\sqrt{2}}\cdot\frac{5}{12}\left(1+\sqrt{3}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

e $\frac{5\sqrt{2}}{6}\sin\frac{\pi}{12} = \frac{5\left(\sqrt{3}-1\right)}{12}$

$$\Rightarrow \sin\frac{\pi}{12} = \frac{6}{5\sqrt{2}}\cdot\frac{5\left(\sqrt{3}-1\right)}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

f $\tan\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}{6-2} = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)^2$

$$= \frac{1}{4}\left(6-2\sqrt{12}+2\right) = \frac{1}{4}\left(8-4\sqrt{3}\right) = 2-\sqrt{3}$$

Exercise 10D

1 $z_1 = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \sqrt{2}\text{cis}\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

a $z_1^{\,3}z_2^{\,2} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^3\left(2e^{i\frac{2\pi}{3}}\right)^2 = 8\sqrt{2}e^{i\left(\frac{3\pi}{4}+\frac{4\pi}{3}\right)} = 8\sqrt{2}e^{i\frac{25\pi}{12}}$

b $\frac{z_1^{\,5}}{z_2^{\,3}} = \frac{4\sqrt{2}e^{i\frac{5\pi}{4}}}{8e^{i2\pi}} = \frac{\sqrt{2}}{2}e^{i\frac{5\pi}{4}}$

c $\left(z_1^{\,4}\right)^*\left(z_2^{\,*}\right)^5 = \left(2e^{i\pi}\right)^*\left(2e^{-i\frac{2\pi}{3}}\right)^5 = \left(2e^{-i\pi}\right)\left(32e^{-i\frac{10\pi}{3}}\right)$

$$= 64e^{-\frac{13\pi i}{3}}$$

d $\frac{\left(z_2^{\,*}\right)^6}{\left(-z_1^{\,*}\right)^3} = \frac{\left(2e^{i\frac{\pi}{3}}\right)^6}{\left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^3} = \frac{2^6e^{i2\pi}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = 16\sqrt{2}e^{i\frac{-\pi}{4}}$

2 $z_1 = \sqrt{2}\left(\cos\frac{\pi}{3}-\cos\left(\frac{5\pi}{6}\right)i\right) = \sqrt{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right); z_2 = 2\left(\sin\frac{5\pi}{6}-i\sin\frac{\pi}{3}\right) = 2\left(coa\frac{-\pi}{3}+i\sin\frac{-\pi}{3}\right)$

$$\frac{z_1^{\,3}}{z_2^{\,5}} = \frac{\left(\sqrt{2}\,e^{i\frac{\pi}{3}}\right)^3}{\left(2\,e^{i\frac{-\pi}{3}}\right)^5} = \frac{2\sqrt{2}}{32}\frac{e^{i\pi}}{e^{i\frac{\pi}{3}}} = \frac{\sqrt{2}}{16}e^{i\frac{2\pi}{3}}$$

3 $\left(\frac{\sin\theta+i\cos\theta}{\cos\theta-i\sin\theta}\right)^{2019} = i^{2019}\left(\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}\right)^{2019} = i^{2019}$

$$= i^{2016}i^3 = \left(i^4\right)^{504}\left(-i\right) = -i$$

4 a $r=\frac{1+3i}{2+i}=\frac{(1+3i)(2-i)}{(2+i)(2-i)}=\frac{5+5i}{5}=1+i=\sqrt{2}e^{i\frac{\pi}{4}}$

b □ $(2+i)\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^8=16(2+i)=32+16i$

c $S_9=(2+i)\frac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^9-1}{\sqrt{2}e^{i\frac{\pi}{4}}-1}=(2+i)\frac{16\sqrt{2}e^{i\frac{\pi}{4}}-1}{\sqrt{2}e^{i\frac{\pi}{4}}-1}$

$=(2+i)\frac{\left(16\sqrt{2}e^{i\frac{\pi}{4}}-1\right)\left(\sqrt{2}e^{-i\frac{\pi}{4}}-1\right)}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}-1\right)\left(\sqrt{2}e^{i\frac{\pi}{4}}-1\right)}=(2+i)\frac{32-16\sqrt{2}e^{i\frac{\pi}{4}}-\sqrt{2}e^{-i\frac{\pi}{4}}+1}{2-\sqrt{2}\left(e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}}\right)+1}$

$=(2+i)\frac{33-16\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)-\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)}{3-\sqrt{2}\left(2\frac{1}{\sqrt{2}}\right)}$

$=(2+i)\frac{33-16(1+i)-(1-i)}{1}$

$=(2+i)(16-15i)$

$=47-14i$

Exercise 10E

1 a $\omega^3+1=0$

$\Rightarrow(\omega+1)(\omega^2-\omega+1)=0$

But $\omega\neq-1$ so it must be the case that $\omega^2-\omega+1=0$

b $(\omega^*)^2-\omega^*+1=(\omega^2)^*-\omega^*+1^*=(\omega^2-\omega+1)^*=0^*=0$

c $\omega^{2019}=(\omega^3)^{673}=1$

$\omega^{2019}+\omega^{2020}(1-\omega+\omega^2)=1+0=1$

2 $(1+\omega+\ldots+\omega^6)(\omega-1)=0\Rightarrow\omega^7-1=0;\omega\neq1$

$\omega_k=e^{i\frac{2\pi k}{7}},k=1,2,\ldots,6$

$1+\omega+\ldots\omega^6$ can be factorised

$(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)(\omega-\omega_4)(\omega-\omega_5)(\omega-\omega_6)=$

since $\omega^*=\omega_6,\omega_2^*=\omega_5,\omega_3^*=\omega_4$

$$(\omega-\omega_1)(\omega-\omega_1^*)(\omega-\omega_2)(\omega-\omega_2^*)(\omega-\omega_3)(\omega-\omega_3^*)=$$
$$(\omega^2-2\,\mathrm{Re}(\omega_1)\omega+1)(\omega^2-2\,\mathrm{Re}(\omega_2)\omega+1)(\omega^2-2\,\mathrm{Re}(\omega_3)\omega+1)=$$
$$\left(\omega^2-2\cos\frac{2\pi}{7}\omega+1\right)=\left(\omega^2-2\cos\frac{4\pi}{7}\omega+1\right)=\left(\omega^2-2\cos\frac{6\pi}{7}\omega+1\right)=$$
$$(\omega^2-1.25\omega+1)(\omega^2+0.445\omega+1)(\omega^2+1.80\omega+1)$$

3 a $\left(5^4 e^{i\pi+2ik\pi}\right)^{\frac{1}{4}}=5e^{i\pi\left(\frac{1}{4}+\frac{k}{2}\right)}$

b $\left(\sqrt{3}-i\right)^{\frac{1}{5}}=2^{\frac{1}{5}}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{\frac{1}{5}}=2^{\frac{1}{5}}\left(e^{-i\frac{\pi}{6}+2k\pi}\right)^{\frac{1}{5}}=2^{\frac{1}{5}}e^{i\pi\left(2k-\frac{1}{30}\right)}$

c $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{\frac{1}{6}}=\left(e^{i\frac{2\pi}{3}+2k\pi}\right)^{\frac{1}{6}}=e^{i\pi\left(\frac{1}{9}+\frac{k}{3}\right)}$

4 a $z_1=1.18+0.334i\Rightarrow \text{Polar}\,1.22\,\mathrm{e}^{0.277i}$

$z_2=1.22\,\mathrm{e}^{i\left(0.277+\frac{\pi}{2}\right)}=-0.334+1.18i$
$z_3=1.22\,\mathrm{e}^{i(0.277+\pi)}=-1.18-0.334i$
$z_4=1.22\,\mathrm{e}^{i\left(0.277+\frac{3\pi}{2}\right)}=0.334-1.18i$

b $z_1=1.40-0.106i\Rightarrow \text{Polar}\,1.40\mathrm{e}^{-0.671i}$

$z_{k+1}=1.40\mathrm{e}^{\left(-0.671i+\frac{2\pi k}{5}\right)}, k=1,2,3,4$
$z_2=0.533+1.30i; z_3=-1.07+0.907i;$
$z_4=-1.19-0.735i; z_5=0.330-1.36i$

c $z_1=1.40+0.287i\Rightarrow \text{Polar}\ 1.43\,\mathrm{e}^{0.202i}$

$z_{k+1}=1.43\,\mathrm{e}^{\left(0.202i+\frac{k\pi}{3}\right)}, k=1,2,3,4,5$
$z_2=0.453+1.36i; z_3=-0.949;$
$z_4=-z_1; z_5=-z_2; z_6=-z_3$

5 a $8\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=8\left(e^{i\frac{3\pi}{4}+2ik\pi}\right)$

b $\mathrm{Re}\left(z^{\frac{1}{6}}\right)=\frac{z^{\frac{1}{6}}+(z^*)^{\frac{1}{6}}}{2}=\frac{8^{\frac{1}{6}}\left(e^{i\frac{\pi}{8}}+e^{-i\frac{\pi}{8}}\right)}{2}=\sqrt{2}\cos\frac{\pi}{8}$

$\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}-1=\frac{1}{\sqrt{2}}$

$\Rightarrow \cos\frac{\pi}{8}=\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}=\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}=\sqrt{\frac{2+\sqrt{2}}{4}}$

$\therefore \mathrm{Re}\left(z^{\frac{1}{6}}\right)=\sqrt{2}\sqrt{\frac{2+\sqrt{2}}{4}}=\sqrt{\frac{2\sqrt{2}+4}{4}}=\frac{\sqrt{2\sqrt{2}+4}}{2}$

Exercise 10F

1 $P(n): \ (\text{cis}\theta)^n = \text{cis}n\theta$

The statement $P(1)$ is true:
$\text{cis}\theta = \text{cis}\theta$
Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$
i.e. $(\text{cis}\theta)^k = \text{cis}k\theta$
Then,
$(\text{cis}\theta)^{k+1} = (\text{cis}\theta)^k(\text{cis}\theta) = (\text{cis}k\theta)(\text{cis}\theta)$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$
$= \cos(k\theta+\theta) + i\sin(k\theta+\theta)$ using the compound angle formula
$= \cos((k+1)\theta) + i\sin((k+1)\theta)$
so $P(k) \Rightarrow P(k+1)$

Therefore it has been shown that $P(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Thus, $P(n)$ is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction

2 **a** $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + (4\cos^3\theta\sin\theta)i - 6\cos^2\theta\sin^2\theta + (-4\cos\theta\sin^3\theta)i + \sin^4\theta$
$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

b $z^4 = \cos 4\theta + i\sin 4\theta$

Comparing these with the answers found in part a,

i $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

ii $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

c $\tan 4\alpha = \dfrac{\sin 4\alpha}{\cos 4\alpha} = \dfrac{4\cos^3\alpha\sin\alpha - 4\cos\alpha\sin^3\alpha}{\cos^4\alpha - 6\cos^2\alpha\sin^2\alpha + \sin^4\alpha} = \dfrac{4\dfrac{\sin\alpha}{\cos\alpha} - 4\dfrac{\sin^3\alpha}{\cos^3\alpha}}{1 - 6\dfrac{\sin^2\alpha}{\cos^2\alpha} + \dfrac{\sin^4\alpha}{\cos^4\alpha}} =$

$= \dfrac{4\tan\alpha - 4\tan^3\alpha}{1 - 6\tan^2\alpha + \tan^4\alpha}$

3 **a** $\left(z+\dfrac{1}{z}\right)^4 = z^4 + \dfrac{4z^3}{z} + \dfrac{6z^2}{z^2} + \dfrac{4z}{z^3} + \dfrac{1}{z^4}$

$= z^4 + 4z^2 + 6 + \dfrac{4}{z^2} + \dfrac{1}{z^4}$

b $\left(z+\dfrac{1}{z}\right)^4 = \left(z^4 + \dfrac{1}{z^4}\right) + 4\left(z^2 + \dfrac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Also,

$\left(z + \frac{1}{z}\right)^4 = 16\cos^4\theta$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

c $\therefore \int \cos^4 x\,dx = \int\left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right)dx$

$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + C$

4 a $\omega^6 - 1 = (\omega^2 - 1)(\omega^4 + \omega^2 + 1) = 0$

$\omega^2 \neq 1$ so it must be that $1 + \omega^2 + \omega^4 = 0$

b $\omega^{102} = (\omega^6)^{17} = 1$

$\omega^{1004} = (\omega^6)^{167}\omega^2 = \omega^2$

$\omega^{20008} = (\omega^6)^{20004}\omega^4 = \omega^4$

$\therefore 1 + \omega^{102} + \omega^{1004} + \omega^{20008} = 1 + (1 + \omega^2 + \omega^4) = 1$

5 Let $z = e^{i\frac{\pi}{8}}$

Let $S = 1 + e^{i\frac{\pi}{8}} + e^{i\frac{\pi}{4}} + \ldots + e^{i\pi}$

$u_1 = 1, r = e^{i\frac{\pi}{8}}$

$$S = \frac{1 - \left(e^{i\frac{\pi}{8}}\right)^9}{1 - e^{i\frac{\pi}{8}}}$$

$$\Rightarrow S = \frac{1 - e^{i\pi}e^{i\frac{\pi}{8}}}{1 - e^{i\frac{\pi}{8}}} = \frac{1 + e^{i\frac{\pi}{8}}}{1 - e^{i\frac{\pi}{8}}} = \frac{\left(1 + e^{i\frac{\pi}{8}}\right)\left(1 - e^{-i\frac{\pi}{8}}\right)}{\left(1 - e^{i\frac{\pi}{8}}\right)\left(1 - e^{-i\frac{\pi}{8}}\right)} = \frac{1 + \left(e^{i\frac{\pi}{8}} - e^{-i\frac{\pi}{8}}\right) - 1}{1 - \left(e^{i\frac{\pi}{8}} + e^{i\frac{\pi}{8}}\right) + 1}$$

$$= \frac{2i\sin\frac{\pi}{8}}{2 - 2\cos\frac{\pi}{8}} = \frac{4i\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{4\sin^2\frac{\pi}{16}} = i\cot\frac{\pi}{16}$$

Chapter review

1 $z = 6e^{-\frac{3\pi}{4}i} = 6\left(\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right)$

$= 6\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -3\sqrt{2} + 3i\sqrt{2}$

so $\mathrm{Re}(z) = -3\sqrt{2},\ \mathrm{Im}(z) = 3\sqrt{2}$

2 $|z_2| = \sqrt{5^2 + (-12)^2} = 13$

$\therefore |z_1{}^2 z_2| = 13r^2 = 52 \Rightarrow r^2 = 4 \Rightarrow r = 2 \ (r \geq 0)$

3 $\frac{1}{1+z} = \frac{1+z^*}{(1+z)(1+z^*)} = \frac{1+z^*}{1+(z+z^*)+|z|^2}$

$= \frac{1+z^*}{2+2\text{Re}(z)} = \frac{1}{2}\left(\frac{1+\cos\theta - i\sin\theta}{1+\cos\theta}\right) = \frac{1}{2}\left(1 - i\frac{\sin\theta}{1+\cos\theta}\right)$

$= \frac{1}{2}\left(1 - i\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) = \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$

4 a $z^5 = 1 = e^{2in\pi}$

$\Rightarrow z = e^{\frac{2in\pi}{5}} \quad (e.g.\ n = 0,1,2,3,4)$

$\Rightarrow z = 1,\ z = e^{\frac{2\pi i}{5}},\ z = e^{\frac{4\pi i}{5}},\ z = e^{\frac{6\pi i}{5}},\ z = e^{\frac{8\pi i}{5}}$

b The five roots above can be written as

$1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4$ i.e. the fifth roots of unity
As a consequence of the fact that the roots of unity sum to zero,
$\text{Re}(1+\omega+\omega^2+\omega^3+\omega^4) = 0$

$\Rightarrow 1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{4} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = 0$

$\Rightarrow \cos\frac{2\pi}{5} + \cos\frac{4\pi}{4} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = -1$

5 a $z^n + \frac{1}{z^n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$
$= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$
$= 2\cos n\theta$

b $\left(z+\frac{1}{z}\right)^6 = z^6 + \frac{6z^5}{z} + \frac{15z^4}{z^2} + \frac{20z^3}{z^3} + \frac{15z^2}{z^4} + \frac{6z}{z^5} + \frac{1}{z^6}$

$= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

c Using part **a**,

$\left(z+\frac{1}{z}\right)^6 = \left(z^6+\frac{1}{z^6}\right) + 6\left(z^4+\frac{1}{z^4}\right) + 15\left(z^2+\frac{1}{z^2}\right) + 20$
$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$
but also
$\left(z+\frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

$\therefore 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$\Rightarrow \cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$

d $\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^6 x dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x + \frac{15}{32}\cos 2x + \frac{5}{16}\right)dx$

$= \left[\frac{1}{192}\sin 6x + \frac{3}{64}\sin 4x + \frac{15}{64}\sin 2x + \frac{5x}{16}\right]_{\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{5\pi}{32} = \left(-\frac{1}{192} + \frac{15}{64} + \frac{5\pi}{64}\right)$

$= \frac{5\pi}{64} - \frac{11}{48}$

6 $\theta = -\arctan\left(\frac{41}{23}\right) = -1.0595656... = -1.06 \ (3\text{s.f.})$

7 $3z_1 - 2z_2 + \frac{1}{2}z_3$

$= \left(9 - 10\cos\frac{17\pi}{83} + \sqrt{2}\cos\left(-\frac{\pi}{4}\right)\right) + i\left(3 - 10\sin\frac{17\pi}{83} + \sqrt{2}\sin\left(-\frac{\pi}{4}\right)\right)$

$= 1.9997458... - (3.9996610....)i$

$= 2.00 - (4.00)i \quad (\text{to } 3\text{s.f.})$

8 $\lim_{x\to 0}\frac{\sin 4x}{\sin x} = \lim_{x\to 0}\frac{4\cos 4x}{\cos x} = 4$ using L'Hopital's Rule

9 The distance from the centre to each vertex is 2

Therefore split the pentagon into five isosceles triangles, and using the formula

Triangle area = $\frac{1}{2}ab\sin C$, we have

Pentagon area = $\frac{5}{2}(2)^2\left(\sin\frac{2\pi}{5}\right) = 9.51056... = 9.51$ to 2d.p.

10 $\omega^2 = (a + 2i)^2 = a^2 + 4ai - 4 = (a^2 - 4)4ai$

$\arg(\omega^2) = \arctan\left(\frac{4a}{a^2 - 4}\right) = 1$

$\frac{4a}{ah2} - 4 = \tan(1)$

$a^2\tan(1) - 4a - 4\tan(1) = 0$

$a = 3.66$

Exam-style questions

11 a $z_1z_2 = 4\operatorname{cis}\left(-\frac{\pi}{3}\right)3\operatorname{cis}\left(\frac{5\pi}{6}\right) = 12\operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)$ (1 mark)

$= 12\operatorname{cis}\left(\frac{\pi}{2}\right)$ (1 mark)

$= 12\text{i}$ (1 mark)

b $\frac{z_1}{z_2} = \frac{4\,\text{cis}\left(-\frac{\pi}{3}\right)}{3\,\text{cis}\left(\frac{5\pi}{6}\right)} = \frac{4}{3}\text{cis}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)$ (1 mark)

$= \frac{4}{3}\text{cis}\left(-\frac{7\pi}{6}\right) = \frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)$ (1 mark)

So $\left(\frac{z_1}{z_2}\right)^3 = \left(\frac{4}{3}\text{cis}\left(\frac{5\pi}{6}\right)\right)^3 = \frac{64}{27}\text{cis}\left(\frac{15\pi}{6}\right)$ (1 mark)

$= \frac{64}{27}\text{cis}\left(\frac{\pi}{2}\right)$ (1 mark)

$= \frac{64}{27}\text{i}$ (1 mark)

c $z_1^{\,2} = 16\,\text{cis}\left(-\frac{2\pi}{3}\right)$

So $\left(z_1^{\,2}\right)^* = 16\,\text{cis}\left(\frac{2\pi}{3}\right)$ (1 mark)

$= 16\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right)$

$= 16\left(-\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right)$ (1 mark)

$= -8 + 8\sqrt{3}\text{i}$ (1 mark)

12 $|1+\text{i}| = \sqrt{2}$ (1 mark)

$\arg(1+\text{i}) = \frac{\pi}{4}$ (1 mark)

$1 + \text{i} = \sqrt{2}\,\text{cis}\frac{\pi}{4}$

$(1+\text{i})^{10} = \left(\sqrt{2}\right)^{10}\text{cis}\frac{10\pi}{4}$ by de Moivre's theorem (1 mark)

$= 2^5\,\text{cis}\frac{5\pi}{2}$ (1 mark)

$= 2^5\,\text{cis}\frac{\pi}{2}$

$= 32\text{i}$ (1 mark)

13 a $|z| = \sqrt{1 + \left(\sqrt{3}\right)^2} = 2$ (1 mark)

$\arg z = -\frac{\pi}{3}$ (1 mark)

$z = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ (1 mark)

b $z^n = 2^n\,\text{cis}\left(-\frac{n\pi}{3}\right)$ (1 mark)

$z^n \in \mathbb{R} \Rightarrow -\frac{n\pi}{3} = 2\pi k$ (1 mark)

So $n = 6$ (1 mark)

c $\left(1 - \text{i}\sqrt{3}\right)^{15} = 2^{15}\,\text{cis}\left(-\frac{15\pi}{3}\right)$ (1 mark)

$= 2^{15}\,\text{cis}(-5\pi)$

$= 2^{15}\,\text{cis}(\pi)$ (1 mark)

$= -2^{15}\,(= -32768)$ (1 mark)

14 a $(\cos\theta + \text{i}\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(\text{i}\sin\theta) + 10\cos^3\theta(\text{i}\sin\theta)^2$

$+10\cos^2\theta(\text{i}\sin\theta)^3 + 5\cos\theta(\text{i}\sin\theta)^4 + (\text{i}\sin\theta)^5$ (2 marks)

$= \cos^5\theta + 5\text{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\text{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \text{i}\sin^5\theta$ (1 mark)

b By de Moivre's theorem, $(\cos\theta + \text{i}\sin\theta)^5 = \cos 5\theta + \text{i}\sin 5\theta$ (1 mark)

Equating real parts of each expression: (1 mark)

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta\left(1 - \cos^2\theta\right)^2$ (1 mark)

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta\left(1 - 2\cos^2\theta + \cos^4\theta\right)$ (1 mark)

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

15 a Let $z^3 = -27\text{i}$

$\left|z^3\right| = 27$ (1 mark)

$\arg\left(z^3\right) = -\frac{\pi}{2}$ (1 mark)

$z^3 = 27\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$ (1 mark)

$z^3 = 27\left(\cos\left(-\frac{\pi}{2} + 2\pi k\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2\pi k\right)\right)$ (1 mark)

$z^3 = 27\left(\cos\left(\frac{4\pi k - \pi}{2}\right) + \text{i}\sin\left(\frac{4\pi k - \pi}{2}\right)\right)$

$z = 3\left(\cos\left(\frac{4\pi k - \pi}{6}\right) + \text{i}\sin\left(\frac{4\pi k - \pi}{6}\right)\right)$ (1 mark)

Choosing $k=1,2,3$ (or equivalent)

$z_1 = 3\operatorname{cis}\frac{\pi}{2}$ (1 mark)

$z_2 = 3\operatorname{cis}\frac{7\pi}{6}$ (1 mark)

$z_3 = 3\operatorname{cis}\frac{11\pi}{6}$ (1 mark)

b Area $= 3\times\left(\frac{1}{2}\times 3\times 3\times \sin\frac{2\pi}{3}\right)$ (2 marks)

$= 3\times\left(\frac{1}{2}\times 3\times 3\times \frac{\sqrt{3}}{2}\right)$

$= \frac{27\sqrt{3}}{4}$ (1 mark)

16 a $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (1 mark)

$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$ (2 marks)

So $z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$ (1 mark)

$= 2\cos n\theta$

b $\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$ (2 marks)

$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$ (1 mark)

$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ (1 mark)

$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$ (1 mark)

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Now $\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$ (1 mark)

Therefore $\cos^4\theta \equiv \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$

c $\int_0^{\frac{\pi}{6}} \cos^4\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{6}} (2\cos 4\theta + 8\cos 2\theta + 6)\, d\theta$ (1 mark)

$= \frac{1}{16}\left[\frac{1}{2}\sin 4\theta + 4\sin 2\theta + 6\theta\right]_0^{\frac{\pi}{6}}$ (1 mark)

$$= \frac{1}{16}\left[\frac{1}{2}\sin\frac{2\pi}{3} + 4\sin\frac{\pi}{3} + \pi\right]_0^{\frac{\pi}{6}}$$ (1 mark)

$$= \frac{1}{16}\left(\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right) + \pi\right)$$ (1 mark)

$$= \frac{1}{16}\left(\frac{\sqrt{3}}{4} + 2\sqrt{3} + \pi\right)$$

$$= \frac{1}{16}\left(\frac{9\sqrt{3}}{4} + \pi\right)$$ (1 mark)

$$= \frac{\pi}{16} + \frac{9\sqrt{3}}{64}$$

17 a $$\left|\frac{\sqrt{3}+i}{\sqrt{3}-i}\right| = \frac{\left|\sqrt{3}+i\right|}{\left|\sqrt{3}-i\right|} = \frac{2}{2} = 1$$ (2 marks)

So $r = 1$

$$\arg\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right) = \arg\left(\sqrt{3}+i\right) - \arg\left(\sqrt{3}-i\right)$$ (1 mark)

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$ (1 mark)

So $\theta = \frac{\pi}{3}$ (1 mark)

$$\frac{\sqrt{3}+i}{\sqrt{3}-i} = e^{i\frac{\pi}{3}}$$

b i $$\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = \left(2\operatorname{cis}\frac{\pi}{6}\right)^n + \left(2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right)^n$$ (1 mark)

$$= 2^n \operatorname{cis}\frac{n\pi}{6} + 2^n \operatorname{cis}\left(-\frac{n\pi}{6}\right)$$ (1 mark)

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$ (1 mark)

$$= 2^n\left(\cos\frac{n\pi}{6} + \cos\frac{n\pi}{6}\right)$$ (1 mark)

$$= 2^n\left(2\cos\frac{n\pi}{6}\right)$$ (1 mark)

$$= 2^{n+1}\cos\left(\frac{n\pi}{6}\right)$$

ii $\left(\sqrt{3}+\mathrm{i}\right)^8+\left(\sqrt{3}-\mathrm{i}\right)^8=2^9\cos\left(\frac{8\pi}{6}\right)$ (1 mark)

$=2^9\cos\left(\frac{4\pi}{3}\right)$

$=2^9\left(-\frac{1}{2}\right)$ (1 mark)

$=-2^8=-256$ (1 mark)

18 a $\omega^*=\omega^2$ (1 mark)

$\left(1+\omega+\omega^*\right)^2=\left(1+\omega+\omega^2\right)^2$

$=\left(\frac{1-\omega^3}{1-\omega}\right)^2$ (1 mark)

$=0^2=0$ (1 mark)

b $\left(1+\omega+3\omega^2\right)^2=\left(1+\omega+\omega^2+2\omega^2\right)^2$ (1 mark)

$=\left(2\omega^2\right)^2=4\omega^4$ (1 mark)

$=4\omega$ (1 mark)

c $\left(1+2\omega+3\omega^2\right)\left(1+3\omega+2\omega^2\right)=\left(1+\omega+\omega^2+\omega+2\omega^2\right)\left(1+\omega+\omega^2+2\omega+\omega^2\right)$ (1 mark)

$=\left(\omega+2\omega^2\right)\left(2\omega+\omega^2\right)$ since $1+\omega+\omega^2=0$ (1 mark)

$=2\omega^2+\omega^3+4\omega^3+2\omega^4$ (1 mark)

$=2\omega^2+5\omega^3+2\omega^4$

$=2\omega^2+5+2\omega$ (1 mark)

$=2\left(1+\omega+\omega^2\right)+3$ (1 mark)

$=2\times0+3$

$=3$ (1 mark)

19 $\mathrm{i}=\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$ (1 mark)

$=\cos\left(\frac{\pi}{2}+2\pi k\right)+\mathrm{i}\sin\left(\frac{\pi}{2}+2\pi k\right)$ (1 mark)

$=\cos\left(\frac{4\pi k+\pi}{2}\right)+\mathrm{i}\sin\left(\frac{4\pi k+\pi}{2}\right)$

So $z-2\mathrm{i}=\cos\left(\frac{4\pi k+\pi}{6}\right)+\mathrm{i}\sin\left(\frac{4\pi k+\pi}{6}\right)$ (1 mark)

$k = 0 \Rightarrow z - 2\text{i} = \cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}$ (1 mark)

$k = 1 \Rightarrow z - 2\text{i} = \cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}$ (1 mark)

$k = 2 \Rightarrow z - 2\text{i} = \cos\frac{9\pi}{6} + \text{i}\sin\frac{9\pi}{6}$ (1 mark)

$z - 2\text{i} = \frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$

$z - 2\text{i} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}$

$z - 2\text{i} = -\text{i}$ (3 marks)

So roots are $z_1 = \frac{\sqrt{3}}{2} + \frac{5}{2}\text{i}$, $z_2 = -\frac{\sqrt{3}}{2} + \frac{5}{2}\text{i}$ and $z_3 = \text{i}$ (1 mark)