5 Analysing data and quantifying randomness: statistics and probability

Skills check

1 a Mean $= \dfrac{720+750+690+975+700+710+720+680+695+645}{10} = \dfrac{1457}{2} = 728.5\text{ kg}$

The number that occurs most often is 720

Median $= \left(\dfrac{n+1}{2}\right)^{th} = \left(\dfrac{11}{2}\right)^{th} = \dfrac{700+710}{2} = 705$

b Range $= 975 - 645 = 330\text{ kg}$

Q_1 is the median of the first half of the list, 690 kg

Q_3 is the median of the second half of the list, 720 kg

$IQR = Q_3 - Q_1 = 720 - 690$ kg

2 Mean $= \dfrac{(2.5\times 5)+(7.5\times 2)+(12.5\times 6)+(17.5\times 8)+(22.5\times 4)+(27.5\times 5)+(32.5\times 8)}{5+2+6+8+4+5+8} = 19.21$ litres

The data is bimodal and the modal classes are $15 < x \le 20$ and $30 < x \le 35$

Median $= \left(\dfrac{n+1}{2}\right)^{th} = \left(\dfrac{38+1}{2}\right)^{th} = 17.5$ litres

Exercise 5A

1 a The target population is "all celery sticks grown in a certain US state"

b The sampling unit is "each celery stick"

c The sample frame is "a list of all celery sticks from the state"

d The sample variable is "the length of the celery stick"

e The sampling values are "the positive real numbers"

2 a The target population is "all ball bearings manufactured by a company"

b The sampling unit is "each ball bearing"

c The sample frame is "a list of all ball bearings enumerated"

d The sample variable is "the weight of the ball bearing"

e The sampling values are "the positive real numbers"

3 a The target population is "all 1 litre soda bottles from a soft drink factory"

b The sampling unit is "each 1 litre soda bottle"

c The sample frame is "all soda bottles enumerated in a list"

d The sample variable is "the volume of the 1 litre soda bottle"

e The sampling values are "the natural numbers"

4 a The target population is "all crates of 50 oranges"

b The sampling unit is "each crate of 50 oranges"

c The sample frame is "an enumerated list of all crates"

d The sample variable is "the weight of a crate of 50 oranges"

e The sampling values are "the positive real numbers"

Exercise 5B

1 List and enumerate all books, generate a random number x then take books $x, x+50, \ldots$

2 **a** Generate a random number x and then sample x bases from each region

b individual response

3 **a** The number of samples of three from $0,1,2,3,4$ is equal to $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$

b The 10 possible samples are $(0,1,2)$, $(0,1,3)$, $(0,1,4)$, $(0,2,3)$, $(0,2,4)$, $(0,3,4)$, $(1,2,3)$, $(1,2,4)$, $(1,3,4)$, $(2,3,4)$

c The means are:

$$\frac{0+1+2}{3} = 1$$

$$\frac{0+1+3}{3} = \frac{4}{3} = 1.3333$$

$$\frac{0+1+4}{3} = \frac{5}{3} = 1.6667$$

$$\frac{0+2+3}{3} = \frac{5}{3} = 1.6667$$

$$\frac{0+2+4}{3} = 2$$

$$\frac{0+3+4}{3} = \frac{7}{3} = 2.3333$$

$$\frac{1+2+3}{3} = 2$$

$$\frac{1+2+4}{3} = \frac{7}{3} = 2.3333$$

$$\frac{1+3+4}{3} = \frac{8}{3} = 2.6667$$

$$\frac{2+3+4}{3} = 3$$

d Mean of population: $\frac{0+1+2+3+4}{5} = \frac{10}{5} = 2$

Mean of sample means:

$$\frac{1+1.3333+1.6667+1.6667+2+2.3333+2+2.3333+2.6667+3}{10} = \frac{20}{10} = 2$$

4 The variable is "whether the envelope is sealed correctly"
The sample is the "batch of selected envelopes"
The population is "all the envelopes"
The variable is discrete

5 It is not possible to wait 4000 years to see if they will last that long

6 This is stratified sampling

7 Pick 12.5 students from each grade (13 from two and 11 from another two).

Exercise 5C

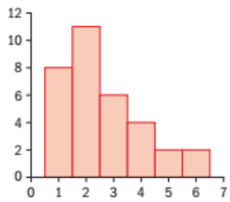
1 **a** The classes are:

$0.5 < x \le 1.5$
$1.5 < x \le 2.5$
$2.5 < x \le 3.5$
$3.5 < x \le 4.5$
$4.5 < x \le 5.5$
$5.5 < x \le 6.5$

b

Number of people	Frequency	Interval on histogram
1	8	$0.5 < x \le 1.5$
2	11	$1.5 < x \le 2.5$
3	6	$2.5 < x \le 3.5$
4	4	$3.5 < x \le 4.5$
5	2	$4.5 < x \le 5.5$
6	2	$5.5 < x \le 6.5$

c



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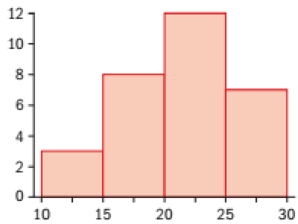

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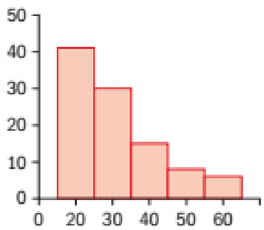


4 **a** The data is continuous

b This plot may show the distribution of lengths of ants in mm

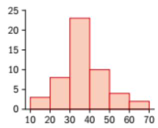
5 a

Hours	Days
4	4
5	5
6	9
7	8
8	4

b $\dfrac{4\times4+5\times5+6\times9+7\times8+8\times4}{4+5+9+8+4}=\dfrac{183}{30}=6.1$ hours

Exercise 5D

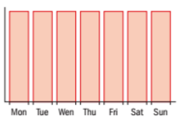
1



Shape: the distribution is unimodal, most koalas had a mass of $30 < x \le 40$ kg
Centre: the midpoint would fall in the $30 < x \le 40$ class.
Spread: the mass of the koalas varies from 17 kg to 61 kg

2 a The data is qualitative therefore, a bar chart is preferable. Each bar would represent each day of the week and would summarise the data very clearly

b

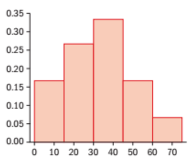


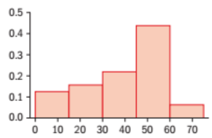
3 a A relative frequency histogram is necessary here as we wish to compare the distributions of two samples from different populations

b

Time spent per day	Male Relative Freq	Female Relative Freq
$0 \le x < 15$	0.1667	0.125
$15 \le x < 30$	0.2667	0.1563
$30 \le x < 45$	0.3333	0.2188
$45 \le x < 60$	0.1667	0.4375
$60 \le x < 75$	0.06667	0.625

c





d The male distribution is symmetric unimodal. The female distribution is right distorted unimodal.

e On average, females spent more time per day on the phone than men.

4 **a** $8+16+11+7+3=43$ families were interviewed

b $\dfrac{(150\times8)+(160\times16)+(170\times11)+(180\times7)+(190\times3)}{43}=\dfrac{7460}{43}=165.\dot{7}=\$166.78$

c The data is left skewed

5 $0.25+0.1875+0.125\times\dfrac{15-11}{16-11}=0.5375,\ 32\times0.53125=17.2$ items

6 **a** Skewed, unimodal, contains an outlier

b Skewed, multimodal, no outliers

c Symmetric, unimodal, no outliers

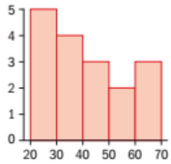
Exercise 5E

1 **a** The mean number of children is $\dfrac{(0\times5)+(1\times10)+(2\times6)+(3\times3)+(4\times1)}{25}=\dfrac{35}{25}=1.4$

b On average, women from Australia have more children

2 **a** Mean $=\dfrac{(25\times5)+(35\times4)+(45\times3)+(55\times2)+(65\times3)}{5+4+3+2+3}=\dfrac{705}{17}=41.4706\approx41.5$ years

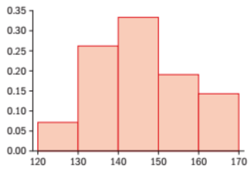
b

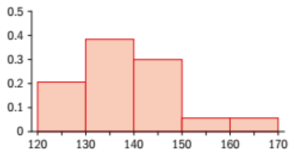


c Left skewed, younger teachers appear to move schools more frequently.

3 **a** A relative frequency histogram is necessary here as we wish to compare the distributions of two samples of different sizes from different populations

b

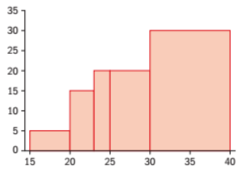




c On average, students from Peru are shorter

4 **a** A standard histogram

b



c Mean $= \dfrac{(17.5\times 5)+(21.5\times 15)+(24\times 20)+(27.5\times 20)+(35\times 30)}{5+15+20+20+30} = \dfrac{2490}{90} = 27.6667$ years

Modal class: $30 < A \le 40$

Exercise 5F

1 a Mean $= \dfrac{2+3+3+4+4+5+5+6+6+6}{10} = \dfrac{44}{10} = 4.4$

Standard deviation

$= \sqrt{\dfrac{2^2+3^2+3^2+4^2+4^2+5^2+5^2+6^2+6^2+6^2}{10} - 4.4^2} = \sqrt{21.2-19.36} = 1.3565 = 1.36$

b Mean $= \dfrac{21+21+24+25+27+29}{6} = \dfrac{147}{6} = 24.5$ kg

Standard deviation

$= \sqrt{\dfrac{21^2+21^2+24^2+25^2+27^2+29^2}{6} - 24.5^2} = \sqrt{608.833-600.25} = 2.92968 = 2.93$ kg kg

c Mean $= \dfrac{3\times 2+4\times 3+5\times 2}{2+3+2} = \dfrac{28}{7} = 4$

Standard deviation $= \sqrt{\dfrac{3^2\times 2+4^2\times 3+5^2\times 2}{2+3+2} - 4^2} = \sqrt{16.5714-16} = 0.75591 = 0.756$

d Mean $= \dfrac{3\times 2+8\times 4+13\times 4+18\times 5+23\times 2}{2+4+4+5+2} = \dfrac{226}{17} = 13.2941 \approx 13.3$

Standard deviation

$= \sqrt{\dfrac{3^2\times 2+8^2\times 4+13^2\times 4+18^2\times 5+23^2\times 2}{2+4+4+5+2} - 13.2941^2} = \sqrt{213.412-176.733} \approx 6.06$

2 Mean $= \dfrac{\Sigma fx}{\Sigma f} = \dfrac{563}{20} = 28.15$

Standard deviation $= \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \left(\dfrac{\Sigma fx}{\Sigma f}\right)^2} = \sqrt{\dfrac{16143}{20} - \left(\dfrac{563}{20}\right)^2} = 3.83764 \approx 3.84$

3 a Mean $= \dfrac{196+197+199+200+200+200+202+203+203+205}{10} = \dfrac{2005}{10} = 200.5$ g

Standard deviation

$= \sqrt{\dfrac{196^2+197^2+199^2+200^2+200^2+200^2+202^2+203^2+203^2+205^2}{10} - 200.5^2}$

$= \sqrt{40207.3-40200.25} = 2.6551 \approx 2.65$ g

b The mean from part a and $\sqrt{\dfrac{10}{9}}$ times the standard deviation from part a

4 a Mean $= \dfrac{6.3+9.6+12.2+12.3+10.3+12.1+10.3+8.4+9.2+4.3}{10} = \dfrac{95}{10} = 9.5$

Standard deviation

$= \sqrt{\dfrac{6.3^2+9.6^2+12.2^2+12.3^2+10.3^2+12.1^2+10.3^2+8.4^2+9.2^2+4.3^2}{10} - 9.5^2}$

$= \sqrt{96.426-90.25} \approx 2.49$

b There is grounds for investigation because the mean amount of lead per litre is within 1 standard deviation of the level that is deemed dangerous

Exercise 5G

1

$$s_{n-1}=\sqrt{\frac{10}{10-1}\left(\frac{10^2+12^2+5^2+0^2+14^2+2^2+5^2+8^2+9^2+6^2}{10}-\left(\frac{10+12+5+0+14+2+5+8+9+6}{10}\right)^2\right)}$$

$$=\sqrt{\frac{10}{9}(67.5-50.41)}=4.35762\approx 4.36$$

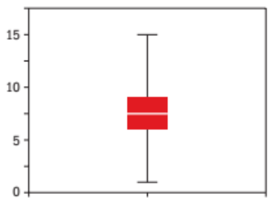
2 $\text{Mean}=\dfrac{11504}{25}=460.16=460\text{ kg kg}$

$$s_{n-1}=\sqrt{\frac{25}{25-1}\left(\frac{5304823}{25}-\left(\frac{11504}{25}\right)^2\right)}=18.72538=18.7\text{ kg}$$

3 $\text{Mean}=\dfrac{\Sigma x}{n}=\dfrac{38750}{25}=1550$

$$s_{n-1}=\sqrt{\frac{n}{n-1}\left(\frac{\Sigma x^2}{n}-\left(\frac{\Sigma x}{n}\right)^2\right)}=\sqrt{\frac{25}{25-1}\left(\frac{60100000}{25}-\left(\frac{38750}{25}\right)^2\right)}=39.5285$$

4 **a**



b The data is symmetric

c $\text{Mean}=\dfrac{\Sigma x}{n}=\dfrac{392}{50}=7.84$

$$s_{n-1}=\sqrt{\frac{50}{50-1}\left(\frac{3470}{50}-\left(\frac{392}{50}\right)^2\right)}=2.8454\approx 2.85$$

d Individual response

Exercise 5H

1 **a** 80

b a is lower quartile mark, a = 55

b is upper quartile mark, b = 75

2 **a** $\dfrac{k-3+k+k+2+k+5}{4}=\dfrac{4k+4}{4}=k+1$

b k + 1 − 3 = k − 2

3 **a** 63

b **i** 87 **ii** 73

4 **a** **i** 1.18m

ii IQR = UQ − LQ = 1.22m − 1.13m = 0.09m

b

Class	**Frequency**
$1.00 \le h < 1.05$	5
$1.05 \le h < 1.10$	8
$1.10 \le h < 1.15$	14
$1.15 \le h < 1.20$	24
$1.20 \le h < 1.25$	18
$1.25 \le h < 1.30$	11

c **i** $\dfrac{5+8+14}{80} = 0.3375 = 0.34$

ii $\dfrac{5+8+14\left(\dfrac{0.02}{0.05}\right)}{5+8+14} = 0.69$

5 **a** **i** Mean $= \dfrac{a+2a+3a+\ldots+na}{n}$

$= \dfrac{a(1+2+3\ldots+n)}{n} = \dfrac{a}{n}\dfrac{n(n+1)}{2} = \dfrac{a(n+1)}{2}$

ii $\dfrac{4n(n+1)}{2} > 100 + \dfrac{4(n+1)}{2}$

$2n^2 + 2n > 100 + 2n + 2$
$2n^2 > 102$
$n^2 > 51$
$n > 7$
$n = 8$

b **i** $M = \dfrac{m(0)+n(1)}{n+m} = \dfrac{n}{n+m}$

$S = \sqrt{\dfrac{n - \dfrac{(m+n)n}{m+n}}{m+n}} = \dfrac{\sqrt{mn}}{m+n}$

ii $M = S \Rightarrow \dfrac{n}{n+m} = \dfrac{\sqrt{nm}}{m+n}$

$n = \sqrt{nm}$
$n^2 = nm$
$n = m$

As there are the same number of x and y points, median is the average of the two values

$$\frac{1+0}{2} = 0.5$$

Exercise 5I

1 **a** No association

b moderate, positive, linear

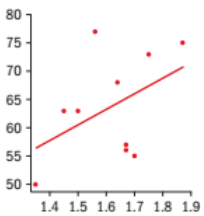
c strong, positive, linear

d moderate, negative, linear

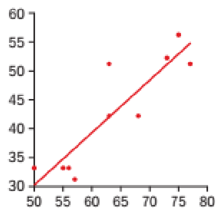
e strong, negative, linear

2

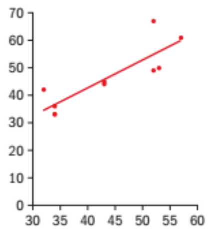
a **i**



b **i**



c **i**

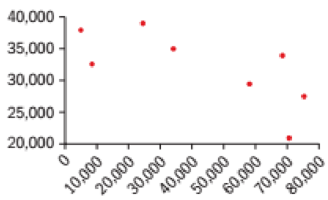


ii 65kg

iii 47s

iv The graphs giving the most accurate predictions are the ones where the data is close to the line of best fit. Graphs b and c are better than graph a.

3



x - Distance (km)	y - Price ($)	x^2	y^2	xy
4895	37900	23961025	1436410000	185520500
75256	27495	5663465536	755975025	2069163720
8563	32595	73324969	1062434025	279110985
24495	38995	600005025	1520610025	955182525
68562	33895	4700747844	1148871025	2323908990
58200	29495	3387240000	869955025	1716609000
34011	34995	1156748121	1224650025	1190214945
70568	21000	4979842624	441000000	1481928000
$\Sigma x = 344550$	$\Sigma y = 256370$	$\Sigma x^2 = 20585335144$	$\Sigma y^2 = 8459905150$	$\Sigma xy = 10201638665$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 20585335144 - \frac{344550^2}{8} = \frac{11491994663}{2}$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 8459905150 - \frac{256370^2}{8} = \frac{488416075}{2}$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 10201638665 - \frac{344550 \times 256370}{8} = -\frac{1679793545}{2}$$

$$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{-\frac{1679793545}{2}}{\sqrt{\frac{11491994663}{2} \times \frac{488416075}{2}}} = -0.709$$

There is moderate negative correlation

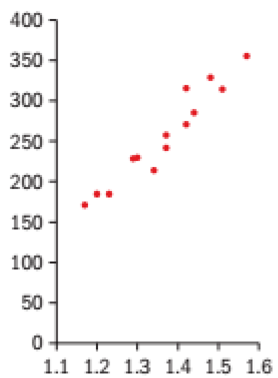
$y = a + bx$, where $b = \frac{S_{xy}}{S_{xx}} = \frac{-\frac{1679793545}{2}}{\frac{11491994663}{2}} = -0.1462$ and

$a = \bar{y} - b\bar{x} = \frac{256370}{8} + 0.1462 \times \frac{344550}{8} = 38341.6$ so $y = 38341.6 - 0.1462x$

$y = 38341.6 - 0.1462 \times 50000 = \$31031.60$

The make of car or price when new would be important information

4 a



b

x - **Height (m)**	y - **Weight (kg)**	x^2	y^2	xy
1.48	329	2.1904	108241	486.92
1.51	314	2.2801	98596	474.14
1.23	185	1.5129	34225	227.55
1.57	356	2.4649	126736	558.92

1.29	228	1.6641	51984	294.12
1.30	230	1.69	52900	299
1.37	257	1.8769	66049	352.09
1.17	171	1.3689	29241	200.07
1.2	185	1.44	34225	222
1.34	214	1.7956	45796	286.76
1.42	315	2.0164	99225	447.3
1.42	271	2.0164	73441	384.82
1.37	242	1.8769	58564	331.54
1.44	285	2.0736	81225	410.4
$\Sigma x = 19.11$	$\Sigma y = 3582$	$\Sigma x^2 = 26.2671$	$\Sigma y^2 = 960448$	$\Sigma xy = 4975.63$

$$S_{xx} = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$$

$$S_{xx} = 26.2671 - \frac{19.11^2}{14} = 0.18195$$

$$S_{yy} = \Sigma y^2 - \frac{(\Sigma y)^2}{n}$$

$$S_{yy} = 960448 - \frac{3582^2}{14} = 43967.7$$

$$S_{xy} = \Sigma xy - \frac{(\Sigma x)(\Sigma y)}{n}$$

$$S_{xy} = 4975.63 - \frac{19.11 \times 3582}{14} = 86.2$$

$y = a + bx$, where $b = \frac{S_{xy}}{S_{xx}} = \frac{86.2}{0.18195} = 473.757$ and

$a = \bar{y} - b\bar{x} = \frac{3582}{14} - 473.757 \times \frac{19.11}{14} = -390.821$ so $y = 473.757x - 390.821$

c $$r = \frac{S_{xy}}{\sqrt{(S_{xx}S_{yy})}}$$

$$r = \frac{86.2}{\sqrt{0.18195 \times 43967.7}} = 0.96375$$

there is a strong positive correlation

d $y = 473.757 \times 1.38 - 390.821 = 262.964$

Chapter review

1 Pick 15 students at random from each MYP class and 15 students at random from the DP group

2 a Pick 12 students at random from the whole medical school

b Pick 1.71 students from each year group at random

c Pick 2.4 students at random from year one and 1.6 students at random from each of the other year groups

d Ask for volunteers and pick the first 12

3 a Make sure the questions are clear

b Make sure the questions are not leading

c Ensure that the possible answers are applicable to everybody and no options are missed

4 a Number of pages

b Height of page

5 a Qualitative, continuous

b Quantitative, continuous

c Quantitative, discrete

d Quantitative, continuous

6

7 a Mean

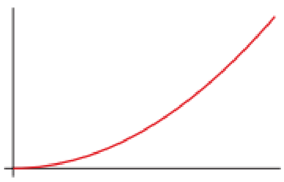
$$= \frac{(5.5\times 13)+(15.5\times 16)+(25.5\times 146)+(35.5\times 139)+(45.5\times 84)+(55.5\times 32)+(65.5\times 20)}{13+16+146+139+84+32+20}$$

$$= \frac{15885}{450} = 35.3$$

b Use mean and median from part a, and use the table to estimate the other points.

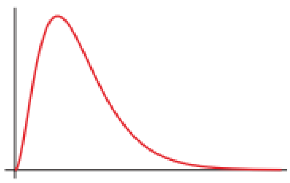
8

a i



a ii

b



9 a

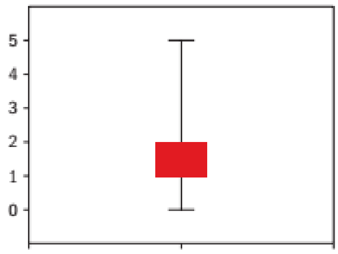
Number of siblings	Frequency
0	14
1	28
2	11
3	5
4	0
5	2

The data is left skewed.

b Mean $= \frac{0\times14+1\times28+2\times11+3\times5+5\times2}{60} = \frac{75}{60} = \frac{5}{4} = 1.25$

standard deviation $= \sqrt{\frac{0^2\times14+1^2\times28+2^2\times11+3^2\times5+5^2\times2}{60}} = \sqrt{\frac{167}{60}-\frac{25}{16}} = 1.105 = 1.11$

c

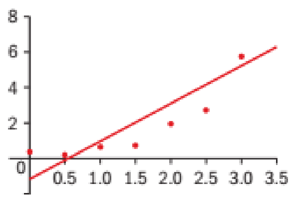


d Notice that the mean of all students is equal to the mean of the original 60 plus the mean of the new 32. Need to find x when $\frac{5}{4}+x=1.25x$, so $\frac{5}{4}=0.25x \Rightarrow$

10 a

x	0	0.5	1	1.5	2	2.5	3	3.5
y	0.6	0.45	0.8	0.85	1.4	1.65	2.4	2.85
y^2	0.36	0.2025	0.64	0.7225	1.96	2.7225	5.75	8.1225

b



c $y^2 = 2.13357x - 1.1725 \Rightarrow y = \sqrt{2.13357x-1.1725}$

11 $y - y_0 = -0.5(x - x_0)$,

$y - 8 = -0.5(x-1)$

$y = 8.5 - 0.5x$

so $y = 8.5 - 0.5x = 8.5 - 0.5\times7 = 5$

Exam-style questions

12 a As the mode is 5 there must be at least another 5. (1 mark)

So we have 1, 3, 5, 5, 6 with another number to be placed in order (1 mark)

The median will be the average of the 3^{rd} and 4^{th} pieces of data. (1 mark)

For this to be 4.5 the missing piece of data must be a 4.

Thus $a = 5$, $b = 4$ (2 marks)

b $\bar{x} = \frac{1+3+4+5+5+6}{6} = \frac{24}{6} = 4$ (2 marks)

13 a $\frac{\sum x}{10} = 70 \Rightarrow \sum x = 700$ (1 mark)

Let Steve's mass be s. $\frac{\sum x + s}{11} = 72$ (1 mark)

$700 + s = 792$ (1 mark)

So $s = 92$ kg (1 mark)

b IQR = 10 (1 mark)

$76 + 1.5 \times IQR = 76 + 15 = 91$ (1 mark)

So Steve's mass of 92 is greater than $1.5 \times IQR$, so is an outlier. (1 mark)

14 a 200 (1 mark)

b 35 (1 mark)

c Using mid-points 5, 15, 25… as estimates for each interval, (1 mark)

i estimate for mean is 22.25 (2 marks)

ii estimate for standard deviation is 11.6 (3 s.f.). (2 marks)

d Median is approximately the 100th piece of data

which lies in the interval $20 < h \le 30$. (1 mark)

Will be 15 pieces of data into this interval

Estimate is $20 + \frac{15}{50} \times 10 = 23$ (2 marks)

15 a **i** 7.5 (1 mark)

ii 6.125 (2 marks)

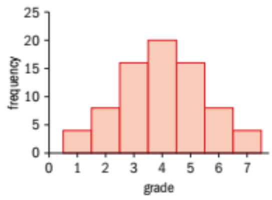
b **i** 6 (1 mark)

ii 6.9 (2 marks)

c Sally's had the greater median (1 mark)

d Rob's had the greater mean (1 mark)

16 a



(1 for scale, 1 for correctly drawn graph)

b **i** 4 **ii** 4 **iii** 4 (3 marks)

c The values of the median and the mean are the same due to the symmetry of the bar chart. (2 marks)

17 a $100 = 70m + c$

$140 = 100m + c$

$40 = 30m$ $m = \frac{4}{3}$ $c = \frac{20}{3}$ (3 marks)

b Positive (1 mark)

c Line goes through $(\bar{x}, \bar{y})$ (1 mark)

$\bar{y} = \frac{4}{3} \times 90 + \frac{20}{3} = \frac{380}{3}$ (2 marks)

d Estimate is $\frac{4}{3} \times 60 + \frac{20}{3} = \frac{260}{3}$ (2 marks)

18 a

x	13	14	15	16	16	17	18	18	19	19
y	2	0	3	1	4	1	1	2	1	2

(5 correct: 2 marks; all correct: 3 marks)

b $r = -0.0695(3sf)$ (2 marks)

c Very weak (negative) correlation so line of best fit is almost meaningless (1 mark)

It would be extrapolation to use this data to predict for a 25-year-old. (1 mark)

19 a **i** no change; $r = 0.87$ (1 mark)

ii no change; 15 (1 mark)

iii the scatter diagram has just been translated up by 5 and to the left by 4, so the PMCC and the gradient of y on x line of best fit are unchanged. (1 mark)

iv Strong, positive (2 marks)

b **i** no change; $r = 0.87$ (1 mark)

ii $2 \times 15 = 30$ (1 mark)

iii the scatter diagram has been stretched vertically by scale factor 2, so PMCC remains unchanged, but gradient of y on x line of best fit is doubled. (1 mark)

c **i** $r = -0.87$ (1 mark)

ii $\frac{15}{-3} = -5$ (1 mark)

iii the scatter diagram has been stretched horizontally by a factor of 3 and then reflected in the y-axis, so gradient becomes −5, but PMCC is unchanged. (2 marks)

iv Strong, negative (2 marks)

20 a i 0.849 (3sf) (2 marks)

ii strong, positive (2 marks)

iii $y = 0.937x + 0.242$ (2 marks)

b i 0.267 (3sf) (2 marks)

ii weak, positive (2 marks)

iii the Pearson product moment correlation coefficient is too small to make the line of best fit particularly meaningful when making predictions. (1 mark)

21 a $r = 0.979$ (3sf) (2 marks)

b Strong, positive (2 marks)

c i $y = 1.23x - 21.3$ (2 marks)

ii $x = 0.776y + 20.8$ (2 marks)

d $1.23 \times 105 - 21.3 = 108$ (1 mark)

e $0.776 \times 95 + 20.8 = 95$ (1 mark)

f It is extrapolation (1 mark)

6 Relationships in space: geometry and trigonometry

Skills check

1 $25^2 = (2x)^2 + x^2$

$25^2 = 5x^2$

$x^2 = 125$

$x = \sqrt{125}$

Area $= \sqrt{125} \times 2\sqrt{125} = 250\text{ cm}^2$

2 $\angle ACB = \angle PAQ$

As AB and PQ are parallel, lines BP and AQ meet AB and PQ at the same angle.

Therefore $\angle ABP = \angle BPQ$ and $\angle BAQ = \angle PQA$

All three angles are identical therefore, triangles are similar.

Exercise 6A

1 **a** $(3, 0, 0)$ **b** $(3, 4, 0)$ **c** $(3, 0, 2)$ **d** $(3, 4, 2)$

e Midpoint of OF: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{0+3}{2}, \frac{0+4}{2}, \frac{0+2}{2}\right) = (1.5, 2, 1)$

f Distance of OF $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \sqrt{(3-0)^2 + (4-0)^2 + (2-0)^2}$

$= \sqrt{9+16+9} = \sqrt{29} \approx 5.4$

2 **a** $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-4+5}{2}, \frac{4-1}{2}, \frac{3+3}{2}\right) = (0.5, -1.5, 3)$

b $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-4-2}{2}, \frac{4+2}{2}, \frac{5+9}{2}\right) = (-3, 3, 7)$

c $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{5-4}{2}, \frac{2-3}{2}, \frac{-4-8}{2}\right) = (0.5, -0.5, -6)$

d $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) = \left(\frac{-5.1+1.4}{2}, \frac{-2+1.7}{2}, \frac{9+11}{2}\right) = (-1.85, 0.15, 10)$

3 **a** $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \sqrt{(4-2)^2 + (3-3)^2 + (1-5)^2} = \sqrt{4+0+16} = \sqrt{20} \approx 4.47$

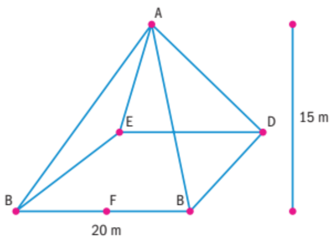
b $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \sqrt{(2+3)^2 + (4-7)^2 + (-1-2)^2} = \sqrt{25+9+9} = \sqrt{43} \approx 6.56$

c $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} = \sqrt{(1+1)^2 + (-3-3)^2 + (4+4)^2}$

$= \sqrt{4+36+64} = \sqrt{104} \approx 10.2$

d $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(-2-2)^2 + (1+1)^2 + (3-3)^2} = \sqrt{16+4+0} = \sqrt{20} \approx 4.47$

4 a



b $\tan\theta = \dfrac{AO}{CO}$

$EC = \sqrt{DC^2 + ED^2} = \sqrt{20^2 + 20^2} = \sqrt{800} = 28.3$

Then

$CO = \dfrac{1}{2}EC = \dfrac{1}{2}28.3 = 14.14$

$\tan\theta = \dfrac{15}{14.14}$

$\theta = \tan^{-1}\dfrac{15}{14.14} = 46.7°$

c $\tan\theta = \dfrac{AO}{OM}$

$OM = \dfrac{1}{2}20 = 10$

$\tan\theta = \dfrac{15}{10}$

$\theta = \tan^{-1}\dfrac{15}{10} = 56.3°$

5 $BD = \sqrt{AB^2 + AD^2} = \sqrt{5^2 + 12^2} = \sqrt{25+144} = \sqrt{169} = 13$

$\tan\theta = \dfrac{FB}{BD} = \dfrac{4}{13}$

$\theta = \tan^{-1}\dfrac{4}{13} = 17°$

6 a $AC^2 = AB^2 + CB^2 = 4^2 + 4^2 = 32\text{cm}$

$AG = \sqrt{AC^2 + CG^2} = \sqrt{32+64} = 9.8\text{cm}$

b $\tan\theta = \dfrac{GC}{AC} = \dfrac{8}{\sqrt{32}}$

$\theta = \tan^{-1}\dfrac{8}{\sqrt{32}} = 55°$

c $\tan\theta = \frac{GE}{AE} = \frac{\sqrt{32}}{8}$

$\theta = \tan^{-1}\frac{\sqrt{32}}{8} = 35°$

$(90° - 55°)$

d $\sin\theta = \frac{4}{\sqrt{96}}$

$\theta = \sin^{-1}\frac{4}{\sqrt{96}} = 24°$

7 **a** $AC = \sqrt{8^2 + 8^2} = \sqrt{128} = 11.3\text{cm}$

b AM is the midpoint of AC, so

$AM = \frac{1}{2} \times AC = 5.7\text{cm}$

c $EA = \sqrt{EM^2 + AM^2} = \sqrt{64 + 32} = \sqrt{96} = 9.8\text{cm}$

d $\tan\theta = \frac{8}{\frac{1}{2}\sqrt{128}}$

$\theta = \tan^{-1}\frac{8}{\frac{1}{2}\sqrt{128}} = 54.7°$

Exercise 6B

1 **a** $V = \frac{1}{3}A_{base}h = \frac{1}{3} \times 12 \times 12 \times 12 = 576\text{cm}^3$

$h_{face} = \sqrt{6^2 + 12^2} = 13.4$

$SA = A_{base} + 4A_{face} = (12 \times 12) + 4 \times \left(\frac{1}{2} \times 12 \times 13.4\right) = 466\text{cm}^2$

b $V = \frac{1}{3}A_{base}h = \frac{1}{3} \times 4 \times 5 \times 6 = 40\text{cm}^3$

$h_{face} = \sqrt{2^2 + 6^2} = 6.32$

$SA = A_{base} + 4A_{face} = (4 \times 5) + 4 \times \left(\frac{1}{2} \times 5 \times 6.32\right) = 83.2\text{cm}^2$

c $V = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi \times 3^2 \times 9 = 85\text{cm}^3$

$s = \sqrt{9^2 + 3^2} = 9.5$

$SA = \pi r^2 + \pi rs = \pi \times 3^2 + \pi \times 3 \times 9.5 = 117.8\text{cm}^2$

d $V = \frac{1}{3}\pi r^2 = \frac{1}{3}\pi \times 1 \times 3 = \pi = 3.14\text{cm}^3$

$s = \sqrt{1^2 + 3^2} = 3.16$

$SA = \pi r^2 + \pi rs = \pi \times 1 + \pi \times 1 \times 3.16 = 13.1\text{cm}^2$

e $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{16.4}{2}\right)^3 = 2310\text{cm}^3$

$SA = 4\pi r^2 = 4\pi\left(\frac{16.4}{2}\right)^2 = 845\text{cm}^2$

f $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \times \left(\frac{6}{2}\right)^3 = 113.1\text{cm}^3$

$SA = 4\pi r^2 = 4\pi(3)^2 = 113.1\text{cm}^2$

2 $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{2}\left(\frac{4}{3}\pi\left(\frac{5.6}{2}\right)^3\right) = 46\text{cm}^3$

3 $V_{cyl} + V_{cone} = \pi r^2 h_{cyl} + \frac{1}{3}\pi r^3 h_{cone} = \pi \times 3.2^2 \times 9.1 + \frac{1}{3}\pi \times 3.2^2 \times 6.2 = 359\text{cm}^3$

4 $V_{clay} = V_{h1} - V_{h2} = \frac{1}{2} \times \frac{4}{3}\pi \times 9^3 - \frac{1}{2} \times \frac{4}{3}\pi \times 8^3 = 454\text{cm}^3$

5 $12\left(\frac{4\pi(3)^2}{3}\right) \div \pi(10)^2 = 1.44\text{ cm}$

6 $V_L = \frac{1}{3}\pi r_L^2 = \frac{1}{3}\pi \times \left(\frac{15}{2}\right)^2 (15 + 45) = 3534\text{cm}^3$

$V_s = \frac{1}{3}\pi r_s^2 = \frac{1}{3}\pi \times \left(\frac{5}{2}\right)^2 (15) = 98.17\text{cm}^3$

$V_f = V_L - V_s = 3436\text{cm}^3$

Exercise 6C

1 **a** $45° = \frac{45°\pi}{180°} = \frac{\pi}{4}$

b $60° = \frac{60°\pi}{180°} = \frac{\pi}{3}$

c $270° = \frac{270°\pi}{180°} = \frac{3\pi}{2}$

d $360° = 2\pi$

e $18° = \frac{18°\pi}{180°} = \frac{\pi}{10}$

f $225° = \frac{225°\pi}{180°} = \frac{5\pi}{4}$

g $80° = \frac{80°\pi}{180°} = \frac{4\pi}{9}$

h $200° = \frac{200°\pi}{180°} = \frac{10\pi}{9}$

i $120° = \frac{120°\pi}{180°} = \frac{2\pi}{3}$

j $135° = \frac{135°\pi}{180°} = \frac{3\pi}{4}$

2 **a** $\frac{\pi}{6}\frac{180°}{\pi}=30°$

b $\frac{\pi}{10}\frac{180°}{\pi}=18°$

c $\frac{5\pi}{6}\frac{180°}{\pi}=120°$

d $3\pi\frac{180°}{\pi}=540°$

e $\frac{7\pi}{20}\frac{180°}{\pi}=63°$

f $\frac{4\pi}{5}\frac{180°}{\pi}=144°$

g $\frac{7\pi}{4}\frac{180°}{\pi}=315°$

h $\frac{14\pi}{9}\frac{180°}{\pi}=280°$

i $\frac{5\pi}{3}\frac{180°}{\pi}=300°$

j $\frac{13\pi}{4}\frac{180°}{\pi}=585°$

3 **a** $10°=\frac{10°\pi}{180°}=0.174$

b $40°=\frac{40°\pi}{180°}=0.698$

c $25°=\frac{25°\pi}{180°}=0.436$

d $300°=\frac{300°\pi}{180°}=5.24$

e $110°=\frac{110°\pi}{180°}=1.92$

f $75°=\frac{75°\pi}{180°}=1.31$

g $85°=\frac{85°\pi}{180°}=1.48$

h $12.8°=\frac{12.8°\pi}{180°}=0.223$

i $37.5°=\frac{37.5°\pi}{180°}=0.645$

j $1°=\frac{1°\pi}{180°}=0.01$

4 **a** $1\times\frac{180°}{\pi}=57.3°$

b $2\times\frac{180°}{\pi}=115°$

c $0.63\times\frac{180°}{\pi}=36.1°$

d $1.41\times\frac{180°}{\pi}=80.7°$

e $1.55\times\frac{180°}{\pi}=88.8°$

f $3\times\frac{180°}{\pi}=172°$

g $0.36\times\frac{180°}{\pi}=20.6°$

h $1.28\times\frac{180°}{\pi}=73.3°$

i $0.01\times\frac{180°}{\pi}=0.573°$

j $2.15\times\frac{180°}{\pi}=123°$

5 **a** $1+2A=\pi$

$A=\frac{\pi-1}{2}$

b $V+2=\pi$

$V=\pi-2$

Exercise 6D

1 **a** **i** $l=r\theta=14\times\frac{\pi}{2}=7\pi\,\text{cm}$

ii $l=r\theta=12\times\frac{3\pi}{4}=9\pi\,\text{m}$

iii $l = r\theta = 3\times\frac{5\pi}{6} = \frac{5}{2}\pi\,\text{m}$

iv $l = r\theta = 15\times\frac{14\pi}{9} = \frac{70}{3}\pi\,\text{cm}$

b i $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 14^2\times\frac{\pi}{2} = 49\pi\,\text{cm}^2$

ii $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 12^2\times\frac{3\pi}{4} = 54\pi\,\text{m}^2$

iii $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 3^2\times\frac{5\pi}{6} = \frac{15}{4}\pi\,\text{m}^2$

iv $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 15^2\times\frac{14\pi}{9} = 175\pi\,\text{cm}^2$

2 $A = \frac{1}{2}r^2\frac{\pi}{12} = 3\pi$

$r^2 = 3\pi\times 2\times\frac{12}{\pi} = 72$

$r = \sqrt{72}\,\text{cm}$

3 a $A = \frac{1}{2}12^2\theta = 36\pi$

$\theta = \frac{36\pi}{12^2}\times 2 = \frac{\pi}{2}$

b $P = 2r + l$

$l = r\theta = 12\times\frac{\pi}{2} = 6\pi$

$P = 2\times 12 + 6\pi = (24 + 6\pi)\,\text{cm}$

4 $A_{shaded} = A_{sector} - A_{triangle}$

$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2}\times 10^2\times 1.5 = 75$

$A_{triangle} = \frac{1}{2}r^2\sin\theta = \frac{1}{2}\times 10^2\sin 1.5 = 49.9$

$A_{shaded} = 75 - 49.9 = 25.1$

5 Arclength is travelled in one second, so we need $60l$ for total distance travelled.

$l = r\theta = 4\times\frac{\pi}{12} = \frac{\pi}{3}$

Then $60l = 20\pi\,\text{m}$

6 $p = 2r + l = 2r + r\theta$

$r(2+\theta) = p$

$r = \frac{p}{2+\theta}$

7 $A_{quad} = \frac{1}{2}r_1^2\theta_1 = \frac{1}{2}\times 10^2\times\frac{\pi}{2} = 78.53$

$A_{semi} = A_{quad} = \frac{1}{2}r_2^2\theta_2 = \frac{1}{2}r_2^2\pi = 78.53$

$r_2^2 = 78.53 \times \frac{2}{\pi} = 50$

$r_2 = \sqrt{50} = 7.07\text{cm}$

$p_{semi} = 2r + l = 2 \times 7.07 + 7.07 \times \pi = 36.36\text{cm}$

Exercise 6E

1 **a** Quadrant II, $\theta = 180 - (-215 + 360) = 35°$

b Quadrant II, $\theta = \pi - \left(\frac{-5\pi}{4} + 2\pi\right) = \frac{\pi}{4}$

c Quadrant IV, $\theta = 4\pi - \frac{7\pi}{2} = \frac{\pi}{2}$

d Quadrant IV, $\theta = 2\pi - \frac{11\pi}{6} = \frac{\pi}{6}$

e Quadrant III, $\theta = 564° - 360° - 180° = 24°$

f Quadrant IV, $\theta = 22°$

g Quadrant II, $\theta = 3\pi - \frac{8\pi}{3} = \frac{\pi}{3}$

2

	$\sin\theta$	$\cos\theta$	$\tan\theta$	θ
A	0.5	–0.866	$\frac{0.5}{-0.866} = -0.577$	$\frac{5\pi}{6}$
B	$-\sin\frac{\pi}{6} = -0.5$	$\cos\frac{\pi}{6} = 0.866$	$-\frac{0.5}{0.866} = -0.577$	$\frac{11\pi}{6}$
C	0.3907	$\frac{0.3907}{-0.4245} = -0.9204$	–0.4245	2.74
D	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\frac{\pi}{4}$
E	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	–1	$\frac{3\pi}{4}$

3 **a** $\theta = \sin^{-1} 0.6 = 36.8°, 143°$

b θ is undefined

c $\theta = \tan^{-1} - 2.36 = 293°, 112°$

4 $\cos\theta = \frac{3}{5}, \theta$ is in QII

$\sin\theta = -\frac{4}{5}$ and $\tan\theta = -\frac{4}{3}$

Exercise 6F

1 $\theta = \cos^{-1} 0.45 = 1.104$ and $\theta = 2\pi - 1.104 = 5.18$

2 $\theta = \tan^{-1} - 0.56 = 5.77$ and $\theta = \pi - (2\pi - 5.77) = 2.63$

3 $\theta = \sin^{-1} 0.23 = 0.23$ and $\theta = \pi - 0.23 = 2.91$

4 theta is undefined

5 $\cos^2 \theta + \cos \theta = 0$

$\cos\theta(\cos \theta + 1) = 0$

Then

$\cos \theta = 0$

gives that $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

and

$\cos \theta = -1$

gives that $\theta = \pi$

6 $2\cos^2 \theta - 3\cos \theta + 1 = 0$

$(2\cos \theta - 1)(\cos \theta - 1) = 0$

Then

$\cos \theta = \frac{1}{2}$

so $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ and

$\cos \theta = 1$

so $\theta = 0, 2\pi$

7 $4\sin^2 \theta = 1$

$\sin \theta = \left(\frac{1}{4}\right)^{1/2} = \pm\frac{1}{2}$

then

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

8 $\frac{3\theta}{2} = \sin^{-1} - 0.62 = 5.61$

Note that $0 < \frac{3\theta}{2} < 3\pi$, so

$\frac{3\theta}{2} = 5.61, 3.81$

and so

$\theta = 3.74, 2.54$

9 $\tan 2\theta = -0.45555$

so $0 < 2\theta < 4\pi$ and

$\tan^{-1} - 0.45555 = 2\theta$

then

$2\theta = 5.855, 2.714, 8.997, 12.1382$

where the first two angles are the angles for the negative tangent, and the last two are an added rotation to them $(+2\pi)$

Exercise 6G

1 **a** $p^2 = q^2 + r^2 - 2qr\cos P = 8^2 + 5^2 - 2 \times 8 \times 5\cos 30° = 19.72$

so

$p = \sqrt{19.72} = 4.44$

$\cos Q = \dfrac{r^2 + p^2 - q^2}{2rp} = \dfrac{5^2 + 4.44^2 - 8^2}{2 \times 5 \times 4.44} = -0.434$

$Q = \cos^{-1} - 0.434 = 115°$

and so

$R = 180° - 115° - 30° = 34.25°$

b $y^2 = x^2 + z^2 - 2xz\cos Y = 4^2 + 5^2 - 2 \times 4 \times 5\cos 95° = 44.49$

$y = \sqrt{44.49} = 6.67$

and so

$\cos X = \dfrac{y^2 + z^2 - x^2}{2yz} = \dfrac{6.67^2 + 5^2 - 4^2}{2 \times 6.67 \times 5} = 0.802$

$X = \cos^{-1} 0.802 = 36.7°$

and

$Z = 180° - 36.7° - 95° = 48.3°$

c $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{8^2 + 4^2 - 5^2}{2 \times 8 \times 4} = 0.8594$

so $A = 30.8°$

$\cos b = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{5^2 + 4^2 - 8^2}{2 \times 5 \times 4} = -0.575$

so $B = 125°$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 4^2}{2 \times 8 \times 5} = 0.9125$$

so $C = 24.2°$

2 Largest angle is opposite the longest side

$$\cos\theta = \frac{3.9^2 + 2.3^2 - 4.5^2}{2 \times 2.3 \times 3.9}$$

$$\theta = \cos^{-1}\left(\frac{3.9^2 + 2.3^2 - 4.5^2}{2 \times 2.3 \times 3.9}\right) = 89.2015°$$

3 We have that

$3PQ = 2QR = 4RP$

so the smallest side is RP. Then

$$\cos PQR = \frac{PQ^2 + QR^2 - PR^2}{2PQQR} = \frac{\left(\frac{4}{3}RP\right)^2 + (2RP)^2 - RP^2}{2 \times \frac{4}{3} \times 2 \times RP^2} = \frac{43}{48}$$

$$\cos^{-1}\frac{43}{48} = 26.38°$$

4 $\cos 60° = \dfrac{5^2 + x^2 - (2x-1)^2}{2 \times 5x}$

or equivalently

$$2 \times 5x \times \frac{1}{2} = 25 + x^2 - 4x^2 + 4x - 1$$

or equivalently

$-3x^2 - x + 24 = 0$

which has solutions $x = -3, \dfrac{8}{3}$. We take the positive value as it is a distance. So

$$x = \frac{8}{3}$$

and $b = \dfrac{8}{3}$, $a = 2 \times \dfrac{8}{3} - 1 = \dfrac{13}{3}$.

Then we calculate the remaining angles as

$$\cos ABC = \frac{5^2 + \left(\frac{13}{3}\right)^2 - \frac{8}{3}}{2 \times 5 \times \frac{13}{3}}$$

so

$$ABC = \cos^{-1}\frac{37}{39} = 18.42°$$

and so

$BCA = 180° - 18.42° - 60° = 101.6°$

5 Note that $DAB = BCD$ and $CDA = CBA$. Then

$2DAB + 2CDA = 360°$

$CDA = 180° - DAB$

We use the cosine rule to get the relationships

$$\cos DAB = \frac{b^2 + a^2 - q^2}{2ba}$$

and

$$\cos CDA = \frac{a^2 + b^2 - p^2}{2ab}$$

Note that we also have that $\cos CDA = \cos(180° - DAB) = -\cos DAB$, so

$$\frac{b^2 + a^2 - q^2}{2ba} = \frac{-a^2 - b^2 + p^2}{2ab}$$

which rearranges to

$$p^2 + q^2 = 2(b^2 + a^2)$$

Exercise 6H

1 **a** $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\frac{\sin 30°}{10} = \frac{\sin 125°}{b}$

$b = \frac{\sin 125°}{\sin 30°} 10 = 16.4 \text{ cm}$

Then $C = 180° - 125° - 30° = 25°$

$\frac{\sin 30°}{10} = \frac{\sin 25°}{c}$

$c = \frac{\sin 25°}{\sin 30°} 10 = 8.45 \text{ cm}$

b $Q = 180° - 45° - 40° = 95°$

$\frac{\sin 45°}{7} = \frac{\sin 40°}{p}$

$p = \frac{\sin 40°}{\sin 45°} 7 = 6.36 \text{ cm}$

and

$\frac{\sin 45°}{7} = \frac{\sin 95°}{q}$

$$q = \frac{\sin 95^\circ}{\sin 45^\circ}7 = 9.86\,\text{cm}$$

c $$\frac{\sin 40^\circ}{9} = \frac{\sin A}{7}$$

$$\sin A = 7\frac{\sin 40^\circ}{9} = 0.4999$$

$$A = \sin^{-1} 0.49999 = 30^\circ$$

Then

$$B = 180^\circ - 40^\circ - 30^\circ = 110^\circ$$

and

$$\frac{\sin 40^\circ}{9} = \frac{\sin 110^\circ}{b}$$

and

$$b = 9\frac{\sin 110^\circ}{\sin 40^\circ} = 13.15$$

2 $$\frac{\sin 15^\circ}{150} = \frac{\sin Q}{80}$$

$$\sin Q = 80\frac{\sin 15^\circ}{150} = 0.138$$

$$Q = \sin^{-1} 0.138 = 7.93^\circ$$

Then $R = 180^\circ - 7.93^\circ - 15^\circ = 157.1^\circ$

$$\frac{\sin 15^\circ}{150} = \frac{\sin 157.1^\circ}{r}$$

$$r = 150\frac{\sin 157.1^\circ}{\sin 15^\circ} = 225.5\,\text{km}$$

The error course took $80 + 150 = 230\text{km}$, which took

$$\frac{230}{400} = 0.575\text{h}$$

and the correct course would take

$$\frac{225.5}{400} = 0.56375\text{h}$$

Then the time difference is

$$0.575 - 0.56375 = 0.01125\text{h}$$

which in seconds is

$$0.01125 \times 3600 = 41\,\text{seconds}$$

3 The distance between the end of the lake and the balloon is given by

$$\tan 32° = \frac{250}{x_1}$$

$x_1 = 400.1\text{m}$

The distance from the balloon to the beginning of the lake is

$$\tan(90° - 68°) = \frac{x_2}{250}$$

$x_2 = 250\tan 22° = 101\text{m}$

Then the length of the lake is the difference between the two lengths,

$x_1 - x_2 = 400.1 - 101 = 300\text{m}$

4 $CBM = 180° - 64° = 116°$

$BMC = 180° - 116° - 23° = 41°$

$$\frac{\sin 41°}{15} = \frac{\sin 116°}{MC}$$

$$MC = 15\frac{\sin 116°}{\sin 41°} = 20.5\text{m}$$

$$\frac{\sin 116°}{20.5} = \frac{\sin 23°}{MB}$$

$$MB = 20.5\frac{\sin 23°}{\sin 116°} = 8.91\text{m}$$

and

$$\frac{\sin 64°}{MA} = \frac{\sin 90°}{8.91}$$

$MA = 8.91\sin 64° = 8\text{m}$

5 $$\frac{\sin 55°}{27} = \frac{\sin ACB}{31}$$

$$\sin ACB = 31\frac{\sin 55°}{27} = 0.9405$$

$ACB = \sin^{-1} 0.9405 = 70°$

Then for this case, the third angle is $BAC = 180° - 55° - 70° = 55°$

Alternatively we can take the obtuse angle,

$ACB = 180° - 70.14° = 110°$

and so the other triangle has angle

$BAC = 180° - 55° - 110° = 15°$

Exercise 6I

1 $A_{total} = A_{PQR} + A_{PRS}$

$$A_{PQR} = \frac{1}{2} \times 10 \times 13 \times \sin 125° = 53.24$$

$$PR^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \times \cos 125° = 119.87$$

$$PR = 20.45$$

Then

$$A_{RPS} = \frac{1}{2} \times 20.45 \times 15 \times \sin 70° = 144.1$$

Then

$$A_{total} = 53.24 + 144.1 = 197.4$$

2 $\frac{\sin B}{104} = \frac{\sin 20°}{52}$

$$B = \sin^{-1}\left(104\frac{\sin 20°}{52}\right) = 43.16°$$

In this case, the other angle is $C = 180° - 43.16° - 20° = 117°$, so

$$A_1 = \frac{1}{2} \times 52 \times 104 \times \sin 117° = 2409\,\text{cm}^2$$

We can also take the obtuse angle for B, and so

$$B = 180° - 43.16° = 137°$$

Then the other angle is $C = 180° - 20° - 137° = 23°$, so

$$A_2 = \frac{1}{2} \times 52 \times 104 \times \sin 23° = 1057\,\text{cm}^2$$

3 $\cos YOZ = \frac{5^2 + 5^2 - 7^2}{2 \times 5 \times 5} = 0.02$

$$YOZ = \cos^{-1} 0.02 = 88.8°$$

$$A_{YZO} = \frac{1}{2} 5 \times 5 \times \sin 88.8° = 12.5\,\text{cm}^2$$

$$\cos XOY = \frac{5^2 + 5^2 - 3^2}{2 \times 5 \times 5} = 0.82\,.$$

$$XOY = \cos^{-1} 0.82 = 34.9°$$

$$A_{XOY} = \frac{1}{2} \times 5 \times 5 \times \sin 34.9° = 7.15$$

Then

$$A_{total} = A_{YZO} + A_{XOY} = 12.5 + 7.15 = 19.7\,\text{cm}^2$$

4 $\tan 60° = \frac{12}{CA}\,.$

$$CA = \frac{12}{\tan 60°} = 6.93$$

$\tan 55° = \frac{12}{DA}$

$DA = \frac{12}{\tan 55°} = 8.4$

$\cos CAD = \frac{CA^2 + DA^2 - CD^2}{2 \times CA \times DA} = \frac{6.93^2 + 8.4^2 - 15^2}{2 \times 6.93 \times 8.4} = -0.914$

$CAD = \cos^{-1} - 0.914 = 156.1°$

$A_{CAD} = \frac{1}{2} \times CA \times DA \times \sin CAD = \frac{1}{2} \times 6.93 \times 8.4 \times \sin 156.1° = 11.8\text{m}^2$

5 **a** $A_{POQ} = \frac{1}{2} \times r^2 \times \sin\left(\frac{\pi}{4} + \frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{1}{2} r^2 \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4} r^2$

and $A_{ROS} = \frac{1}{2} r^2 \sin\frac{\pi}{6} = \frac{1}{4} r^2$

b $A_{sectorPQ} = \frac{1}{2} r^2\theta = \frac{1}{2} r^2 \left(\frac{\pi}{4} + \frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\pi r^2}{3}$

$A_{minorPQ} = A_{sectorPQ} - A_{POQ} = \frac{\pi r^2}{3} - \frac{\sqrt{3}}{4} r^2 = r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

c $A_{sectorRS} = \frac{1}{2} r^2\theta = \frac{1}{2} r^2 \left(\frac{\pi}{6}\right) = \frac{\pi r^2}{12}$

$A_{minorRS} = A_{sectorRS} - A_{ROS} = \frac{\pi r^2}{12} - \frac{1}{4} r^2 = r^2\left(\frac{\pi}{12} - \frac{1}{4}\right)$

d Shaded area should not include minor of RS (otherwise it's just $A_{minorPQ}$)

$A_{shaded} = A_{minorPQ} - A_{minorRS} = r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right) - r^2\left(\frac{\pi}{12} - \frac{1}{4}\right) = \frac{r^2}{4}\left(\pi + 1 - \sqrt{3}\right)$

6 $Yacht : 24 \times 5 = 120$ nautical miles

$Catamaran : 15 \times 5 = 75$ nautical miles

$\theta = 139° - 37° = 102°$

$d^2 = 75^2 + 120^2 - 2 \times 75 \times 120 \cos 102° = 23767.4$

$d = \sqrt{23767.4} = 154.2$ nautical miles

The bearing is given by the relationship

$360° - 143° = 217°$

and this this the angle complementary to the bearing from the yacht to the origin. Then we find the other angle of the triangle as

$\frac{\sin 102°}{154} = \frac{\sin A}{75}$

$A = 151.6°$

where we chose the obtuse angle. Then the complementary angle to the bearing we are looking for is

$\theta = 217° - 151.6° = 65.4°$

Finally, the angle we are searching for is

$\phi = 180° - 65.4° = 115°$

Exercise 6J

1 $\sec\frac{\pi}{6} - \cot\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{6}} - \frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} - \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$

2 $\csc\left(-\frac{2\pi}{3}\right) + 2\tan\frac{7\pi}{6} = \frac{1}{\sin\left(-\frac{2\pi}{3}\right)} + 2\frac{\sin\frac{7\pi}{6}}{\cos\frac{7\pi}{6}} = -\frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} = 0$

3 **a** $\cos\theta\tan\theta = \cos\theta\frac{\sin\theta}{\cos\theta} = \sin\theta$

b $\cot\theta\sec\theta = \frac{\cos\theta}{\sin\theta}\frac{1}{\cos\theta} = \frac{1}{\sin\theta} = \csc\theta$

c $\csc\theta\tan\theta = \frac{1}{\sin\theta}\frac{\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta$

d $\cos\theta\sec^2\theta\sin\theta = \cos\theta\frac{1}{\cos^2\theta}\sin\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

e $\frac{\tan\theta\cot\theta}{\sin\theta} = \frac{\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{\sin\theta}}{\sin\theta} = \frac{1}{\sin\theta} = \csc\theta$

f $\tan\theta\csc\theta = \frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta} = \frac{1}{\cos\theta} = \sec\theta$

4 $\csc\theta = \frac{1}{\sin\theta} = \frac{13}{5}$

Then

$\sin\theta = \frac{5}{13}$

This comes from a triangle of sides 5, 13 and $c = \sqrt{13^2 - 5^2} = 12$, then

$\cos\theta = \frac{-12}{13}$

and

$\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{\frac{12}{13}}{\frac{5}{13}} = -\frac{12}{5}$

5 $\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{4}$

$\cos\theta = -\frac{4}{5}$

Then the third side can be calculated as $c = \sqrt{25-16} = 3$, and so

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$

with $\sin\theta = -\frac{3}{5}$ which is negative to satisfy $\pi < \theta < 2\pi$

Exercise 6K

1 **a** $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

b $\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

c $\frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

d $\frac{\cos\theta}{1-\cos^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$

e $\sqrt{1+t^2} = \sqrt{1+\tan^2\theta} = \sqrt{1+\sec^2\theta-1} = \sec\theta$

f $\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

2 **a** $a^2\sin^2\theta - a^2 = a^2\left(\sin^2\theta - 1\right) = a^2\cos^2\theta$

b $b\cot\theta\sqrt{b^2+b^2\cot^2\theta} = b\cot\theta\sqrt{b^2(1+\cot^2\theta)} = b^2\cot\theta\sqrt{\csc^2\theta} = b^2\cot\theta\csc\theta$

$b^2\frac{\cos\theta}{\sin\theta}\frac{1}{\sin\theta} = b^2\frac{\cos\theta}{\sin^2\theta}$

c $\frac{b}{\sqrt{b^2+y^2}} = \frac{b}{\sqrt{b^2+b^2\cot^2\theta}} = \frac{b}{b\sqrt{1+\cot^2\theta}} = \frac{1}{\csc\theta} = \sin\theta$

d $\frac{\sqrt{a^2\sec^2\theta+a^2}}{a\sec\theta} = \frac{a\sqrt{\tan^2\theta-1+1}}{a\sec\theta} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

3 **a** $3 - 3\cos\theta = 2\sin^2\theta$

$3 - 3\cos\theta = 2(1-\cos^2\theta)$

$2\cos^2\theta - 3\cos\theta + 1 = 0$

$(2\cos\theta - 1)(\cos\theta - 1) = 0$

Then

$\cos\theta = \frac{1}{2}$

so $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ and

$\cos\theta = 1$

so $\theta = 0, 2\pi$

b $\sec\theta = \frac{1}{\cos\theta} = 2$

so

$\cos\theta = \frac{1}{2}$

then $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

c $\cos^2\theta + \sin\theta + 1 = 0$

$1 - \sin^2\theta + \sin\theta + 1 = 0$

$\sin^2\theta - \sin\theta - 2 = 0$

$(\sin\theta - 2)(\sin\theta + 1) = 0$

Then

$\sin\theta = 2$

cannot be a solution as it is outside the domain of the sine function. We also have

$\sin\theta = -1$

so $\theta = \frac{3\pi}{2}$

d $\sec^2\theta = 1 + \tan\theta$

$\tan^2\theta + 1 - 1 - \tan\theta = 0$

$\tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$

Then $\tan\theta = 0$, then $\theta = 0, 2\pi$ and

$\tan\theta = 1$

so $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

e $3\tan^2\theta - 5\sec\theta + 1 = 0$

$3(\sec^2\theta - 1) - 5\sec\theta + 1 = 0$

$$3\sec^2\theta - 5\sec\theta - 2 = 0$$

$$(\sec\theta - 2)(3\sec\theta + 1) = 0$$

So we have

$$\sec\theta = \frac{1}{\cos\theta} = 2$$

$$\cos\theta = \frac{1}{2}$$

Then $= \frac{\pi}{3}, \frac{5\pi}{3}$. The second equation gives

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{3}$$

$$\cos\theta = -3$$

which is outside of the domain of cosine, so it gives no solutions.

f $2\cot\theta + 8 = 7\csc\theta$

$$2\frac{\cos\theta}{\sin\theta} + 8 = 7\frac{1}{\sin\theta}$$

We multiply both sides by $\sin\theta$ and get

$$2\cos\theta + 8\sin\theta = 7$$

4 a $\csc^2\theta - 1 = \cot^2\theta$

b $\dfrac{1 + 2\sin\theta\cos\theta}{\sin\theta + \cos\theta} = \dfrac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{\sin\theta + \cos\theta} = \dfrac{(\cos\theta + \sin\theta)^2}{\sin\theta + \cos\theta} = \cos\theta + \sin\theta$

c $\dfrac{2\tan\theta + \sec^2\theta}{(\sin\theta + \cos\theta)^2} = \dfrac{2\tan\theta + \tan^2\theta + 1}{(\sin\theta + \cos\theta)^2} = \left(\dfrac{\tan\theta + 1}{\sin\theta + \cos\theta}\right)^2 = \dfrac{\frac{\sin\theta}{\cos\theta} + 1}{\sin\theta + \cos\theta}$

$$= \frac{\dfrac{\sin\theta + \cos\theta}{\cos\theta}}{\dfrac{\sin\theta + \cos\theta}{1}} = \frac{1}{\cos\theta} = \sec\theta$$

Exercise 6L

1 $\sin(A + (-B)) = \sin A\cos(-B) + \sin(-B)\cos A = \sin A\cos B - \sin B\cos A$

2 a Using the construction on pg. 38

$$\cos(A + B) \equiv \frac{OT}{OR} \equiv \frac{OP - TP}{OR} \equiv \frac{OP - SQ}{OR} \equiv \frac{OP}{OR} - \frac{SQ}{OR}$$

$$\equiv \frac{OP}{OQ}\frac{OQ}{OR} - \frac{SQ}{RQ}\frac{RQ}{OR} \equiv \cos B\cos A - \sin B\sin A$$

b $\cos(A + (-B)) = \cos A\cos(-B) - \sin A\sin(-B) = \cos A\cos B + \sin A\sin B$

3 **a** $\tan(A+B)=\dfrac{\sin(A+B)}{\cos(A+B)}=\dfrac{\sin A\cos B+\sin B\cos A}{\cos A\cos B-\sin A\sin B}$

$$=\dfrac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{1-\dfrac{\sin A}{\cos A}\dfrac{\sin B}{\cos B}}=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$$

b $\tan(A+(-B))=\dfrac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

4 **a** $\sin 75°=\sin(45°+30°)=\sin 45°\cos 30°+\sin 30°\cos 45°$

$$=\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{2}\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$$

b $\tan 105°=\tan(60°+45°)=\dfrac{\tan 60°+\tan 45°}{1-\tan 60°\tan 45°}=\dfrac{\dfrac{1}{\sqrt{3}}+1}{1-\dfrac{1}{\sqrt{3}}}=\sqrt{3}+2$

$105°$ is obtuse, so we take the negative value

$\tan 105°=-\left(\sqrt{3}+2\right)$

c $\sin 33°\cos 3°-\cos 33°\sin 3°=\sin(33°-3°)=\sin 30°=\dfrac{1}{2}$

d $\cos 75°\cos 15°+\sin 75°\sin 15°=\cos(75°-15°)=\cos(60°)=\dfrac{1}{2}$

5 **a** Note that the missing side of the triangle with angle A is $c=\sqrt{5^2-3^2}=4$, so $\cos A=\dfrac{-4}{5}$ as A is obtuse, and the side of the triangle with angle B is $c=\sqrt{13^2-5^2}=12$, so $\cos B=\dfrac{12}{13}$ as B is acute. Then

$$\cos(A-B)=\cos A\cos B+\sin A\sin B=-\frac{4}{5}\times\frac{12}{13}+\frac{3}{5}\times\frac{5}{13}=-\frac{33}{65}$$

b Note that

$$\tan A=\frac{\sin A}{\cos A}=\frac{\frac{3}{5}}{-\frac{4}{5}}=-\frac{3}{4}$$

and $\tan B=\dfrac{\sin B}{\cos B}=\dfrac{\frac{5}{13}}{\frac{12}{13}}=\dfrac{5}{12}$

Then $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{-\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}}=-\dfrac{16}{63}$

6 **a** $\cot(A+B)=\dfrac{\cos(A+B)}{\sin(A+B)}=\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\sin B\cos A}$

Divide by $\sin A\sin B$ and get

$$=\dfrac{\dfrac{\cos A\cos B-\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B+\sin B\cos A}{\sin A\sin B}}=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$$

b $\dfrac{\sin(A+B)}{\cos A\cos B}=\dfrac{\sin A\cos B+\sin B\cos A}{\cos A\cos B}=\tan A+\tan B$

c $\sec A+\tan A=\dfrac{1}{\cos A}+\dfrac{\sin A}{\cos A}=\dfrac{1+\sin A}{\cos A}$

d $\tan A+\cot A=\dfrac{\sin A}{\cos A}+\dfrac{\cos A}{\sin A}=\dfrac{\sin^2 A+\cos^2 A}{\cos A\sin A}=\dfrac{1}{\cos A\sin A}=\sec A\csc A$

e $\sec^2\theta+\csc^2\theta=\dfrac{1}{\cos^2\theta}+\dfrac{1}{\sin^2\theta}=\dfrac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}=\dfrac{1}{\sin^2\theta\cos^2\theta}=\sec^2\theta\csc^2\theta$

f $\dfrac{\csc\theta-\cot\theta}{1-\cos\theta}=\dfrac{\dfrac{1}{\sin\theta}-\dfrac{\cos\theta}{\sin\theta}}{1-\cos\theta}=\dfrac{\dfrac{1-\cos\theta}{\sin\theta}}{1-\cos\theta}=\dfrac{1}{\sin\theta}=\csc\theta$

g $\csc x-\sin x=\dfrac{1}{\sin x}-\sin x=\dfrac{1-\sin^2 x}{\sin x}=\dfrac{\cos^2 x}{\sin x}=\cos x\cot x$

h $1+\cos^4 x-\sin^4 x=1+\cos^4 x-\left(\sin^2 x\right)^2=1+\cos^4 x-\left(1-\cos^2 x\right)^2$

$1+\cos^4 x-\left(1-2\cos^2 x+\cos^4 x\right)=2\cos^2 x$

i $\sec\theta+\tan\theta=\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{1+\sin\theta}{\cos\theta}$

We multiply numerator and denominator by $1-\sin\theta$ and get

$$=\dfrac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}=\dfrac{\cos^2\theta}{\cos\theta(1-\sin\theta)}=\dfrac{\cos\theta}{1-\sin\theta}$$

j $\dfrac{\sin A\tan A}{1-\cos A}=\dfrac{\sin A\dfrac{\sin A}{\cos A}}{1-\cos A}=\dfrac{\dfrac{\sin^2 A}{\cos A}}{1-\cos A}=\dfrac{\dfrac{1-\cos^2 A}{\cos A}}{1-\cos A}=\dfrac{\dfrac{(1-\cos A)(1+\cos A)}{\cos A}}{1-\cos A}=\dfrac{1+\cos A}{\cos A}=1+\sec A$

7 **a** $\tan\left(A+(B+C)\right)=\dfrac{\tan A+\tan(B+C)}{1-\tan A\tan(B+C)}=\dfrac{\tan A+\dfrac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\dfrac{\tan B+\tan C}{1-\tan B\tan C}}$

$$=\dfrac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}$$

b Substituting into our formula above

$$\tan(A+B+C)=\dfrac{\frac{1}{2}-\frac{1}{2}\times\frac{1}{5}\times\frac{1}{8}+\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}-\frac{1}{2}\times\frac{1}{5}-\frac{1}{2}\times\frac{1}{8}}=1$$

Hence $A+B+C=\frac{\pi}{4}$

c If A, B and C form the angles of a triangle, then $\tan\left(A+B+C\right)=0$, so

$$\frac{\tan A-\tan A\tan B\tan C+\tan B+\tan C}{1-\tan B\tan C-\tan A\tan B-\tan A\tan C}=0$$

or equivalently

$$\tan A-\tan A\tan B\tan C+\tan B+\tan C=0$$

or equivalently

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

Exercise 6M

1 a $\tan A+\cot A=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}=\frac{1}{\cos A\sin A}=\frac{2}{2\cos A\sin A}=\frac{2}{\sin 2A}=2\csc 2A$

b $\frac{\sin 2A+\cos 2A+1}{\sin 2A-\cos 2A+1}=\frac{2\sin A\cos A+2\cos^2 A-1+1}{2\sin A\cos A-1+2\sin^2 A+1}=\frac{2\cos A\ (\sin A+\cos A)}{2\sin A\left(\cos A+\sin A\right)}=\cot A$

c $\frac{\cos 3X-\sin 3X}{1-2\sin 2X}=\frac{\cos\left(2X+X\right)-\sin\left(2X+X\right)}{1-2\sin 2X}=$

$$=\frac{\cos 2X\cos X-\sin 2X\sin X-\sin 2X\cos X-\sin X\cos 2X}{1-4\sin X\cos X}$$

$$=\frac{\cos 2X(\cos X\sin X)-\sin 2X\left(\sin X+\cos X\right)}{1-4\sin X\cos X}$$

$$=\frac{(\cos^2 X-\sin^2 X)\left(\cos X-\sin X\right)-2\sin X\cos X(\sin X+\cos X)}{1-4\sin X\cos X}$$

$$=\frac{(\cos X+\sin X)(\cos X-\sin X)(\cos X-\sin X)-2\sin X\cos X(\sin X+\cos X)}{1-4\sin X\cos X}$$

$$=\frac{(\cos X-\sin X)^2\left(\sin X+\cos X\right)-2\sin X\cos X(\sin X+\cos X)}{1-4\sin X\cos X}$$

$$=\frac{(\sin X+\cos X)\ (\cos^2 X-2\sin X\cos X+\sin^2 X-2\sin X\cos X)}{1-4\sin X\cos X}$$

$$=\frac{\left(\sin X+\cos X)\ (\cos^2 X-4\sin X\cos X+\sin^2 X\right)}{1-4\sin X\cos X}$$

$$=\frac{\left(\sin X+\cos X)\ (1-4\sin X\cos X\right)}{1-4\sin X\cos X}=\sin X+\cos X$$

d $\cot x-\csc 2x=\frac{\cos x}{\sin x}-\frac{1}{\sin 2x}=\frac{\cos x}{\sin x}-\frac{1}{2\sin x\cos x}=\frac{2\cos^2 x-\cos^2 x-\sin^2 x}{2\sin x\cos x}$

$$=\frac{\cos^2 x-\sin^2 x}{2\sin x\cos x}=\frac{\cos 2x}{\sin 2x}=\cot 2x$$

2 $\frac{\sin 4A}{\sin A} = \frac{2\sin 2A\cos 2A}{\sin A} = \frac{4\sin A\cos A\cos 2A}{\sin A} = \frac{4\sin A\cos A\ (2\cos^2 A - 1)}{\sin A}$

$= 8\cos^3 A - 4\cos A$

3 $\sin 2A\sec 2A = \frac{\sin 2A}{\cos 2A} = \tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} = \frac{24}{7}$

4 $\cos 3X = \cos(2X + X) = \cos 2X\cos X - \sin 2X\sin X$

$\left(2\cos^2 X - 1\right)\cos X - 2\sin X\cos X\sin X = 2\cos^3 X - \cos X - 2\sin^2 X\cos X$

$= 2\cos^3 X - \cos X - 2\left(1-\cos^2 X\right)\cos X$

$= 4\cos^3 X - 3\cos X$

$\sin 3X = \sin(2X + X) = \sin 2X\cos X + \sin X\cos 2X$

$= 2\sin X\cos^2 X + \sin X\ (1 - 2\sin^2 X)$

$= 2\sin X\ (1-\sin^2 X) + \sin X - 2\sin^3 X$

$= -4\sin^3 X + 3\sin X$

5 $\cos 4A = 2\cos^2 2A - 1 = 2\left(2\cos A - 1\right)^2 - 1 = 8\cos^4 A - 8\cos^2 A + 1$

6 $\tan 4A = \frac{2\tan 2A}{1-\tan^2 2A} = \frac{2\left(\frac{2\tan A}{1-\tan^2 A}\right)}{1-\left(\frac{2\tan A}{1-\tan^2 A}\right)^2} = \frac{\frac{4\tan A}{1-\tan^2 A}}{\frac{(1-\tan^2 A)^2 - 4\tan^2 A}{(1-\tan^2 A)^2}}$

$\frac{4\tan A(1-\tan^2 A)}{(1-\tan^2 A)^2 - 4\tan^2 A}$

7 **a** We use the formula obtained in 4 to express $\sin 3x$

$-4\sin^3 x + 3\sin x = \sin^2 x$

or equivalently

$\sin x(4\sin^2 x + \sin x - 3) = 0$

Then $\sin x = 0$ gives $x = 0, 2\pi$ and

$(4\sin^2 x + \sin x - 3) = \left(\sin x + 1\right)\left(4\sin x - 3\right)$

then $\sin x = -1$ is given by $x = \frac{3\pi}{2}$ and $\sin x = \frac{3}{4}$ gives $x = 0.848, 2.29$

b $\cot 2x = 2 + \cot x$

Note that

$\cot 2x = \frac{1}{\tan 2x} = \frac{1-\tan^2 x}{2\tan x}$

Then the equation becomes

$$\frac{1-\tan^2 x}{2\tan x} = 2 + \frac{1}{\tan x}$$

which simplifies into

$$1-\tan^2 x - 4\tan x - 2 = 0$$

or equivalently

$$\tan^2 x + 4\tan x + 1 = 0$$

We use the quadratic formula to get that

$$\tan x = -2 \pm \sqrt{3}$$

so $x = \frac{23\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{12}$

c We use the formula obtained in 4 to express $\cos 3x$

$$4\cos^3 x - 3\cos x - 3\cos x = 2\cos^2 x - 1 + 1$$

which simplifies into

$$4\cos^3 x - 2\cos^2 x - 6\cos x = 0$$

or equivalently

$$2\cos x\ (2\cos^2 x - \cos x - 3) = 0$$

so $2\cos x = 0$ gives $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and

$$(2\cos^2 x - \cos x - 3) = (2\cos x - 3)(\cos x + 1) = 0$$

which gives for $\cos x = \frac{3}{2}$ no real results as it is outside of the domain of the cosine function.

Finally, $\cos x = -1$, gives $x = \pi$

Exercise 6N

1 Reasoning

$\csc\theta = 1$ when $\sin\theta = 1$, so at $\theta = \frac{\pi}{2} \pm 2n\pi, n \in \mathbb{Z}$

$\csc\theta = -1$ when $\sin\theta = -1$, so at $\theta = \frac{3\pi}{2} \pm 2n\pi, n \in \mathbb{Z}$

$\csc\theta$ is unidentified when $\sin\theta = 0$, so at $\theta = 0 \pm n\pi, n \in \mathbb{Z}$ we have vertical asymptotes

$\csc\theta \to \infty$ as $\sin\theta \to 0^-$

$\csc\theta \to -\infty$ as $\sin\theta \to 0^+$

2 Reasoning

$\cot\theta = 1$ when $\tan\theta = 1$, so at $\theta = \frac{\pi}{4} \pm n\pi, n \in \mathbb{Z}$

$\cot\theta = -1$ when $\tan\theta = -1$, so at $\theta = \frac{3\pi}{4} \pm n\pi, n \in \mathbb{Z}$

$\cot\theta$ is unidentified when $\tan\theta = 0$, so at $\theta = 0 \pm n\pi, n \in \mathbb{Z}$ we have vertical asymptotes

$\cot\theta \to \infty$ as $\tan\theta \to 0^{+}$

$\cot\theta \to -\infty$ as $\tan\theta \to 0^{-}$

3 **a** $f(x) = \sin x$

Then $g(x) = 3f(4x)$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 3 parallel to the y-axis

We stretch the function $y = 3\sin x$ by a scale factor $\frac{1}{4}$ parallel to the x-axis

The period of the new function is therefore $\frac{2\pi}{4} = \frac{\pi}{2}$

The amplitude of $g(x)$ is 3 and the period of $g(x)$ is $\frac{\pi}{2}$

b The graph will the same as for $\sin(x)$ with a vertical shift of $\frac{3\pi}{2}$

c Let $f(x) = \sin x$, and $g(x) = -2f(\pi x)$. We apply the following transformations to $f(x)$:

Then we reflect $f(x)$ with respect to the x axis, and apply a vertical stretch of scale factor 2 parallel to the y axis.

We stretch the function $y = -2\sin x$ by a scale factor $1/\pi$ parallel to the x-axis

The period of the new function is therefore $\frac{2\pi}{\pi} = 2$

The amplitude of $g(x)$ is 2 and the period is 2

d Let $f(x) = \sin x$, then $g(x) = 2f\left(4\left(x + \frac{\pi}{4}\right)\right) - 1$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 2 parallel to the y-axis.

We stretch the function by a scale factor $\frac{1}{4}$ parallel to the x-axis

We shift horizontally by $\frac{\pi}{4}$

We shift vertically downwards by 1

4 **a** Let $f(x) = \cos x$ and $g(x) = 2f(x) + 2$. We apply the following transformations to $f(x)$:

We apply a vertical stretch of scale factor 2 parallel to the y axis.

We shift the function vertically upwards by 2.

b Let $f(x) = \cos x$ and $g(x) = f(3x) - 1$. We apply the following transformations to $f(x)$:

We stretch the function by a factor of $\frac{1}{3}$ parallel to the x axis

We shift vertically downwards by 1

The new period is $\frac{2\pi}{3}$

c Let $f(x) = \cos x$ and $g(x) = -2f(3x)$. We apply the following transformations to $f(x)$:

We reflect along the x axis.

We stretch the function by a factor of 2 parallel to the y axis

We stretch the function by a factor of $\frac{1}{3}$ parallel to the x axis

The new period is $\frac{2\pi}{3}$

d Let $f(x) = \cos x$ and $g(x) = 3f(2x) + 3$. We apply the following transformations to $f(x)$:

We stretch the function by a factor of 3 parallel to the y axis

We stretch the function by a factor of $\frac{1}{2}$ parallel to the x axis

The new period is $\frac{2\pi}{2} = \pi$

We shift the function vertically upwards by 3

5 There is one solution in the interval $0 \le x \le \pi$ as there is only one intersection between $f(x)$ and $g(x)$ in that interval.

Exercise 6O

1 **a** The amplitude is 5

the vertical shift is $+1$

the horizontal/phase shift is $\frac{\pi}{12}$

the period is now $\frac{2\pi}{3}$

$5 = \frac{max - min}{2}$ and $1 = \frac{\max + min}{2}$, then

the maximum value is 6, as

the minimum value is -4

b The amplitude is 2

the vertical shift is -2

We rewrite the argument as $3\left(x+\frac{5\pi}{6}\right)$, so the horizontal/phase shift is $\frac{5\pi}{6}$

the period is now $\frac{2\pi}{3}$

$5=\frac{max-min}{2}$ and $1=\frac{\text{max}+min}{2}$, then

the maximum value is 6, as

the minimum value is -4

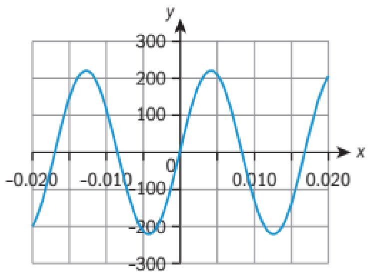
2 **a** The maximum value is 220, taken from the amplitude

b The minimum value is -220, as there is no vertical shift

c The amplitude of V is 220

d The period is given by $\frac{2\pi}{120\pi}=\frac{1}{110}$

e



3 **a** $a=\frac{\text{max}-\text{min}}{2}=\frac{14.4-1.2}{2}=6.6\text{m}$

vertical shift

$d=\frac{\text{max}+\text{min}}{2}=\frac{14.4+1.2}{2}=7.8\text{m}$

period of the function

$\frac{2\pi}{b}=12\Rightarrow b=\frac{2\pi}{12}$

Then our function looks like

$h(t)=6.6\sin\left(\frac{2\pi}{12}(t+c)\right)+7.8$

Note that $08:15$ is equivalent to $t=8.25$ hours. At $t=8.25$, $h=14.4$. Substituting into our equation for $h(t)$, gives us

$$6.6\sin\left(\frac{2\pi}{12}(8.25+c)\right)+7.8=14.4$$

$$\sin\left(\frac{2\pi}{12}(8.25+c)\right)=1$$

$$\left(\frac{2\pi}{12}(8.25+c)\right)=\frac{\pi}{2}$$

$$8.25+c=3$$

$$c=-5.25$$

Then

$$h(t)=6.6\sin\left(\frac{2\pi}{12}(t-5.25)\right)+7.8$$

b sketch of the graph, first minimum occurs at $t=2.25$

c The time intervals during which the boat could enter or leave the harbour on that particular day are calculated by plotting $y=5$ along with $h(t)$, and obtain the intervals for which $h(t)>y$ over a period of 24 hours. This gives $4.41<t<12.1$ and $16.1<t<24$.

4 **a** The minimum value is -3.5 and the maximum value is -2.5, this is a cosine function. We calculate the vertical shift and the amplitude as

$$d=\frac{-2.5-3.5}{2}=-3$$

$$a=\frac{-2.5+3.5}{2}=\frac{1}{2}$$

so

$$f(x)=\frac{1}{2}\cos x-3$$

b Horizontal shift, $\frac{\pi}{2}$, and we choose a cosine function. The amplitude is

$$a=\frac{7-3}{2}=2$$

and

$$d=\frac{7+3}{2}=5$$

is the vertical shift. Then the function is

$$f(x)=2\cos\left(x-\frac{\pi}{2}\right)+5$$

c We choose a cosine function. The amplitude is

$$a=\frac{2+4}{2}=3$$

and the vertical shift is

$$d = \frac{2-4}{2} = -1$$

The period is π, so $b = \frac{2\pi}{\pi} = 2$. Then the function is

$$f(x) = 3\cos(2x) - 2$$

d We choose a sine function, reflected along the x axis. The amplitude is

$$a = \frac{2+2}{2} = 2$$

and there is no vertical shift. The period is $\frac{2\pi}{3}$, so $b = \frac{2\pi}{\frac{2\pi}{3}} = 3$. Then the function is

$$f(x) = -2\sin 3x$$

e This is a tangent function, shifted horizontally by $\frac{\pi}{4}$, so $f(x) = \tan\left(x - \frac{\pi}{4}\right)$

f This is a secant plot, shifted upwards by 1, where the asymptotes are at $x = -\pi, \pi$, etc… This corresponds to

$$f(x) = \sec\frac{1}{2}x + 1$$

Exercise 6P

1 **a** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ **c** $\theta = \pi$ **d** $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$

e $\theta = 0, 2\pi$ **f** $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$ **g** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

2 $\cos^{-1} x = \frac{\pi}{2} - \frac{2\pi}{9} = \frac{5\pi}{18}$

and

$$\tan^{-1} x = \tan\left(\sin\frac{2\pi}{9}\right) = 0.748$$

3 **a** $\tan x < 0$ for $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$

b $\sec 2x < 0$ for $\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $\frac{5\pi}{4} < x < \frac{7\pi}{4}$

c $\sin 4x > 3$ never happens as it is outside of the domain of sine

4 **a** We solve

$$y = \sin 2x$$

for x, giving

$$x = \frac{\sin^{-1} y}{2}$$

Then

$$f^{-1}(x) = \frac{\sin^{-1} x}{2}$$

is defined for $x \in (0,1)$

In the case of $g(x)$,

$$g^{-1}(x) = 2x.$$

and it is also well defined for all x

b $g^{-1}\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1$

and

$$f^{-1}g\left(\frac{\pi}{6}\right) = \frac{\sin^{-1} g\left(\frac{\pi}{6}\right)}{2} = \frac{\sin^{-1}\frac{\pi}{3}}{2}$$

which has no real results

5 **a** $5 = \frac{2\pi}{period}$

so the period is $\frac{2\pi}{5}$

b e amplitude is 6

c The sine function is symmetric about the origin

d We stretch the function by a factor of 6 parallel to the y axis

We stretch the function by a factor of $\frac{1}{5}$ parallel to the x axis

The new period is $\frac{2\pi}{5}$

There are no vertical shifts

Exercise 6Q

1 **a** For all of these, we graphically show the plot for the left hand side and the right hand side and find the points of intersection

$$\csc^2\theta = 3\cot\theta - 1$$

$$\frac{1}{\sin^2\theta} = 3\frac{\cos\theta}{\sin\theta} - 1$$

$$1 = 3\cos\theta\sin\theta - \sin^2\theta$$

$\cos^2\theta - 3\cos\theta\sin\theta + 2\sin^2\theta = 0$

$(\cos\theta - 2\sin\theta)(\cos\theta - \sin\theta) = 0$

so we are searching for the solutions of

$\cos\theta = 2\sin\theta$

or equivalently

$2\tan\theta = 1$

$\tan\theta = \frac{1}{2}$

which has solutions $\theta = 0.463, 3.605$, and

$\cos\theta = \sin\theta$

which is true for $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

b $2\tan\theta = 3 + 5\cot\theta$

$2\tan\theta = 3 + \frac{5}{\tan\theta}$

$2\tan^2\theta = 3\tan\theta + 5$

$2\tan^2\theta - 3\tan\theta - 5 = 0$

$(2\tan\theta - 5)(\tan\theta + 1) = 0$

Then we get that

$\tan\theta = \frac{5}{2}$

which gives $\theta = 1.19, 4.33$ and

$\tan\theta = -1$

which gives $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

c $2\sec^2\theta - 3 + \tan\theta = 0$

$2\frac{1}{\cos^2\theta} - 3 + \frac{\sin\theta}{\cos\theta} = 0$

$2 - 3\cos^2\theta + \sin\theta\cos\theta = 0$

$2\sin^2\theta - \cos^2\theta + \sin\theta\cos\theta = 0$

We divide by $\cos^2\theta$ and get

$2\tan^2\theta + \tan\theta - 1 = 0$

or equivalently

$(\tan\theta + 1)(\tan\theta - \frac{1}{2}) = 0$

so $\tan\theta = -1$ gives $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ and $\tan\theta = 1/2$ gives $\theta = 0.464, 3.61$

d $5\csc\theta + \cot\theta = 2\tan\theta$

$$\frac{5}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = 2\frac{\sin\theta}{\cos\theta}$$

$$5\cos\theta + \cos^2\theta = 2\left(1 - \cos^2\theta\right)$$

$$3\cos^2\theta + 5\cos\theta - 2 = 0$$

$$(\cos\theta + 2)(3\cos\theta - 1) = 0$$

Then $\cos\theta = -2$ has no real solutions, and

$$\cos\theta = \frac{1}{3}$$

then $\theta = 1.231, 5.052$

Exercise 6R

1 a $\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos(x)}{h}$

$$= \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= \lim_{h\to 0}\left(\cos x\frac{(\cos h - 1)}{h} - \sin x\frac{\sin h}{h}\right)$$

$$= \cos x\lim_{h\to 0}\frac{(\cos h - 1)}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h}$$

$$= 0 - \sin x(1) = -\sin x$$

b $\frac{d}{dx}(\sin 2x) = \lim_{h\to 0}\frac{\sin\left(2x + \frac{2h}{2}\right) - \sin(2x)}{\frac{h}{2}}$

$$= \lim_{h\to 0}\frac{\sin 2x\cos h + \sin 2\cos 2x - \sin 2x}{\frac{h}{2}}$$

$$= \lim_{h\to 0}(\sin 2x\frac{\cos h - 1}{\frac{h}{2}} + \cos 2x\frac{\sin h}{\frac{h}{2}})$$

$$= \sin 2x\lim_{h\to 0}\frac{\cos h - 1}{\frac{h}{2}} + \cos 2x\lim_{h\to 0}\frac{\sin h}{\frac{h}{2}}$$

$$= \sin 2x\cdot 0 + 2\times\cos 2x = 2\cos 2x$$

c $\dfrac{d}{dx}(\sin\dfrac{x}{3}) = \lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{x}{3}+\dfrac{3h}{3}\right)-\sin\left(\dfrac{x}{3}\right)}{3h}$

$= \lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{x}{3}\right)\cos h + \sin h\cos\left(\dfrac{x}{3}\right)-\sin\left(\dfrac{x}{3}\right)}{3h}$

$= \lim\limits_{h\to 0}\left(\sin\left(\dfrac{x}{3}\right)\dfrac{\cos h - 1}{3h} + \cos\left(\dfrac{x}{3}\right)\dfrac{\sin h}{3h}\right)$

$= \sin\left(\dfrac{x}{3}\right)\lim\limits_{h\to 0}\dfrac{\cos h - 1}{3h} + \cos\left(\dfrac{x}{3}\right)\lim\limits_{h\to 0}\dfrac{\sin h}{3h}$

$= \sin\left(\dfrac{x}{3}\right)\cdot 0 + \dfrac{1}{3}\cos\left(\dfrac{x}{3}\right)\cdot 1 = \dfrac{1}{3}\cos\left(\dfrac{x}{3}\right)$

d $\dfrac{d}{dx}(\sin(2x+3)) = \lim\limits_{h\to 0}\dfrac{\sin\left(2x+3+2\left(\dfrac{h}{2}-\dfrac{3}{2}\right)+3\right)-\sin(2x+3)}{\dfrac{h}{2}-\dfrac{3}{2}}$

$= \lim\limits_{h\to 0}\dfrac{\sin(2x+3)\cos h + \sin h\cos(2x+3) - \sin(2x+3)}{\dfrac{h-3}{2}}$

$= \sin(2x+3)\lim\limits_{h\to 0}\dfrac{\cos h - 1}{\dfrac{h-3}{2}} + \cos(2x+3)\lim\limits_{h\to 0}\dfrac{\sin h}{\dfrac{h-3}{2}}$

$= 0\cdot\sin(2x+3) + 2\cos(2x+3)\cdot 1 = 2\cos(2x+3)$

2 a $\dfrac{d}{dx}(\tan x) = \lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan x}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{\tan x + \tan h}{1-\tan x\tan h}-\tan x}{h}$

$= \lim\limits_{h\to 0}\dfrac{\tan h\ (1+\tan^2 x)}{h} = \sec^2 x\lim\limits_{h\to 0}\dfrac{\tan h}{h} = \sec^2 x\cdot 1 = \sec^2 x$

b $\dfrac{d}{dx}(\cot x) = \lim\limits_{h\to 0}\dfrac{\cot(x+h)-\cot(x)}{h} = \lim\limits_{h\to 0}\dfrac{\dfrac{\cos(x+h)}{\sin(x+h)}-\dfrac{\cos x}{\sin x}}{h}$

$= \lim\limits_{h\to 0}\dfrac{\dfrac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}}{h}$

$= \lim\limits_{h\to 0}\dfrac{\dfrac{\sin(x-x-h)}{\sin x\sin(x+h)}}{h} = -1\lim\limits_{h\to 0}\dfrac{1}{\sin x\sin(x+h)} = \dfrac{-1}{\sin x}\lim\limits_{h\to 0}\dfrac{1}{\sin(x+h)} = \dfrac{-1}{\sin^2 x} = -\csc^2 x$

c $\dfrac{d}{dx}(\tan 3x) = \lim\limits_{h\to 0}\dfrac{\tan\left(3x+\dfrac{h}{3}\right)-\tan 3x}{\dfrac{h}{3}}$

$$= \lim_{h\to 0}\frac{\dfrac{\tan 3x + \tan\frac{h}{3}}{1-\tan 3x\tan\frac{h}{3}} - \tan 3x}{\frac{h}{3}} = \lim_{h\to 0}\frac{\tan\frac{h}{3}\,(1+\tan^2 33x)}{\frac{h}{3}}$$

$$= \sec^2 3x \lim_{h\to 0}\frac{\tan\frac{h}{3}}{\frac{h}{3}} = \sec^2 3x\cdot 3 = 3\sec^2 3x$$

Exercise 6S

1 a $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\frac{d}{dx}(\cos x)\cdot\sin x - \frac{d}{dx}(\sin x)\cdot\cos x}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$

b $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -1\times(\sin x)^{-2}\times\cos x = \frac{-\cos x}{\sin^2 x} = -\cot x\csc x$

2 a $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x) = \cos 2x\times 2 = 2\cos 2x$

b $\frac{dy}{dx} = \frac{d}{dx}(\cos(2x+1)) = -\sin(2x+1)\times 2 = -2\sin(2x+1)$

c $\frac{dy}{dx} = \frac{d}{dx}(\cos(8-3x)) = -\sin(8-3x)\times -3 = 3\cos(8-3x)$

d $\frac{dy}{dx} = \frac{d}{dx}(\cot\left(\frac{7-2x}{13}\right)) = \frac{d}{dx}\left(\frac{\cos\left(\frac{7-2x}{13}\right)}{\sin\left(\frac{7-2x}{13}\right)}\right)$

$$= \frac{-\sin\left(\frac{7-2x}{13}\right)\times\frac{-2}{13}\times\sin\left(\frac{7-2x}{13}\right) - \cos\left(\frac{7-2x}{13}\right)\times\frac{-2}{13}\times\cos\left(\frac{7-2x}{13}\right)}{\sin^2\left(\frac{7-2x}{13}\right)}$$

$$\frac{\frac{2}{13}(\sin^2\left(\frac{7-2x}{13}\right) + \cos^2\left(\frac{7-2x}{13}\right))}{\sin^2\left(\frac{7-2x}{13}\right)} = \frac{2}{13}\csc^2\left(\frac{7-2x}{13}\right)$$

3 a $\frac{df(x)}{dx} = -\sin\left(x^5-3\right)\times 5x^4 = -5x^4\sin\left(x^5-3\right)$

b $\frac{df(x)}{dx} = \frac{d(\sin(x^2+1))^{-1}}{dx} = -1\times\sin^{-2}\left(x^2+1\right)\times\cos\left(x^2+1\right)\times 2x$

$$= \frac{-2x\cos\left(x^2+1\right)}{\sin^2\left(x^2+1\right)} = -2x\cot\left(x^2+1\right)\csc\left(x^2+1\right)$$

c $\frac{df(x)}{dx} = \frac{d}{dx}(\cos\left(4x^3-2x^2+7x+17\right))^{-1}) =$

$$-1\times(\cos(4x^3-2x^2+7x+17))^{-2}\times-\sin(4x^3-2x^2+7x+17)\times(12x^2-4x+7)$$

$$=\frac{(12x^2-4x+7)\sin(4x^3-2x^2+7x+17)}{\cos^2(4x^3-2x^2+7x+17)}$$

$$=(12x^2-4x+7)\tan(4x^3-2x^2+7x+17)\sec(4x^3-2x^2+7x+17)$$

d $$\frac{df(x)}{dx}=\frac{d}{dx}\left(\frac{1}{\cos\left(\sqrt{e^x+1}\right)}\right)=$$

$$-1\times(\cos\left(\sqrt{e^x+1}\right))^{-2}\times-\sin\left(\sqrt{e^x+1}\right)\times\left(\frac{1}{2}\right)(e^x+1)^{-1/2}\times e^x$$

$$=\left(\frac{1}{2}\right)\frac{e^x\sin\left(\sqrt{e^x+1}\right)}{\sqrt{e^x+1}\cos^2\left(\sqrt{e^x+1}\right)}=\frac{1}{2\sqrt{e^x+1}}e^x\tan\sqrt{e^x+1}\sec\sqrt{e^x+1}$$

e $$\frac{df(x)}{dx}=\cos(\cos(\tan x))\times-\sin(\tan x)\times\sec^2 x=-\sec^2 x\sin(\tan x)\cos(\cos(\tan x))$$

Exercise 6T

1 a $y'=\cos x(2x+1)+2\sin x$

b $y'=-2(x+x^2)\sin 2x+(1+2x)\cos 2x$

c $$y'=\frac{-\sin x\times x-\cos x}{x^2}=\frac{-\cos x-x\sin x}{x^2}$$

d $$y'=\frac{2\times\sin 2x-2(2x+3)\cos 2x}{\sin^2 2x}=\frac{2\sin 2x-(4x+6)\cos 2x}{\sin^2 2x}$$

e $$y'=\frac{\sec^2 x\ (\sqrt{2-x)}-\tan x\times\frac{1}{2}\times(2-x)^{-\frac{1}{2}}\times-1}{2-x}$$

$$=\frac{\sec^2 x}{\sqrt{2-x}}+\frac{1}{2}\times\frac{\tan x}{(2-x)^{3/2}}$$

2 a $y'=3\cos 3x$

Then evaluate at the point,

$$y'\left(\frac{\pi}{3}\right)=3\cos\left(\frac{3\pi}{3}\right)=-3$$

b $y'=-2\sin 2x$

Then evaluate at the point,

$$y'\left(\frac{5\pi}{4}\right) = -2\sin(2\frac{5\pi}{4}) = -2$$

c $y' = \sin x + (x-2)\cos x$

Then evaluate at the point

$y'(0) = 0 - 2(1) = -2$

d $y' = -3\cos x + 3x\sin x$

Then evaluate at the point

$$y'\left(\frac{\pi}{2}\right) = -2\cos\frac{\pi}{2} + \frac{3\pi}{2}\sin\frac{\pi}{2} = 0 + \frac{3\pi}{2} = \frac{3\pi}{2}$$

e $y' = 3x^2\tan x + x^3\sec^2 x$

Then evaluate at the point

$$y'\left(\frac{3\pi}{4}\right) = 3\left(\frac{3\pi}{4}\right)^2\tan\left(\frac{3\pi}{4}\right) + \left(\frac{3\pi}{4}\right)^3\sec^2\left(\frac{3\pi}{4}\right) = \frac{27\pi^3}{32} - \frac{27\pi^2}{16}$$

3 **a** $\sin^2\alpha + \cos^2\alpha = 1,$ so the gradient is 0.

b $\dfrac{\tan\beta}{\sin\beta} = \sec\beta,$ so the gradient is $\tan\beta\sec\beta$

c The gradient is $\dfrac{\sin^2 x + 2\sin 2x + 3\sin x\sin 2x + \cos^2 x + 2\cos^2 2x + 3\cos 2x\cos x}{(\cos x + \cos 2x)^2}$

$$= \frac{3(\cos x + 1)}{(\cos x + \cos 2x)^2}$$

Exercise 6U

1 **a** Let $f(x) = \cos x$, then $f^{-1}(x) = \cos^{-1} x = y$, then $f(y) = \cos y$ and so

$$\frac{dy}{dx} = \frac{1}{\frac{df}{dy}} = \frac{1}{-\sin y} = \frac{-1}{\sin(\cos^{-1} x)} = \frac{-1}{\sqrt{1-\cos^2((\cos^{-1} x))}} = \frac{-1}{\sqrt{1-x^2}}$$

Then, we use the chain rule

$$\frac{d}{dx}(\arccos 2x) = \frac{-1}{\sqrt{1-4x^2}} \times 2 = \frac{-2}{\sqrt{1-4x^2}}$$

b We use the form obtained in Investigation 8, and find

$$\frac{d}{dx}(\arcsin\frac{3}{2}x) = \frac{1}{\sqrt{1-x^2}} \times \frac{3}{2} = \frac{3}{2\sqrt{1-x^2}}$$

c Let $f(x) = \tan x$, then $f^{-1}(x) = \tan^{-1} x = y$, then $f(y) = \tan y$ and so

$$\frac{dy}{dx} = \frac{1}{\frac{df}{dy}} = \frac{1}{\sec^2 y} = \cos^2 y = \cos^2(\tan^{-1} x) = \frac{1}{1+x^2}$$

Then

$$\frac{d}{dx}(\tan^{-1}(2x+1)) = \frac{1}{1+(2x+1)^2}\times 2 = \frac{2}{4x^2+4x+2} = \frac{1}{2x^2+2x+1}$$

2 a $\frac{dy}{dx} = \frac{d}{dx}(2x\arccos x) = 2\arccos x + 2x\left(\frac{-1}{\sqrt{1-x^2}}\right) = 2\arccos x - \frac{2x}{\sqrt{1-x^2}}$

b $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\arccos x}{2x}\right) = \frac{\frac{-1}{\sqrt{1-x^2}}\times 2x - 2\arccos x}{4x^2} = -\frac{\frac{2x}{\sqrt{1-x^2}} + 2\arccos x}{4x^2}$

$\frac{-1}{2x\sqrt{1-x^2}} - \frac{\arccos x}{2x^2}$

c $\frac{dy}{dx} = 2x\arctan 3x + 3\times\frac{x^2-1}{1+9x^2}$

3 a $\frac{d}{dx}(\arcsin x + \arccos x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

This is valid because we are calculating two angles that add up to π in a right-angled triangle.

b $\frac{d}{dx}(\arctan x + \arctan(-x)) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$

Here both inverse tangents correspond to the same angle, in different quadrants (due to the negative sign). So the rate of change between both is zero.

c $\frac{d}{dx}\left(2\arctan x - \arcsin\left(\frac{2x}{x^2+1}\right)\right) = \frac{2}{x^2+1} - \frac{1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}}\times\frac{2(x^2+1)-2x(2x)}{(x^2+1)}$

$$= \frac{2}{x^2+1} + \frac{1}{\sqrt{\frac{(x^2-1)^2}{(x^2+1)^2}}}\times\left(\frac{-2(x^2-1)}{x^2+1}\right) = \frac{2}{x^2+1} - \frac{2}{x^2+1} = 0$$

Trigonometric functions give angles as outputs, so we can call

$\theta_1 = \arctan x$

and

$\theta_2 = \arcsin\left(\frac{2x}{x^2+1}\right)$

Then we want to show that

$2\theta_1 = \theta_2$

so we check

$$\sin(2\theta_1) = 2\sin\theta_1\cos\theta_1 = 2\times\frac{x}{\sqrt{x^2+1}}\times\frac{1}{\sqrt{x^2+1}} = \frac{2x}{x^2+1} = \sin\theta_2$$

Exercise 6V

1 **a** $f(x) = \sin(3x + \pi)$

The tangent is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + 0$$

We calculate

$$f'(x) = 3\cos(3x + \pi)$$

Then $f'\left(\frac{\pi}{3}\right) = 3\cos(\pi + \pi) = 1$

So the tangent equation is

$$y = x - \frac{\pi}{3}$$

The normal is

$$y = \frac{1}{f'\left(\frac{\pi}{3}\right)}\left(x - \frac{\pi}{3}\right) + 0$$

$$y = -\left(x - \frac{\pi}{3}\right)$$

b $f(x) = \arccos 2x$

We calculate the derivative $f'(x) = -\frac{2x}{\sqrt{1 - 4x^2}}$

Then $f'(0.05) = -0.101$

and $f(0.05) = 1.147$

The tangent equation is $y = -0.101(x - 0.05) + 1.147$

and the normal is $y = -\frac{1}{-0.101}(x - 0.05) + 1.147$

or equivalently $y = 9.9(x - 0.05) + 1.147$

c $f(x) = x\sin 2x$

We calculate $f'(x) = \sin 2x + 2x\cos 2x$

and so $f'(-0.5) = -1.382$

and $f(0.5) = 0.421$

Then the tangent equation is $y = -1.382(x + 0.5) + 0.421$

and the normal is $y = \frac{-1}{1.382}(x+0.5)+0.421$

2 **a** $x\cos 2x = \tan(3x+\pi)-1$

Point of intersection (using GDC) is $(0.298, 0.247)$.

b Normal to $y = x\cos 2x$

$f'(x) = -2x\sin x + \cos 2x$

so $f'(0.298) = 0.653$

Then the equation of the normal is $y = \frac{-1}{0.653}(x-0.298)+0.247$

$y = -1.531(x-0.298)+0.247$

Normal to $y = \tan(3x+\pi)-1$

$f'(x) = 3\sec^2(3x+\pi)$

so $f'(0.298) = 7.65$

Then the equation of the normal is $y = \frac{-1}{7.65}(x-0.298)+0.247$

$y = -0.131(x-0.298)+0.247$

c We calculate the y -intercepts of each of the normal functions, which we call y_1 and y_2 respectively.

$y_1(0) = -1.531(0-0.298)+0.247 = 0.703$

and $y_2(0) = -0.131(0-0.298)+0.247 = 0.286$

The intersection between both function is given by $x = 0.298$, as it is the point that they both share.

Then the two lines form a triangle, with sides

$c = 0.703-0.286 = 0.417$

$a = \sqrt{(0.298-0.703)^2+(0.247-0)^2} = 0.474$

$b = \sqrt{(0.298-0.286)^2+(0.247-0)^2} = 0.247$

Then $\cos C = \frac{0.474^2+0.247^2-0.417^2}{2(0.474)(0.247)} = 0.477$

$C = \cos^{-1} 0.477 = 1.073 = 61.5°$

3 **a** $y' = \frac{(\cos x - x\sin x)(x+\cos x) - x\cos x(1-\sin x)}{(x+\cos x)^2} = \frac{\cos^2 x - x^2\sin x}{(x+\cos x)^2}$

for $x \geq 0$

b Tangent at point $\left(\frac{\pi}{2},0\right)$. We calculate

$$y'\left(\frac{\pi}{2}\right)=\frac{\cos^2\frac{\pi}{2}-\frac{\pi}{2}^2\sin\frac{\pi}{2}}{(\frac{\pi}{2}+\cos\frac{\pi}{2})^2}=-1$$

Then the tangent is given by

$$y=-1\left(x-\frac{\pi}{2}\right)+0=-\left(x-\frac{\pi}{2}\right)$$

and the normal is given by

$$y=-\frac{1}{-1}\left(x-\frac{\pi}{2}\right)+0=x-\frac{\pi}{2}$$

4 a $y'=(8x)\arctan 2x+\frac{4x^2+1}{1+x^2}2$

b $y'(0.5)=8(0.5)\arctan(2\times 0.5)+2=\pi+2$

and $y(0.5)=\left(4(0.5)^2+1\right)\arctan 0.5=0.927$

and so the equation of the tangent at 0.5 is

$$y=(\pi+2)(x-0.5)+0.927=5.14(x-0.5)+0.927$$

Exercise 6W

1 a $\frac{d\theta}{dt}=6\times\frac{1}{60}=0.1\,\text{rot}\,\text{s}^{-1}$

and $\tan\theta=\frac{y}{100}\Rightarrow y=100\tan\theta$

Then $\frac{dy}{dt}=100\sec^2\theta\frac{d\theta}{dt}=10\sec^2\theta$

We are measuring θ with respect to the shoreline, so when they are at right angles with respect to each other is exactly when $\theta=0$, so

$$\left.\frac{dy}{dt}\right|_{\theta=0}=10\,\text{m}\,\text{s}^{-1}$$

b $\tan\theta=\frac{50}{100}\Rightarrow\theta=\tan^{-1}\frac{1}{2}=0.464$

Then $\left.\frac{dy}{dt}\right|_{\theta=0.464}=12.5\,\text{m}\,\text{s}^{-1}$

c When the ray approaches being parallel to the light beam, the velocity of the light beam is increasing and is undetermined at the point where it is exactly parallel.

2 a $\frac{dy}{dt}=90\frac{km}{h}=90\frac{1}{60\text{min}}\times\frac{1\text{min}}{60\text{s}}\times\frac{1000\text{m}}{1\text{km}}=25\,\text{m}\,\text{s}^{-1}$

We measure the angle with respect to the horizontal of the camera 'line of sight'. Then we can write

$$\tan\theta = \frac{y}{30} \Rightarrow 30\tan\theta = y$$

Then $\frac{dy}{dt} = 30\sec^2\theta\frac{d\theta}{dt} \Rightarrow \frac{25}{30}\cos^2\theta = \frac{d\theta}{dt}$

At $\theta = 0$, the bird is directly in front of the camera, so we calculate

$$\frac{25}{30}\times 1 = 0.833 \text{ rot s}^{-1}$$

b We integrate $\frac{d\theta}{dt}$ function with respect to t to find

$$t^2 = \frac{30}{25}\tan\theta$$

then at $t = 0$ is when the bird is directly in front of the camera. So one second later is at $t = 1$, gives $\theta = 0.695$, so we evaluate $\frac{d\theta}{dt}$ at this value of θ and we get $0.491\,\text{rot}\,\text{s}^{-1}$

3 a We use the cosine rule to write a relationship between the decreasing angle θ and the decreasing side $6 - y$ so we are modelling the decrease y as

$$\cos\theta = \frac{5^2 + 5^2 - (6-y)^2}{2\times 5\times 5} = \frac{50-(6-y)^2}{50}$$

Then $(6-y)^2 = 50 - 50\cos\theta$

We differentiate both sides with respect to t and get

$$-2(6-y)\frac{dy}{dt} = 50\sin\theta\frac{d\theta}{dt}$$

or equivalently, substituting with $\frac{dy}{dt} = 0.1\,\text{cm}\,\text{s}^{-1}$ we get

$$(0.004y - 0.028)\csc\theta = \frac{d\theta}{dt}$$

b In this case, we change the 6 for a 5 in the expression above, and write the rate of change as

$$\cos\theta = \frac{5^2 + 5^2 - (5-y)^2}{2\times 5\times 5} = \frac{50-(5-y)^2}{50} = \frac{25 + 10y - y^2}{50}$$

$$50\cos\theta = 25 + 10y - y^2$$

Then differentiating we get

$$-50\sin\theta\frac{d\theta}{dt} = 10\frac{dy}{dt} - 2y\frac{dy}{dt} = 10(0.1) - 2y(0.1) = 1 - 0.2y$$

and so

$$\frac{d\theta}{dt} = \frac{0.2y - 1}{50\sin\theta} = \frac{0.2y-1}{50}\csc\theta$$

4 **a** We have the relationship for the volume of the sphere

$$V = \frac{4}{3}\pi r^3$$

We differentiate with respect to t and get

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

We use that $\frac{dV}{dt} = 3\,\text{cm}^3\,\text{min}^{-1}$ and rearrange to get

$$\frac{3}{4\pi r^2} = \frac{dr}{dt}$$

so evaluating at $r = 10$ gives

$$\frac{3}{4\pi(10)^2} = 0.002387\,\text{cm}\,\text{min}^{-1}$$

b We have the relationship for the surface area of the sphere

$$SA = 4\pi r^2$$

and the relationship between the surface area and the volume as

$$\frac{A}{3}r = V.$$

so differentiating with respect to t gives the relationship

$$\frac{dA}{dt} \times \frac{1}{3}r + \frac{A}{3}\frac{dr}{dt} = \frac{dV}{dt}$$

For $r = 4.5,$ we use the relationship in a to get that

$$\frac{3}{4\pi r^2} = \frac{3}{4\pi(4.5)^2} = \frac{dr}{dt} = 0.0118$$

which we substitute into the rate of change for A, then

$$\frac{dA}{dt} \times \frac{1}{3} \times 4.5 + \frac{4\pi(4.5)^2}{3} 0.0118 = 3$$

Hence

$$\frac{dA}{dt} = 1.333\,\text{cm}^2\,\text{s}^{-1}$$

5 **a** take y to be the horizontal distance, and calculate the expression

$$\tan\theta = \frac{y}{10000}$$

Then

$$y = 10000\tan\theta$$

We convert $\frac{dy}{dt}$ into m s^{-1} to keep units consistent, so

$$\frac{dy}{dy} = 1025\frac{\text{km}}{\text{h}} = 284.72\ \text{ms}^{-1}$$

We differentiate the expression for y to get

$$\frac{dy}{dt} = 10000\sec^2\theta\frac{d\theta}{dt} = \frac{dy}{dt}$$

The angle we're looking at corresponds to

$$\theta = \tan^{-1}\frac{8000}{10000} = 38.7°$$

Then we substitute into expression for the derivatives to get

$$\frac{d\theta}{dt} = \frac{284.72}{10000}\cos^2 38.7° = 0.017\,\text{deg}\,\text{s}^{-1}$$

b When the plane is directly above the radar, $\theta = 0$, so

$$\frac{d\theta}{dt} = \frac{284.72}{10000}\times 1 = 0.028\,\text{deg}\,\text{s}^{-1}$$

6 **a** We can write the distance between the camera z and the train as

$$z = \sqrt{y^2+2^2} = \sqrt{y^2+4}$$

Then we use the chain rule to get

$$\frac{dz}{dt} = \frac{1}{2}\left(y^2+4\right)^{-\frac{1}{2}}(2y)\frac{dy}{dt} = \frac{y}{\sqrt{y^2+4}}\frac{dy}{dt}$$

We evaluate at $z = 4$ so $4 = \sqrt{y^2+4}$, so $y = 3.46$. Substituting into the above equation we get that

$$\frac{dz}{dt} = \frac{3.46}{4}\times 75 = 64.9\frac{\text{km}}{\text{h}}$$

b So we are looking for $\frac{d\theta}{dt}$ when $y = 3.46$, where $\tan\theta = \frac{y}{2}$

so $\theta = \tan^{-1}\frac{3.46}{2} = 60°$. We differentiate with respect to t and get

$$\sec^2\theta\frac{d\theta}{dt} = \frac{dy}{dt}\times\frac{1}{2}$$

which then substituting and converting 75 km/h to 20.8 m/s gives

$$\sec^2 60°\frac{d\theta}{dt} = 20.8\times\frac{1}{2}$$

which gives

$$\frac{d\theta}{dt} = 2.6 \text{ degrees per second.}$$

7 We have that

$$\cos\theta = \frac{y}{3}$$

where y is the horizontal distance in metres. Then

$y = 3\cos\theta$

We convert to meters per second, $\frac{dy}{dt} = \frac{6}{100} = 0.06\text{m/s}$. We differentiate with respect to t and get

$-3\sin\theta\frac{d\theta}{dt} = \frac{dy}{dt}$

The angle corresponding to the horizontal distance of 1 m is

$\sin\theta = \frac{1}{3}$

so $\theta = \sin^{-1}\frac{1}{3} = 0.34$. We substitute all together is

$-3\sin 0.34\frac{d\theta}{dt} = 0.06$

then $\frac{d\theta}{dt} = -0.06$

Chapter review

1 $V_{cube} = 4^3 = 64\text{cm}^3$

we equate to the volume of the sphere

$64 = \frac{4}{3}\pi r^3$

so

$r^3 = \frac{48}{\pi}$

then

$r = 2.48\text{cm}$

2 **a** $V_{tot} = V_{cone} + V_h$

$V_{cone} = \frac{1}{3}\pi r^2 h_{cone} = \frac{1}{3}\pi(6)^2(14-6) = 301.6\text{cm}^3$

$V_h = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3\right) = 452.4\text{cm}^3$

$V_{tot} = 301.6 + 452.4 = 754\text{cm}^3$

b For the surface area we need to be careful to not count the circular base.

The cone slant height is

$s = \sqrt{8^2 + 6^2} = 10$

$SA_{cone} = \pi rs = \pi(6)(10) = 188.5\text{cm}^2$

$SA_h = \frac{1}{2}(4\pi r^2) = \frac{1}{2}\left(4\pi(6)^2\right) = 226.2\text{cm}^2$

Then

$SA_{tot} = 188.5 + 226.2 = 414.7\text{cm}^2$

3 **a** $\cos CAB = \dfrac{3^2 + 5^2 - 3^2}{2 \times 3 \times 5} = 0.833$

then $\cos^{-1} 0.833 = 0.586 = 33.6°$.

b **i** Note that $AY = XB$ as this is an isosceles triangle. Then

$AB - BY = AY$

$5 - 3 = 2$

Then

$AB = AY + XB + XY$

$5 = 2 + 2 + XY$

Then $XY = 1$.

So the length of the perimeter is

$CY + CX + XY = 4.516$

ii $A_{ACX} = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2(0.833) = 3.749\text{cm}^2$

iii Height of triangle ABC is

$h = \sqrt{9 - 2.5^2} = 1.66$

$A_{ABC} = \dfrac{5 \times 1.66}{2} = 4.15\text{cm}^2$

Then

$A_{ABC} - A_{ACX} = 4.15 - 3.749 = 0.401 = A_{CXB} = A_{ACY}$

$A_R = A_{ABC} - 2A_{CXB} = 3.348\text{cm}^2$

4 **a** $6\sin^2 x = 5 + \cos x$

$6(1 - \cos^2 x) = 5 + \cos x$

$6\cos^2 x + \cos x - 1 = 0$

$(3\cos x - 1)(2\cos x + 1) = 0$

and so $\cos x = \frac{1}{3}$ and $\cos x = -\frac{1}{2}$

b $x = \cos^{-1}\frac{1}{3} = 360° - 70.529 = 289.4°, 430.5°$, and $x = \cos^{-1} -\frac{1}{2} = 180° + 60° = 240°, 480°$

5 $4\tan^2 x + 12\sec x + 1 = 0$

$$4\frac{\sin^2 x}{\cos^2 x} + 12\frac{1}{\cos x} + 1 = 0$$

$$4\sin^2 x + 12\cos x + \cos^2 x + 1 = 0$$

$$4\left(1 - \cos^2 x\right) + 12\cos x + \cos^2 x + 1 = 0$$

$$3\cos^2 x - 12\cos x - 5 = 0$$

Then

$\cos x = 4.38$

which is undefined, and

$\cos x = -0.38$

$\cos^{-1} - 0.38 = 112.3°$

and $-112.3°$

6 $\tan 3A = \tan(2A + A) = \dfrac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

$$= \frac{\tan A + \dfrac{2\tan A}{1 - \tan^2 A}}{1 - \tan A \dfrac{2\tan A}{1 - \tan^2 A}} = \frac{\tan A\ (3 - \tan^2 A)}{1 - 3\tan^2 A}$$

7 We use a change of variables. Let $\phi = 3\theta$. Then we can rewrite the equation as

$$\cos\left(\frac{4}{3}\phi\right) = \cos\phi$$

or equivalently

$$\cos\left(2\pi - \frac{4}{3}\phi\right) = \cos\phi$$

$$n\pi + 2\pi = \left(\frac{3}{3} + \frac{4}{3}\right)\phi$$

Then

$$\theta = \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, 0, 2\pi$$

If we substitute into the equation $\cos 4\theta = \cos 3\theta$, and apply the identities for $\cos(2\theta + 2\theta)$ and $\cos(2\theta + \theta)$ we get the equation

$$8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1 = 0$$

so the roots to the equation are precisely $\cos\dfrac{2\pi}{7}, \cos\dfrac{4\pi}{7}, \cos\dfrac{6\pi}{7}$ where $x = \cos\theta$

Substitute the inverse roots to show the required equality.

8 **a** $\sin(x + 60°) = \cos x$

$\sin x\cos 60° + \sin 60°\cos x = \cos x$

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \cos x$

$\frac{1}{2}\tan x + \frac{\sqrt{3}}{2} = 1$

$\tan x = \left(1 - \frac{\sqrt{3}}{2}\right) \times 2$

$x = 15°, 195°$

b $\tan(A - x) = \frac{\tan A - \tan x}{1 - \tan A\tan x} = \frac{2}{3}$

$\frac{3 - \tan x}{1 - 3\tan x} = \frac{2}{3}$

$3(3 - \tan x) = 2(1 - 3\tan x)$

$3\tan x = -7$

$x = \tan^{-1} - \frac{7}{3} = 293°, 113.2°$

9 $\sin 3A = \sin(A + 2A) = \sin A\cos 2A + \sin 2A\cos A$

$= \sin A(1 - 2\sin^2 A) + 2\sin A\ (1 - \sin^2 A)$

$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A = -4\sin^3 A + 3\sin A$

10 We can write the equation for the angles, relabelling $\arctan x = \theta_1$ and $\arctan\frac{x}{2} = \theta_2$

$\theta_1 + \theta_2 = \frac{5\pi}{6}$

Then we calculate the sine on both sides (could be cosine or tangent, any function will do) and get

$\sin(\theta_1 + \theta_2) = \sin(\frac{5\pi}{6})$

$\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 = 1$

We complete the triangle and get that

$\frac{x}{1} \times \frac{x}{2} - \sqrt{1 - \frac{x^2}{4}}\sqrt{1 - x^2} = 1$

or equivalently

$\left(1 - \frac{x^2}{4}\right)\left(1 - x^2\right) = \left(\frac{x^2}{2} - 1\right)^2$

which simplifies into

$x^4 - x^2 = 0$

which has solutions

$$x^2\left(x^2-1\right)=0$$

and so

$$x=0, x=\pm 1$$

11 $$\tan\frac{A}{2}\tan\frac{B-C}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\times\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$

Note that

$$A+B+C=\pi$$

as ABC are the angles of a triangle. Then

$$\frac{A}{2}=\frac{\pi-(B+C)}{2}$$

so

$$\sin\frac{A}{2}=\sin\left(\frac{\pi}{2}-\left(\frac{B+C}{2}\right)\right)=\cos\left(\frac{B+C}{2}\right)$$

and

$$\cos\frac{A}{2}=\cos\left(\frac{\pi}{2}-\left(\frac{B+C}{2}\right)\right)=\sin\left(\frac{B+C}{2}\right)$$

Then we substitute back into the first equation to get

$$\frac{(\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2})(\sin\frac{B}{2}\cos\frac{C}{2}-\sin\frac{C}{2}\cos\frac{B}{2})}{\left(\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{B}{2}\right)\left(\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right)}$$

Note that

$$\sin\frac{B}{2}\cos\frac{B}{2}=\frac{1}{2}\sin B$$

and equivalently for C. We multiply the brackets and substitute with the form for $\sin B$ to get

$$\frac{\sin B(\cos^2\frac{C}{2}+\sin^2\frac{C}{2})-\sin C(\cos^2\frac{B}{2}+\sin^2\frac{B}{2})}{\sin B(\cos^2\frac{C}{2}+\sin^2\frac{C}{2})+\sin C(\cos^2\frac{B}{2}+\sin^2\frac{B}{2})}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

Finally, we have that the sine rule holds, so we rewrite in terms of only $\sin C$ to get

$$\frac{\frac{b}{c}\times\sin C-\sin C}{\frac{b}{c}\sin C+\sin C}=\frac{\frac{b}{c}-1}{\frac{b}{c}+1}=\frac{b-c}{b+c}$$

12 The angle CAB is

$$CAB=247°-36°=211°$$

and the two sides adjacent to it are 120 and 234, so the distance to C corresponds to the third side, calculated with the cosine rule as

$$c^2 = 120^2 + 234^2 - 2\times 120\times 234\cos 211°$$

$$c = 342.5\text{km}$$

The bearing of town B from C is given by the complementary angle to

$$360° - 36° - 211° = 113°$$

so the angle we are searching for is

$$180° - 113° = 67°$$

13 a The acceleration is given by the double derivative of x with respect to t, so (assuming a constant)

$$\frac{dx}{dt} = -2a\cos t\sin t = -a\sin 2t$$

so

$$\frac{d^2x}{dt^2} = -2a\cos 2t$$

b The particle is at rest where the velocity is zero, so where

$$\frac{dx}{dt} = 0 = -a\sin 2t$$

Then we have that $2t = n\pi$, and so $t = \frac{n\pi}{2}$

c First we check where the acceleration is zero, which will give us the turning points of the velocity, as

$$\frac{d^2x}{dt^2} = -2a\cos 2t = 0$$

so $2t = \frac{\pi}{2} + n\pi$ and so $t = \frac{\pi}{4} + \frac{n\pi}{2}$. To find whether it is a maximum or a minimum, we check that the velocity as time approaches $t = \frac{\pi}{4}$ it decreases from both the right and the left, so this is a minimum. For $t = \frac{3\pi}{4}$. we have the opposite behaviour, so this is a maximum. This is for positive a. If we have a negative a, the maximum and the minimum will be reversed.

Exam-style questions

14 a $A = \frac{1}{2}\times 5\times 10\sin 30° = \frac{25}{2}$ (2 marks)

b $BD^2 = 5^2 + 10^2 - 2\times 5\times 10\cos 30°$ (2 marks)

$BD = \sqrt{125 - 50\sqrt{3}}$ (1 mark)

$BD = \sqrt{25\left(5 - 2\sqrt{3}\right)}$ (1 mark)

$\text{BD} = 5\sqrt{5-2\sqrt{3}}$

c $\dfrac{\sin C\hat{D}B}{13} = \dfrac{\sin 45^\circ}{5\sqrt{5-2\sqrt{3}}}$ (2 marks)

$\sin C\hat{D}B = \dfrac{13\sqrt{2}}{10\sqrt{5-2\sqrt{3}}}$ (1 mark)

d The angle $C\hat{D}B$ can either be acute or obtuse (1 mark)

and the two possible values add up to 180º. (1 mark)

15 a $l = \sqrt{5^2+3^2} = 5.83$ cm (2 marks)

$S = 2\times(\pi\times 3\times 5.83\ldots) = 110\ \text{cm}^2$ (2 marks)

b $\dfrac{2\times\frac{1}{3}\times\pi\times 3^2\times 5}{\pi\times 3.1^2\times 10.1}\times 100 = 30.9\%$ (2 marks)

16 $2\cos^2 x = \sin 2x \Rightarrow 2\cos^2 x - 2\sin x\cos x = 0$ (1 mark)

$2\cos x(\cos x - \sin x) = 0$ (1 mark)

$\cos x = 0,\ \ \cos x = \sin x$ (1 mark)

(or $\cos x = 0,\ \ \tan x = 1$)

$\cos x = 0 \Rightarrow x = \dfrac{\pi}{2},\ x = \dfrac{3\pi}{2}$ (1 mark)

$\tan x = 1 \Rightarrow x = \dfrac{\pi}{4},\ x = \dfrac{5\pi}{4}$ (1 mark)

17 a i $-1 \le y \le 3$ (1 mark)

ii 2 (1 mark)

b $a = -2$ (1 mark)

$b = \dfrac{2\pi}{2} = \pi$ (2 marks)

$c = 1$ (1 mark)

c $-2\cos\pi x + 1 = 0 \Rightarrow \cos\pi x = \dfrac{1}{2}$ (1 mark)

$\pi x \in \left\{-\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3},\right\}$ (1 mark)

$x \in \left\{-\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{5}{3}, \dfrac{7}{3},\right\}$ (1 mark)

18 a $\dfrac{\cos x}{1-\sin x} - \tan x = \dfrac{\cos x}{1-\sin x} - \dfrac{\sin x}{\cos x}$ (1 mark)

$= \dfrac{\cos^2 x - \sin x(1-\sin x)}{(1-\sin x)(\cos x)}$ (1 mark)

$= \dfrac{\cos^2 x + \sin^2 x - \sin x}{(1-\sin x)(\cos x)}$ (1 mark)

$= \dfrac{1-\sin x}{(1-\sin x)(\cos x)}$ (1 mark)

$= \dfrac{1}{\cos x}$ (1 mark)

$= \sec x$

b $\dfrac{\cos 2x}{1-\sin 2x} - \tan 2x = \sec 2x$

So $\sec 2x = \sqrt{2}$ (1 mark)

$\cos 2x = \dfrac{1}{\sqrt{2}}$ (1 mark)

$2x = \dfrac{\pi}{4}, \dfrac{7\pi}{4}, \dfrac{9\pi}{4}, \dfrac{15\pi}{4}$ (2 marks)

$x = \dfrac{\pi}{8}, \dfrac{7\pi}{8}, \dfrac{9\pi}{8}, \dfrac{15\pi}{8}$ (2 marks)

19 a $\dfrac{dy}{dx} = \dfrac{1}{1+\left(\frac{1}{x}\right)^2} \times \left(-\dfrac{1}{x^2}\right)$ (2 marks)

$\dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$ (1 mark)

b Valid attempt to apply product rule (1 mark)

$\dfrac{dy}{dx} = 2xe^{\arctan x} + \dfrac{x^2 e^{\arctan x}}{1+x^2}$ (3 marks)

$\left(\dfrac{dy}{dx} = e^{\arctan x}\left(2x + \dfrac{x^2}{1+x^2}\right)\right)$

20 Valid attempt at implicit differentiation (1 mark)

$(\cos y)\dfrac{dy}{dx} = -\sin x\left(\sec^2(\cos x)\right)$ (2 marks)

At $\left(\dfrac{\pi}{2}, 0\right)$: $(\cos 0)\dfrac{dy}{dx} = -\sin\dfrac{\pi}{2}\left(\sec^2\left(\cos\dfrac{\pi}{2}\right)\right)$ (1 mark)

$\dfrac{dy}{dx} = -\sec^2 0$ (1 mark)

$= -1$ (1 mark)

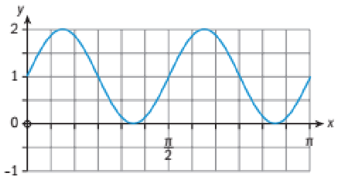
So gradient of normal is $-\dfrac{1}{(-1)} = 1$ (1 mark)

So equation is $y - 0 = 1\left(x - \dfrac{\pi}{2}\right)$, or $y = x - \dfrac{\pi}{2}$ (2 marks)

21 a $S(x) = \underbrace{\sin^2 2x + \cos^2 2x}_{1} + \underbrace{2\sin 2x \cos 2x}_{\sin 4x}$ (3 marks)

$= 1 + \sin 4x$

b

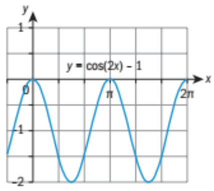


(1 mark) for correct shape, (1 mark) for 2 cycles, (1 mark) for correct max/min

c i $\frac{\pi}{2}$ (1 mark)

ii $0 \le y \le 2$ (1 mark)

d



(3 marks)

e i $k = 2$ (1 mark)

ii $p = -\frac{\pi}{4}$, $q = -2$ (2 marks)

22 a $D = \frac{22+12}{2}$ (1 mark)

$= 17$ (1 mark)

$A = \frac{22-12}{2}$ (1 mark)

$= 5$ (1 mark)

The period is $\frac{360}{B} = 24$ (1 mark)

Therefore $B = 15$ (1 mark)

So $T = 5\sin(15(t-C)) + 17$

At $(3, 12)$, $12 = 5\sin(15(t-C)) + 17$ (1 mark)

$-1 = \sin(15(3-C))$

$15(3-C) = -90$ (1 mark)

$C = 9$ (1 mark)

Therefore $T = 5\sin\left(15\left(t-9\right)\right)+17$

b Solving $T = 5\sin\left(15\left(t-9\right)\right)+17$ and $T = 20$ by GDC (1 mark)

Solutions are $T = 18.54$ and $T = 11.46$ (2 marks)

$18.54 - 11.46 = 7.08$ hours $\left(7 \text{ hours } 5 \text{ minutes}\right)$ (1 mark)

23 $V = \frac{1}{3}\pi r^2 h = \frac{50\pi r^2}{3}$ (1 mark)

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{100\pi r}{3}$ (2 marks)

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ (1 mark)

$2 = \frac{100\pi r}{3} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ (1 mark)

$r = 0.4 \Rightarrow 2 = \frac{40\pi}{3} \times \frac{\mathrm{d}r}{\mathrm{d}t}$

$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{20\pi} \text{ cm min}^{-1}$ (1 mark)