3 Expanding the number system: complex numbers

Skills check

1 **a** $x = \pm 13$ **b** $x = \pm 2\sqrt{7}$ **c** $x = \pm 3\sqrt{11}$

2 **a** $x = 1, -2$ **b** $x = -2, 1$ **c** $x = -\frac{1}{3}, 1$

3 **a** $x < -3$ **b** $-4 \le x \le 2$ **c** $-\frac{1}{4} \le x \le 2$

4 **a** $x = -5, y = -9$ **b** $y = \frac{1}{3} - \frac{2x}{3}$

c $x = \frac{13}{8}, y = -\frac{3}{4}$ **d** No solutions

Exercise 3A

1 $x^2 + 8x + 15 = (x+5)(x+3) = 0$

$\therefore x = -5 \text{ or } x = -2$

2 $x^2 + 5x - 14 = (x+7)(x-2) = 0$

$\therefore x = -7 \text{ or } x = 2$

3 $3x^2 - 7x + 2 = (3x-1)(x-2) = 0$

$\therefore x = \frac{1}{3} \text{ or } x = 2$

4 $4x^2 - 20x + 25 = (2x-5)^2 = 0$

$\therefore x = \frac{5}{2}$

5 $5x^2 - 4x - 12 = (5x+6)(x-2) = 0$

$\therefore x = -\frac{6}{5} \text{ or } x = 2$

Exercise 3B

1 **a** $x^2 + 6x - 7 = (x+3)^2 - 16 = 0$

$\therefore (x+3)^2 = 16 \Rightarrow x = -3 \pm 4$

$\therefore x = -7 \text{ or } x = 1$

b $x^2 - 7x - 30 = \left(x - \frac{7}{2}\right)^2 - \frac{49}{4} - 30 = 0$

$$\therefore \left(x-\frac{7}{2}\right)^2 = \frac{169}{4} \Rightarrow x = \frac{7}{2} \pm \frac{13}{2}$$

$\therefore x = -3$ or $x = 10$

c $x^2 - x - 1 = \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} - 1 = 0$

$$\therefore \left(x-\frac{1}{2}\right)^2 = \frac{5}{4} \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

d $x^2 - \frac{7}{3}x + \frac{2}{3} = \left(x-\frac{7}{6}\right)^2 - \frac{49}{36} + \frac{2}{3} = 0$

$$\Rightarrow \left(x-\frac{7}{6}\right)^2 = \frac{25}{36}$$

$\Rightarrow x = \frac{7 \pm 5}{6} \Rightarrow x = \frac{1}{3}$ or $x = 2$

e $4x^2 + 12x + 5 = 0$

$$x^2 + 3x + \frac{5}{4} = 0$$

$$\therefore \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{5}{4} = 0$$

$$\Rightarrow \left(x+\frac{3}{2}\right)^2 = 1$$

$$\Rightarrow x = -\frac{3}{2} \pm 1$$

$\therefore x = -\frac{5}{2}$ or $x = -\frac{1}{2}$

f $x^2 - 2x + \frac{2}{5} = (x-1)^2 - \frac{3}{5} = 0$

$$(x-1)^2 = \frac{3}{5}$$

$$\therefore x = 1 \pm \sqrt{\frac{3}{5}} = \frac{\sqrt{5} \pm \sqrt{3}}{\sqrt{5}} = 1 \pm \frac{\sqrt{15}}{5}$$

2 a $x^2 + 2x - 1 = (x+1)^2 - 2 = 0$

$$\Rightarrow x = 1 \pm \sqrt{2}$$

$\therefore x = 2.41$ or $x = -0.414$ to 3s.f.

b $x^2 - 3x + 1 = \left(x-\frac{3}{2}\right)^2 + 1 - \frac{9}{4} = \left(x-\frac{3}{2}\right)^2 - \frac{5}{4} = 0$

$$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$$

$\therefore x = 0.382$ or $x = 2.62$ to 3s.f.

c $x^2 - \frac{1}{2}x - \frac{3}{2} = 0$

$$\left(x-\frac{1}{4}\right)^2-\frac{3}{2}-\frac{1}{16}=\left(x-\frac{1}{4}\right)^2-\frac{25}{16}=0$$

$$\Rightarrow x=\frac{1\pm 5}{4}$$

$\therefore x=-1.00$ or $x=1.50$ to 3s.f.

d $x^2+3x+\frac{5}{3}=0$

$$\left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{5}{3}=\left(x+\frac{3}{2}\right)^2-\frac{7}{12}=0$$

$$\Rightarrow x=-\frac{3}{2}\pm\sqrt{\frac{7}{12}}$$

$\therefore x=-2.26$ or -0.736 to 3s.f.

Exercise 3C

1 a $x^2+9x+18=0$

$$\therefore x=\frac{-9\pm\sqrt{9^2-4(1)(18)}}{2(1)}=\frac{-9\pm\sqrt{9}}{2}=\frac{-9\pm 3}{2}$$

$\Rightarrow x=-6$ or $x=-3$

b $x^2-x-30=0$

$$\therefore x=\frac{1\pm\sqrt{(-1)^2-4(1)(-30)}}{2(1)}=\frac{1\pm 11}{2}$$

$\Rightarrow x=-5$ or $x=6$

c $x^2-x+1=0$

$$\therefore x=\frac{1\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}\notin\mathbb{R}$$

d $2x^2-3x-2=0$

$$x=\frac{3\pm\sqrt{(-3)^2-4(2)(-2)}}{2(2)}=\frac{3\pm\sqrt{25}}{4}=\frac{3\pm 5}{4}$$

$\therefore x=2$ or $x=-\frac{1}{2}$

e $\sqrt{2}x^2-11x+\sqrt{3}=0$

$$\therefore x=\frac{11\pm\sqrt{(-11)^2-4\left(\sqrt{2}\right)\left(\sqrt{3}\right)}}{2\sqrt{2}}=\frac{11\pm\sqrt{121-4\sqrt{6}}}{2\sqrt{2}}$$

2 a $x^2+(3-a)x-3a=0$

$$\therefore x = \frac{a-3 \pm \sqrt{(a-3)^2 - 4(-3a)}}{2}$$
$$= \frac{a-3 \pm \sqrt{a^2+6a+9}}{2}$$
$$= \frac{a-3 \pm \sqrt{(a+3)^2}}{2}$$
$$= \frac{a-3 \pm |a+3|}{2}$$

so $x = a$ or $x = -3$

b $2x^2 + (2b-1)x - b = 0$

$$\therefore x = \frac{1-2b \pm \sqrt{(2b-1)^2 - 4(2)(-b)}}{2(2)}$$
$$= \frac{1-2b \pm \sqrt{4b^2+4b+1}}{4}$$
$$= \frac{1-2b \pm \sqrt{(2b+1)^2}}{4}$$
$$= \frac{1-2b \pm |2b+1|}{4}$$

so $x = \frac{1}{2}$ or $x = -b$

c $x^2 + kx - 2k^2 = 0$

$$x = \frac{-k \pm \sqrt{k^2 - 4(1)(-2k^2)}}{2}$$
$$= \frac{-k \pm \sqrt{9k^2}}{2} = \frac{-k \pm 3|k|}{2}$$

so $x = k$ or $x = -2k$

d $p^2x^2 + 2px - 3 = 0$

$$\therefore x = \frac{-2p \pm \sqrt{(2p)^2 - 4p^2(-3)}}{2p^2}$$
$$= \frac{-2p \pm \sqrt{16p^2}}{2p^2} = \frac{-2p \pm 4|p|}{2p^2}$$

so $x = -\frac{3}{p}$ or $x = \frac{1}{p}$

Exercise 3D

1 a $\Delta = 3^2 - 4(1)(-7) = 37 > 0$ so two distinct real roots

b $\Delta = 1^2 - 4(1)(2) = -7 < 0$ so no real roots

c $\Delta = (2)^2 - 4(1)(1) = 0$ so one repeated real root

d $\Delta = \left(\sqrt{3}\right)^2 - 4(5)(2) = -37 < 0$ so no real roots

e $2x^2 - \pi x + 1 = 0$

$\Delta = (-\pi)^2 - 4(2)(1) = \pi^2 - 8 > 3^2 - 8 = 1 > 0$ so two distinct real roots

f $2.25x^2 - 21x + 49 = 0$

$\Delta = (-21)^2 - 4(2.25)(49) = 0$ so one repeated real root

2 **a** $mx^2 + 2x - 5 = 0$

$\Delta = 2^2 - 4m(-5) = 4 + 20m$

Two distinct real roots: $4 + 20m > 0 \Rightarrow m > -\frac{1}{5}$

One repeated real root: $m = -\frac{1}{5}$

No real roots: $m < -\frac{1}{5}$

b $4x^2 - 3x + t - 4 = 0$

$\Delta = (-3)^2 - 4(4)(t-4) = 9 - 16(t-4) = 73 - 16t$

Two distinct real roots: $t < \frac{73}{16}$

One repeated real root: $t = \frac{73}{16}$

No real roots: $t > \frac{73}{16}$

c $(2s+1)x^2 = s(3x-1)$

$(2s+1)x^2 - 3sx + s = 0$

$\Delta = (-3s)^2 - 4s(2s+1) = 9s^2 - 8s^2 - 4s = s(s-4)$

Two distinct real roots: $s < 0$ or $s > 4$

One repeated real root: $s = 0$ or $s = 4$

No real roots: $0 < s < 4$

Exercise 3E

1 $(x+4)(x+2) < 0$

$\therefore x \in]4, 2[$

2 $(x-4)^2 > 0$

$\therefore x \in \mathbb{R} \setminus \{4\}$

3 $(x-15)(x+2) \geq 0$

$\therefore x \in]-\infty, -2] \cup [15, \infty[$

4 $(4x-3)(x-2) \le 0$

$\therefore x \in \left[\frac{3}{4}, 2\right]$

5 $5x^2 - 6x - 8 > 0$

$(5x+4)(x-2) > 0$

$\therefore x \in \left]-\infty, -\frac{4}{5}\right] \cup [2, \infty[$

6 $9x^2 - 12x + 4 \le 0$

$(3x-2)^2 \le 0$ so $x \in \left\{\frac{2}{3}\right\}$

Exercise 3F

1 a $\text{Re}(z) = 0,\ \text{Im}(z) = -4$

b $\text{Re}(z) = 5,\ \text{Im}(z) = 0$

c $\text{Re}(z) = -24,\ \text{Im}(z) = 7$

d $\text{Re}(z) = \frac{5}{13},\ \text{Im}(z) = -\frac{12}{13}$

e $\text{Re}(z) = -\frac{1}{\sqrt{5}},\ \text{Im}(z) = \frac{2}{\sqrt{5}}$

2 a $|z| = \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4$

b $|z| = \sqrt{5^2 + 0^2} = \sqrt{25} = 5$

c $|z| = \sqrt{(-24)^2 + 7^2} = \sqrt{576 + 49} = \sqrt{625} = 25$

d $|z| = \sqrt{\left(\frac{5}{13}\right)^2 + \left(-\frac{12}{13}\right)^2} = \sqrt{\frac{169}{169}} = 1$

e $|z| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(-\frac{1}{\sqrt{5}}\right)^2} = 1$

3 a $z_1 - z_2 + z_3 = (3-5-1) + i(-4-1+2) = -3-3i$

b $2z_1 + 3z_2 - 4z_3 = (6+15+4) + i(-8+3-8) = 25-13i$

c $-\frac{1}{2}z_1+\frac{2}{3}z_2-\frac{1}{4}z_3$

$$-\frac{1}{2}(3-4i)+\frac{2}{3}(5+i)-\frac{1}{4}(-1+2i)$$
$$=\left(-\frac{3}{2}+\frac{10}{3}+\frac{1}{4}\right)+i\left(2+\frac{2}{3}-\frac{1}{2}\right)$$
$$=\frac{25}{12}+\frac{13i}{6}$$

d $\frac{4z_3-5z_1+2z_2}{3}$

$$=\frac{4(-1+2i)-5(3-4i)+2(5+i)}{3}$$
$$=\frac{-4+8i-15+20i+10+2i}{3}$$
$$=\frac{-9+30i}{3}=-3+10i$$

Exercise 3G

1 a $z_1z_2+2z_3{}^*=(1+i)(3-2i)+2(-2-3i)$

$$=5+i-4-6i=1-5i$$

b $\frac{z_1}{z_3}-\frac{z_2}{5}=\frac{z_1z_3{}^*}{|z_3|^2}-\frac{z_2}{5}$

$$=\frac{(1+i)(-2-3i)}{(-2)^2+3^2}-\frac{3-2i}{5}$$
$$=\frac{1-5i}{13}-\frac{3-2i}{5}$$
$$=\frac{5-25i-39+26i}{65}$$
$$=-\frac{34}{65}+\frac{1}{65}i$$

c $z_1{}^2-3z_2z_3=(1+i)^2-3(3-2i)(-2+3i)$

$$=1+2i-1-3(-6+13i+6)$$
$$=-37i$$

d $\frac{z_1z_3}{z_2{}^*}=\frac{z_1z_2z_3}{|z_2|^2}=\frac{(1+i)(3-2i)(-2+3i)}{3^2+(-2)^2}$

$$=\frac{(1+i)(13i)}{13}=i(1+i)=i-1$$

e $\frac{2z_1-4z_2{}^*}{z_3z_2{}^*}=\frac{2(1+i)-4(3+2i)}{(-2+3i)(3+2i)}$

$$= \frac{-10-6i}{-12+5i} = \frac{10+6i}{12-5i} = \frac{(10+6i)(12+5i)}{12^2+(-5)^2}$$

$$= \frac{90+122i}{169} = \frac{90}{169} + \frac{122i}{169}$$

2 a $\frac{1+2i}{i} = 2-i$ so $\mathrm{Re}(z) = 2$ and $\mathrm{Im}(z) = -1$

b $\frac{1}{i} + \frac{2i}{1-i} = -i + \frac{2i(1+i)}{2} = -i + i(1+i) = -1$

so $\mathrm{Re}(z) = -1$ and $\mathrm{Im}(z) = 0$

c $\frac{1+2i}{1-2i} - \frac{1-2i}{1+2i} = \frac{(1+2i)^2-(1-2i)^2}{(1-2i)(1+2i)}$

$$= \frac{(1+2i+1-2i)(1+2i-1+2i)}{5}$$

$$= \frac{8i}{5}$$

so $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = \frac{8}{5}$

3 a $(1+3i)(a+bi) = a-3b+(3a+b)i = 5+5i$

so $a-3b = 5$ and $3a+b = 5$
$\therefore 3a-9b = 15$
$\therefore 10b = -10 \Rightarrow b = -1$
$\Rightarrow a = 5+3b \Rightarrow a = 2$

b $\frac{a+bi}{1+2i} = -3+i$

$\Rightarrow a+bi = (-3+i)(1+2i) = -5-5i$
so $a = -5$ and $b = -5$

4 a $2(z+i) = 3i(z-1)$

$\Rightarrow z(2-3i) = -2i-3i = -5i$

$\therefore z = \frac{-5i}{2-3i} = \frac{-5i(2+3i)}{2^2+(-3)^2}$

$= \frac{-10i+15}{13} = \frac{15}{13} - \frac{10}{13}i$

b □ $\frac{z-2}{1-2i} = \frac{z-i}{2+i}$

$(z-2)(2+i) = (z-i)(1-2i)$
$\Rightarrow z(2+i-1+2i) = 2(2+i) - i(1-2i)$
$\Rightarrow (1+3i)z = 2+i$

$\Rightarrow z = \frac{2+i}{1+3i} = \frac{(2+i)(1-3i)}{1^2+3^2}$

$= \frac{5-5i}{10} = \frac{1-i}{2}$

c $(z+2i)(2+i)=(z-1)(1-i)$

$\therefore z(2+i-1+i)=-2i(2+i)-(1-i)$

$\Rightarrow z(1+2i)=1-3i$

$\Rightarrow z=\dfrac{1-3i}{1+2i}=\dfrac{(1-3i)(1-2i)}{1^2+2^2}=\dfrac{-5-5i}{5}=-1-i$

d $\dfrac{z+1+i}{1-4i}=\dfrac{z-3i+2}{2i+5}$

$\therefore (z+1+i)(2i+5)=(z-3i+2)(1-4i)$

$\Rightarrow z(2i+5-1+4i)=-(i+1)(2i+5)+(-3i+2)(1-4i)$

$\Rightarrow (6i+4)z=-3-7i-10-11i=-13-18i$

$\therefore z=-\dfrac{13+18i}{4+6i}=-\dfrac{(13+18i)(4-6i)}{4^2+6^2}$

$=-\dfrac{52+108-6i}{52}$

$=-\dfrac{40}{13}+\dfrac{3}{26}i$

5 a $\dfrac{a+bi}{2+i}=k\in\mathbb{R}\Rightarrow a+bi=2k+ki$

Comparing imaginary parts $\Rightarrow b=k$
Comparing real parts $\Rightarrow a=2k=2b$
so $a=2b$

b $\dfrac{1-i}{a-bi}=ki$ where $k\in\mathbb{R}$

$\therefore b+ai=\dfrac{1}{k}-\dfrac{1}{k}i$

Comparing real parts $\Rightarrow b=\dfrac{1}{k}$

Comparing imaginary parts $\Rightarrow a=-\dfrac{1}{k}=-b$

so $a=-b$

6 a $|z|+z=1$

$\Rightarrow \sqrt{x^2+y^2}+x+iy=1$
Comparing imaginary parts,
$y=0$
$\therefore \sqrt{x^2}+x-1=0$
$\Rightarrow |x|+x-1=0$
$x<0$ yields no solution $-x+x-1=0\Rightarrow$ false statement
for $x\geq 0$
$2x-1=0$

$\Rightarrow x=\dfrac{1}{2}$

b $|z|-z^*=i$

$\therefore \sqrt{x^2+y^2}-(x-iy)=i$

$\Rightarrow \sqrt{x^2+y^2}-x+iy=i$

Comparing real parts, $y=1$

$\therefore \sqrt{x^2+1}-x=0$

$\Rightarrow x^2+1=x^2 \Rightarrow 1=0$ false statement

Therefore, this has no solutions

c $z^2+z^*=2$

$\Rightarrow (x+iy)^2+(x-iy)=2$

$\Rightarrow x^2+2ixy-y^2+x-iy=2$

$\Rightarrow (x^2+x-y^2)+(2xy-y)i=2$

Comparing imaginary parts,

$2xy-y=0 \Rightarrow y(2x-1)=0$

so $y=0$ or $x=\frac{1}{2}$

If $x=\frac{1}{2}$,

$\left(\frac{1}{2}\right)^2+\frac{1}{2}-y^2=2$

$\Rightarrow y^2=-\frac{5}{4}$ which has no solutions

If $y=0$,

$x^2+x-2=0$

$(x-1)(x+2)=0$

so $x=1$ or $x=-2$

7 a $|z_1z_2|=|(x_1+iy_1)(x_2+iy_2)|=|(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)|$

$=\sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+x_2y_1)^2}$

$=\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_1^2y_2^2+x_2^2y_1^2}$

$=\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$=\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}=|z_1||z_2|$

b Follows from part a: replace z_2 with $\frac{1}{z_2}$

c $|z_1+z_2|^2=(z_1+z_2)(z_1+z_2)^*$

$=|z_1|^2+|z_2|^2+(z_1z_2^*+z_1^*z_2)$

$=|z_1|^2+|z_2|^2+2\text{Re}(z_1z_2^*)$

$\le |z_1|^2+|z_2|^2+2|z_1z_2^*|$

$=|z_1|^2+|z_2|^2+2|z_1||z_2^*|$

$=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$=(|z_1|+|z_2|)^2$

Since $|z_1+z_2|$ and $|z_1|+|z_2|$ are non-negative

$|z_1+z_2|\le|z_1|+|z_2|$

8 **a** $\left(z^*\right)^* = (x - iy)^* = x + iy = z$

b $(z_1 + z_2)^* = \left((x_1 + x_2) + i(y_1 + y_2)\right)^*$

$= (x_1 + x_2) - i(y_1 + y_2)$
$= (x_1 - iy_1) + (x_2 - iy_2)$
$= z_1{}^* + z_2{}^*$

c $(z_1 z_2)^* = \left((x_1 + iy_1)(x_2 + iy_2)\right)^*$

$= \left(x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\right)^*$
$= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)$
$= (x_1 - iy_1)(x_2 - iy_2)$
$= z_1{}^* z_2{}^*$

d $\left(\frac{z_1}{z_2}\right)^* = \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)^* = \left(\frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2{}^2 + y_2{}^2}\right)^*$

$= \left(\frac{x_1x_2 + y_1y_2 + i(y_1x_2 - x_1y_2)}{x_2{}^2 + y_2{}^2}\right)^*$
$= \frac{x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)}{x_2{}^2 + y_2{}^2}$

and

$\frac{z_1{}^*}{z_2{}^*} = \frac{x_1 - iy_1}{x_2 - iy_2} = \frac{(x_1 - iy_1)(x_2 + iy_2)}{x_2{}^2 + y_2{}^2}$
$= \frac{x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)}{x_2{}^2 + y_2{}^2}$

so $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1{}^*}{z_2{}^*}$

e $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |z^*|$

Exercise 3H

1 **a** $i^7 + i^{17} + i^{27} + i^{37}$

$= i^4 i^3 + \left(i^4\right)^4 i + \left(i^4\right)^6 i^3 + \left(i^4\right)^9 i$
$= i^3 + i + i^3 + i = -i + i - i + i = 0$

b $i^{173} - i^{272} + i^{27} + i^{37}$

$= i^{172} i - i^{272} + i^{24} i^3 + i^{36} i$
$= \left(i^4\right)^{43} i - \left(i^4\right)^{68} + \left(i^4\right)^6 i^3 + \left(i^4\right)^9 i$
$= i - i + i^3 + i$
$= i - i - i + i = 0$

c $\left(3 + i^{77}\right)\left(1 - 2i^{93}\right) = \left(3 + i^{76} i\right)\left(1 - 2i^{92} i\right)$

$$= \left(3 + \left(i^4\right)^{19} i\right)\left(1 - 2\left(i^4\right)^{23} i\right)$$
$$= (3 + i)(1 - 2i)$$
$$= 5 - 5i$$

d $$\frac{3i^{2018} + 2i^{2019}}{4i^{2020} - 3i^{2021}} = \frac{3i^{2016}i^2 + 2i^{2016}i^3}{4i^{2020} - 3i^{2020}i} = \frac{3\left(i^4\right)^{504} i^2 + 2\left(i^4\right)^{504} i^3}{4\left(i^4\right)^{505} - 3\left(i^4\right)^{505} i}$$

$$= \frac{-3 - 2i}{4 - 3i} = -\frac{(3 + 2i)(4 + 3i)}{(4 - 3i)(4 + 3i)} = -\frac{6 + 17i}{25} = -\frac{6}{25} - \frac{17}{25}i$$

e $$i + i^2 + i^3 + i^4 = 0 \Rightarrow \sum_{k=1}^{2019} i^k = 0 + 0 + \ldots 0 + i + i^2 + i^3 = -1$$

$$i \times i^2 \times i^3 \times i^4 = -1 \Rightarrow \prod_{k=1}^{2019} i^k = (-1)^{504} \times i \times i^2 \times i^3 = -1$$
$$\frac{-1}{-1} = 1$$

f $$i + i^3 = 0 \Rightarrow \sum_{k=1}^{1010} i^{2k-1} = 0$$

$$i \times i^3 = 1 \Rightarrow \prod_{k=1}^{1010} i^{2k} = 1$$
$$\frac{0}{1} = 0$$

2 **a** $$(3 + 2i)^3 = (3)^3 + 3(3)^2(2i) + 3(3)(2i)^2 + (2i)^3$$

$$= 27 + 54i - 36 - 8i$$
$$= -9 + 46i$$

b $$(1 - 3i)^4 = 1 + 4(-3i) + 6(-3i)^2 + 4(-3i)^3 + (-3i)^4$$

$$= 1 - 12i + 6(-9) + 4(27i) + 81$$
$$= 28 + 96i$$

c $$(1 - 2i)^4 + (1 + 2i)^4$$
$$= \left[1 + 4(-2i) + 6(-2i)^2 + 4(-2i)^3 + (-2i)^4\right] + \left[1 + 4(2i) + 6(2i)^2 + 4(2i)^3 + (2i)^4\right]$$
$$= \left[1 - 4(2i) + 6(2i)^2 - 4(2i)^3 + (2i)^4\right] + \left[1 + 4(2i) + 6(2i)^2 + 4(2i)^3 + (2i)^4\right]$$
$$= 2\left(1 + 6(2i)^2 + (2i)^4\right)$$
$$= 2(1 - 24 + 16) = -14$$

d $$(1 + i)^5 - (1 - i)^5$$
$$= \left(1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5\right) - \left(1 - 5i + 10i^2 - 10i^3 + 5i^4 - i^5\right)$$
$$= 2\left[5i + 10i^3 + i^5\right]$$
$$= 2\left[5i - 10i + i\right]$$
$$= -8i$$

3 **a** $\sqrt{i} = a + bi$

$\Rightarrow i = (a+bi)^2 = a^2 - b^2 + 2abi$

Comparing real part:

$a^2 - b^2 = 0 \Rightarrow a = \pm b$

If $a = b,\ 2ab = 2a^2 = 1 \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$

If $a = -b,\ 2ab = -2a^2 = 1$ no solutions since $a \in \mathbb{R}$

$\therefore \dfrac{1+i}{\sqrt{2}}$ or $\dfrac{-1-i}{\sqrt{2}}\left[\text{or} \pm\left(\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i\right)\right]$

b $\sqrt{-i} = a + bi \Rightarrow -i = (a+bi)^2 = a^2 - b^2 + 2abi$

Comparing real parts, $a^2 - b^2 = 0 \Rightarrow a = \pm b$

Comparing imaginary parts:

$-1 = 2ab$

if $a = b \Rightarrow -1 = 2a^2 \Rightarrow a^2 = -\dfrac{1}{2} \Rightarrow$ no solution

Only a solution if $a = -b$

$\therefore -1 = -2a^2 \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$

$\therefore \dfrac{1-i}{\sqrt{2}}$ or $\dfrac{-1+i}{\sqrt{2}}\left[\text{or} \pm\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)\right]$

c $\sqrt{-21+20i} = a + bi \Rightarrow -21 + 20i = (a+bi)^2 = a^2 - b^2 + 2abi$

$\therefore a^2 - b^2 = -21$ and $ab = 10 \Rightarrow a = \dfrac{10}{b}$

$\therefore \dfrac{100}{b^2} - b^2 = -21$

$\Rightarrow b^4 - 21b^2 - 100 = 0$

$\Rightarrow (b^2 - 25)(b^2 + 4) = 0$

$b \in \mathbb{R} \Rightarrow b = \pm 5$

$\Rightarrow a = \dfrac{10}{\pm 5} = \pm 2$

$\therefore \pm(2 + 5i)$

d $\sqrt{\dfrac{5}{36} - \dfrac{i}{3}} = a + bi$

$\dfrac{5}{36} - \dfrac{i}{3} = (a+bi)^2 = a^2 - b^2 + 2abi$

$\Rightarrow a^2 - b^2 = \dfrac{5}{36}$ and $2ab = -\dfrac{1}{3}$

$\therefore a = -\dfrac{1}{6b}$

$\therefore \dfrac{1}{36b^2} - b^2 = \dfrac{5}{36}$

$\Rightarrow 1 - 36b^4 = 5b^2$

$\Rightarrow 36b^4 + 5b^2 - 1 = 0$

$\Rightarrow (9b^2 - 1)(4b^2 + 1) = 0$

$b \in \mathbb{R} \Rightarrow b = \pm\dfrac{1}{3} \Rightarrow a = \mp\dfrac{1}{2}$

$\therefore \pm\left(\dfrac{1}{2} - \dfrac{1}{3}i\right)$

4 a The statement follows repeated application of the result

given in Exercise 3G Question 7a. Can be (quite trivially) proven formally using this property via induction (in a similar fashion to the below)

b $P(n)$: $(z^*)^n = (z^n)^*$, $n \in \mathbb{Z}^+$

$P(1)$ is true
Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$
Then,
$(z^*)^{k+1} = (z^*)^k z^* = (z^k)^* z^*$
$= (z^k z)^*$ using Exercise 3G Question 8c
$= (z^{k+1})^*$
so $P(k) \Rightarrow P(k+1)$

Therefore it has been shown that $\mathrm{P}(1)$ is true and that if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Thus, the statement is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction

5 a $(1+i)^2 = 2i$

$(1+i)^3 = 1 + 3i - 3 - i = -2 + 2i$
$(1+i)^4 = (1+i)(-2+2i) = -4$
$(1+i)^5 = -4(1+i)$ which is a multiple of $(1+i)$
so it is clear that whenever $n = 4k$ $(k \in \mathbb{Z}^+)$, $(1+i)^n$ is real

b It immediately follows from above that when $n = 4k + 2$ $(k = 0, 1, 2...)$
$(1+i)^2$ is purely imaginary

6 a $(1-i)^{2n} = \left((1-i)^2\right)^n = (-2i)^n$

b $(1-i)^{2n+1} = (1-i)(1-i)^{2n} = (1-i)(-2i)^n$ using part a

Exercise 3I

1 a $q(x) = 2x^2 - 3x + 1$ **b** $q(x) = 3x^3 + x^2 + 2$

c $q(x) = x^3 - x^2 - 2$

2 a $q(x) = 3x^2 - 3x - 2$, $r = -3$ **b** $q(x) = 2x^2 - 5x + 5$, $r(x) = 6x - 15$

c $q(x) = x^2 + x$, $r(x) = -x^2 - x + 1$

Exercise 3J

1 a $q(x) = x^2 + 4x + 5$, $r = 11$ **b** $q(x) = 2x^2 - 3x - 1$, $r = 11$

c $q(x) = 2x^3 + 2x^2 - x + 3$, $r = 1$ **d** $q(x) = 3x^4 + 2x^3 - 2x^2 - x + 13$, $r = -81$

2 a $(x+1)(x-3)\overline{\big)2x^4 - x^3 - 32x^2 + 31x + 60}$ with quotient $2x^2 + 3x - 20$

b $2x^2 + 3x - 20 = (2x-5)(x+4)$

$2x^4 - x^3 - 32x^2 + 31x + 60 = (x+1)(x-3)(2x-5)(x+4)$

Exercise 3K

1 a $q(x) = x^2 - x + 3,\ r = 1$

b $q(x) = 3x^2 + x + 1,\ r = 1$

c $q(x) = x^3 + 2x^2 - x + 1,\ r = 4$

d $q(x) = x^4 - 2x^3 + x^2 - x + 3,\ r = -1$

2 $f(x) = (x^2+2)(2x^2 - 3x + 1) + x - 3$

$= 2x^4 - 3x^3 + x^2 + 4x^2 - 6x + 2 + x - 3$
$= 2x^4 - 3x^3 + 5x^2 - 5x - 1$

3 By factor theorem,

$f(-2) = 6(-2)^5 + 17(-2)^4 - 20(-2)^3 - 35(-2)^2 + 44(-2) + a$
$= 12 + a = 0 \Rightarrow a = -12$

Horner's algorithm

	6	+17	−20	−35	+44	+a
−2	6	5	−30	25	−6	12 + a

$12 + a = 0 \Rightarrow a = -12$

4 By factor theorem,

$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^4 + 5\left(-\frac{1}{2}\right)^3 - 4\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + 1$
$= -\frac{1}{2} - \frac{b}{2} = 0 \Rightarrow b = -1$

Horner's algorithm

	2	5	−4	−b	1
$-\frac{1}{2}$	2	4	−6	3 + b	$-\frac{1}{2} - \frac{1}{2}b$

$-\frac{1}{2} - \frac{1}{2}b = 0 \Rightarrow b = 1$

5 By factor theorem,

$f(1) = 1 + 5 + 5 + a + b = 11 + a + b = 0 \Rightarrow a + b = -11$
$f(-2) = (-2)^4 + 5(-2)^3 + 5(-2)^2 + a(-2) + b = 0$
$\Rightarrow -2a + b = 4$
Eliminating b,
$3a = -15 \Rightarrow a = -5$ and therefore $b = -6$

Horner's algorithm

	1	5	5	a	b
1	1	6	11	11 + a	11 + a + b
−2	1	4	3	5 + a	

$11 + a + b = 0$

$5 + a = 0 \Rightarrow a = -5 \Rightarrow b = -6$

6 By factor theorem,

$$f\left(\frac{1}{2}\right) = \frac{6}{2^5} + \frac{13}{2^4} - \frac{30}{2^3} - \frac{45}{4} + \frac{a}{2} + b = -14 + \frac{a}{2} + b = 0$$

$\Rightarrow a + 2b = 28$

By polynomial remainder theorem,

$f(1) = 6 + 13 - 30 - 45 + a + b = -40 \Rightarrow a + b = 16$

Eliminating a,

$b = 12 \Rightarrow a = 4$

7 $f(x) = (x+2)g(x) + 4$ for some polynomial $g(x)$

$f(-2) = 4; f(5) = -3$

$$f(x) = (x^2 - 3x - 10) \bullet g(x) + \overbrace{ax + b}^{r(x)}$$

$$\left.\begin{aligned} 4 &= a(-2) + b \\ 3 &= a(5) + b \end{aligned}\right\} \text{Elimination}$$

$7 = -7a \Rightarrow a = -1$

$4 = 2 + b \Rightarrow b = 2$

$r(x) = -x + 2$

8 $f(-1) = (-1)^{2019} + (-1)^{2018} + \ldots + 1 = 0$

so in fact $(x+1)$ is factor of $f(x)$

9 $f(x) = (x+2)^{2n} + (x+3)^n - 1$

$x^2 + 5x + 6 = (x+3)(x+2)$ so $f(x)$ is divisible by $x^2 + 5x + 6$ if and only if it is divisible by both $(x+3)$ and $(x+2)$.

$f(-2) = (1)^n - 1 = 0$ so $f(x)$ is divisible by $(x+2)$

$f(-3) = (-1)^{2n} - 1 = 1^{2n} - 1 = 0$ so $f(x)$ is divisible by $(x+3)$

Thus $f(x)$ is divisible by $x^2 + 5x + 6$

10 By polynomial remainder theorem,

$f(x) = \left(x - \frac{b}{a}\right)q(x) + f\left(\frac{b}{a}\right)$ for some polynomial $q(x)$

$\Rightarrow af(x) = (ax - b)q(x) + af\left(\frac{b}{a}\right)$

i.e. the function $af(x)$ leaves remainder $af\left(\frac{b}{a}\right)$ when divided by $(ax - b)$.

Thus $f(x)$ leaves a remainder of $f\left(\frac{b}{a}\right)$ when divided by $(ax - b)$.

Exercise 3L

1 a $f(x) = x(x-2)(x-7) = x^3 - 9x^2 + 14x$

b $f(x) = (x+3)(x+2)(x-1)(x-3)$

$= (x^2 + 5x + 6)(x^2 - 4x + 3)$
$= x^4 + x^3 - 11x^2 - 9x + 18$

c $f(x) = 2(x+1)\left(x + \frac{1}{2}\right)(x-2)(x-5)$

$= (x+1)(2x+1)(x^2 - 7x + 10)$
$= (2x^2 + 3x + 1)(x^2 - 7x + 10)$
$= 2x^4 - 11x^3 + 23x + 10$

2 a $f(x) = (x^2 - 2)(x-2) = x^3 - 2x^2 - 2x + 4$

b $f(x) = 2(x+1)\left(x - \frac{1}{2}\right)(x^3 - 3)$

$= (x+1)(2x-1)(x^3 - 3)$
$= (2x^2 + x - 1)(x^3 - 3)$
$= 2x^5 + x^4 - x^3 - 6x^2 - 3x + 3$

c $f(x) = \left(x - \left(1 - \sqrt{3}\right)\right)\left(x - \left(1 + \sqrt{3}\right)\right)(x^3 - 2)$

$= (x^2 - 2x - 2)(x^3 - 2)$
$= x^5 - 2x^4 - 2x^3 - 2x^2 + 4x + 4$

3 a $f(x) = (x-1)(x^2 - 2x + 2)$

b $f(x) = 3x^3 - x^2 + 2x + 6 = (x+1)(3x^2 - 4x + 6)$

c $f(x) = 2x^4 - 5x^3 + 11x^2 - 3x - 5$

$= (x-1)(2x^3 - 3x^2 + 8x + 5)$
$= (x-1)(2x+1)(x^2 - 2x + 5)$

4 a □ $f(x) = (x+2)^2(ax+b) = (x^2 + 4x + 4)(ax+b)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^3 is 3 $\Rightarrow a = 3$
and $4b = -20 \Rightarrow b = -5$
$\therefore f(x) = (x+2)^2(3x-5)$

b $f(x) = (3x-2)^2(ax+b) = (9x^2 - 12x + 4)(ax+b)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^3 is $9 \Rightarrow a = 1$
and $4b = 16 \Rightarrow b = 4$
$\therefore f(x) = (3x-2)^2(x+4)$

c $f(x) = (x-1)^2(ax^2+bx+c) = (x^2-2x+1)(ax^2+bx+c)$

Now, long division or synthetic division can be used, though it is easier to note the coefficient of x^4 is $1 \Rightarrow a = 1$
and $c = -4$ and the coefficient of x is
$8 = -2c + b \Rightarrow b = 8 + 2c = 0$
$\therefore f(x) = (x^2-2x+1)(x^2-4) = (x-1)^2(x-2)(x+2)$

d $f(x) = (2x+1)^3(ax+b) = (8x^3+12x^2+6x+1)(ax+b)$

Now, long division or synthetic division may be used, but it is easier to note the coefficient of x^4 is $8 \Rightarrow a = 1$
and $b = 1$
$\therefore f(x) = (2x+1)^3(x+1)$

e $f(x) = (x-1)^4(ax+b) = (x^4-4x^3+6x^2-4x+1)(ax+b)$

Now, long division or synthetic division may be used, but it is easier to note the coefficient of x^5 is $5 \Rightarrow a = 5$
and $b = 7$
$\therefore f(x) = (x-1)^4(5x+7)$

5 **a** If $z = 2i$ is a root, then so is $z = -2i$

$\therefore f(z) = (z+2i)(z-2i)(az+b) = (z^2+4)(az+b)$
Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $1 \Rightarrow a = 1$ and $4b = -8 \Rightarrow b = -2$
$\therefore f(z) = (z+2i)(z-2i)(z-2)$
So the remaining roots are $-2i$ and 2

b If $z = 3-2i$ is a zero then so is $z = 3+2i$

$\therefore f(z) = (z-(3-2i))(z-(3+2i))(az+b)$
$= (z^2-6z+13)(az+b)$
Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $2 \Rightarrow a = 2$
and $13b = -13 \Rightarrow b = -1$
$\therefore f(z) = (z^2-6z+13)(2z-1)$
So the remaining roots are $3+2i$ and $\frac{1}{2}$

c If $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a root, then so is $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\therefore f(z) = \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)(az + b)$

$= (z^2 + z + 1)(az + b)$

Now, long division or synthetic division can be used but it is easier to note that the coefficient of z^3 is $3 \Rightarrow a = 3$

and $b = 7$

$\therefore f(z) = \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)(3z + 7)$ □

So the remaining roots are $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{7}{3}$

d If $z = -i$ is a zero then so is $z = i$

$\therefore f(z) = (z - i)(z + i)(az^2 + bz + c)$

$= (z^2 + 1)(az^2 + bz + c)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of z^4 is $1 \Rightarrow a = 1$

and $c = 5$ and the coefficient of z is

$-2 = b$

$\therefore f(z) = (z^2 + 1)(z^2 - 2z + 5)$

$\Rightarrow z^2 - 2z + 5 = 0$

$\Rightarrow (z - 1)^2 = -4$

$\Rightarrow z = 1 \pm 2i$

So the remaining roots are $z = i$ and $z = 1 \pm 2i$

e If $z = -2 - i$ is a zero then so is $z = -2 + i$

$\therefore f(z) = (z - (-2 - i))(z - (-2 + i))(az^2 + bz + c)$

$= (z^2 + 4z + 5)(az^2 + bz + c)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of z^4 is $1 \Rightarrow a = 1$

and $5c = 10 \Rightarrow c = 2$

and the coefficient of z is $13 = 4c + 5b \Rightarrow b = \frac{13 - 8}{5} = 1$

$f(z) = (z^2 + 4z + 5)(z^2 + z + 2)$

so the remaining zeros satisfy

$z^2 + z + 2 = \left(z + \frac{1}{2}\right)^2 = -\frac{7}{4}$

$\Rightarrow z = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

So the remaining roots are $-2 + i$ and $-\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

f If $z = -\frac{1}{3} + \frac{\sqrt{2}}{3}i$ then so is $z = -\frac{1}{3} - \frac{\sqrt{2}}{3}i$

$\therefore f(z) = \left(z - \left(-\frac{1}{3} + \frac{\sqrt{2}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{2}}{3}i\right)\right)\left(az^2 + bz + c\right)$

$= \left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)\left(az^2 + bz + c\right)$

Now, long division or synthetic division may be used, but it is easier to note that the coefficient of z^4 is $3 \Rightarrow a = 3$

and $\frac{c}{3} = 6 \Rightarrow c = 18$

and the coefficient of z is $12 = \frac{2c}{3} + \frac{b}{3} \Rightarrow b = 36 - 2(18) = 0$

$\therefore f(z) = \left(z^2 + \frac{2}{3}z + \frac{1}{3}\right)\left(3z^2 + 18\right)$

$\Rightarrow 3z^2 + 18 = 0 \Rightarrow z = \pm i\sqrt{6}$

so the remaining roots are $z = \frac{-1 - \sqrt{2}i}{3}$ and $z = \pm\sqrt{6}i$

6 a $f(2) = 8 + 4a + 2 - 2 = 0 \Rightarrow a = -2$

$\therefore f(x) = x^3 - 2x^2 + x - 2 = (x - 2)\left(x^2 + 1\right)$

roots of $x^2 + 1 = 0$ can be found by long division, synthetic division, or inspection so the remaining roots are $x = \pm i$

b $f(-5) = -250 + 250 - 5a + 15 = 0 \Rightarrow a = 3$

$f(x) = 2x^3 + 10x^2 + 3x + 15$

$= (x + 5)\left(2x^2 + 3\right)$

roots of $2x^2 + 3 = 0$ can be found by long division, synthetic division, or inspection

so the remaining roots are $x = \pm\sqrt{\frac{3}{2}}i$

c $f(x) = x^4 - 2x^3 + ax^2 + bx + 85$

If $x = \sqrt{5}i$ is a root then so is $x = -\sqrt{5}i$

$\therefore f\left(\sqrt{5}i\right) = 25 + 10\sqrt{5}i - 5a + \sqrt{5}bi + 85 = 0$

Comparing real parts, $25 - 5a + 85 = 0 \Rightarrow a = 22$

Comparing imaginary parts, $10\sqrt{5} + \sqrt{5}b = 0 \Rightarrow b = -10$

$\therefore f(x) = x^4 - 2x^3 + 22x^2 - 10x + 85$

$= \left(x - \sqrt{5}i\right)\left(x + \sqrt{5}i\right)\left(cx^2 + dx + e\right) = \left(x^2 + 5\right)\left(cx^2 + dx + e\right)$

Now, long division or synthetic division can be used, but it is easier to note that the coefficient of x^4 is $1 = c$

and $5e = 85 \Rightarrow e = 17$

and the coefficient of x is $5d = -10 \Rightarrow d = -2$

$\therefore x^2 - 2x + 17 = (x - 1)^2 + 16 = 0 \Rightarrow x = 1 \pm 4i$

So the remaining zeros are

$x = -\sqrt{5}i,\ x = 1 + 4i,\ x = 1 - 4i$

d If $x = 1 - 2i$ is a root then so is $x = 1 + 2i$

$f(1 - 2i) = 2 + a + b - (14 + 2a)i = 0$
Comparing real and imaginary parts,
$\therefore a = -7 \Rightarrow b = 5$
$\therefore f(x) = (x - (1 - 2i))(x - (1 + 2i))(cx^2 + dx + e)$
$= (x^2 - 2x + 5)(cx^2 + dx + e)$
$= 3x^4 - 7x^3 + 18x^2 - 7x + 5$
Now, long division or synthetic division can be used, but it is easier to note that the coefficient of x^4 is $3 \Rightarrow c = 3$
and $5e = 5 \Rightarrow e = 1$
and the coefficient of x is $5d - 2e = -7 \Rightarrow d = -1$
$\therefore 3x^2 - x + 1 = 0 \Rightarrow x = \dfrac{1 \pm i\sqrt{11}}{6}$
so the remaining roots are
$x = 1 + 2i$ and $x = \dfrac{1 \pm i\sqrt{11}}{6}$

Exercise 3M

1 a $x_1 + x_2 + x_3 + x_4 = -\dfrac{-3}{1} = 3$

$x_1x_2x_3x_4 = \dfrac{4}{1} = 4$

b $\displaystyle\sum_{i=1}^{6} x_i = -\frac{-6}{2} = 3$

$\displaystyle\prod_{i=1}^{6} x_i = \frac{0}{2} = 0$

c $\displaystyle\sum_{i=0}^{17} x_1 = -\frac{0}{23} = 0$

$\displaystyle\prod_{i=1}^{17} x_i = (-1)^{17} - \frac{46}{23} = 2$

d $\displaystyle\sum_{i=1}^{2020} x_i = -\frac{-4}{3} = \frac{4}{3}$

$\displaystyle\prod_{i=1}^{2020} x_i = (-1)^{2020} \times -\frac{8}{3} = -\frac{8}{3}$

2 a □ $x_1 + x_2 + x_3 = -\dfrac{-2}{4} = \dfrac{1}{2}$

b $x_1x_2x_3 = (-1)^3 \dfrac{-17}{4} = \dfrac{17}{4}$

c $10x_1 + 10x_2 + 10x_3 = 10(x_1 + x_2 + x_3) = 10\left(-\dfrac{-2}{4}\right) = 5$

d $\dfrac{3}{x_1x_2}+\dfrac{3}{x_1x_3}+\dfrac{3}{x_2x_3}=\dfrac{3(x_1+x_2+x_3)}{x_1x_2x_3}=\dfrac{3\left(\frac{1}{2}\right)}{\frac{17}{4}}=\dfrac{6}{17}$

3 a $x_1+x_2+x_3+x_4=-\dfrac{2}{6}=-\dfrac{1}{3}$

b $x_1x_2x_3x_4=(-1)^4\dfrac{-3}{6}=-\dfrac{1}{2}$

c $3x_1+3x_2+3x_3+3x_4=3(x_1+x_2+x_3+x_4)=3\left(-\dfrac{1}{3}\right)=-1$

d $\dfrac{6}{x_1x_2x_3}+\dfrac{6}{x_1x_2x_4}+\dfrac{6}{x_1x_3x_4}+\dfrac{6}{x_2x_3x_4}=\dfrac{6(x_1+x_2+x_3+x_4)}{x_1x_2x_3x_4}$

$=\dfrac{6\left(-\frac{1}{3}\right)}{-\frac{1}{2}}=4$

Exercise 3N

1 a $x^3+x^2-4x-4=(x-1)(x^2+2x+4)=(x-1)(x+2)^2=0$

$\therefore x=1,\ x=-2$

b $x^3+2x^2-9x-18=(x+2)(x^2-9)=(x+2)(x+3)(x-3)=0$

$\therefore x=-3,\ x=-2,\ x=3$

c $x^3-3x^2+3x-2=(x-2)(x^2-x+1)=0$

$x^2-x+1=0$ exhibits no real solution since $\Delta=1^2-4=-3<0$
$\therefore x=2$

d $x^4+x^3-3x^2-x+2=(x+1)(x^3-3x+2)=(x-1)^2(x-2)=0$

$\therefore x=1,\ x=2$

2 a $(-2)^3+(-2)^2+a(-2)-4=0\Rightarrow a=-4$

b $x^3+x^2-4x-4=(x+2)(x^2-x-2)=(x+2)(x-2)(x+1)=0$

So the remaining roots are $x=2$ and $x=-1$

3 **a** $2x^3 + ax^2 + bx + 9 = (x-3)^2(cx+d) = (x^2 - 6x + 9)(cx+d)$

Coefficient of x^3 : $2 = c$

Units: $9 = 9d \Rightarrow d = 1$

$\therefore 2x^3 + ax^2 + bx + 9 = (x^2 - 6x + 9)(2x+1)$

$= 2x^3 - 11x^2 + 12x + 9$

$\therefore a = -11,\ b = 12$

b From the factorised form, deduce that the remaining root is $x = -\frac{1}{2}$

4 **a** Let the three roots be $-x_1, x_1$ and x_2

$(x - x_1)(x + x_1)(x - x_2 = 0)$

$(x^2 - x_1{}^2)(x - x_2) = 0$

$x^3 - x_2x^2 - x_1{}^2x + x_1{}^2x_2 = 0$

$a = -x_2, b = -x_1{}^2, c = x_1{}^2x_2$

$\Rightarrow ab = c$

b From part a, $x_2 = -a$

Exercise 3O

1 **a** $x^3 + x^2 - 4x - 4 < 0$

$\Rightarrow -1 < x < 2\ ,\ \ x < -2$

b $x^3 + 2x^2 - 9x - 18 > 0$

$\Rightarrow x > 3,\quad -3 < x < -2$

c $x^3 - 3x^2 + 3x - 2 \geq 0$

$\Rightarrow x \leq 2$

d $4x^3 + 8x^2 + x - 3 \geq 0$

$\Rightarrow x \geq \frac{1}{2},\quad -\frac{3}{2} \leq x \leq -1$

e $3x^3 + 4x^2 + 7x + 2 > 0$

$\Rightarrow x > -\frac{1}{3}$

f $12x^3 - 16x^2 - 81x - 35 \geq 0$

$\Rightarrow x > \frac{7}{2},\quad -\frac{5}{3} < x < -\frac{1}{2}$

g $x^4 + x^3 - 3x^2 - x + 2 \geq 0$

$\Rightarrow x \geq 1,\ \ -1 \leq x \leq 1,\ \ x \leq -2$

h $2x^4 - x^3 + x^2 - x - 1 < 0$

$-\frac{1}{2} < x < 1$

2 $f(x) \le g(x)$

$\Rightarrow 2x^3 + 3x^2 - 2x - 5 \le 0$
$\Rightarrow x < 1.17227...$

3 **a** Rearrange to give $x^3 > -x - 2$

Plot $y = x^3$ and $y = -2 - x$ and using a GDC or by inspection deduce $x > -1$

b Rearrange to give $x^2 + 1 \ge 2x^3$

Plot $y = 2x^3$ and $y = x^2 + 1$ and using a GDC or by inspection deduce $x \le 1$

c Rearrange to give $x^4 \le 2 - x^2$

Plot $y = x^4$ and $y = 2 - x^2$ and using a GDC or by inspection deduce $-1 \le x \le 1$

4 **a** $x^5 - 4x^3 + 2x + 1 \ge 0$

$\Rightarrow x > 1.78897...,\quad -1.8947... < x < 1$

b $x^{13} + 4 < 3x^8 + 5x$

$0.746571... < x < 1.27299...,\quad x < -1.09526$

c $x^{15} + 2x^{14} + 5x^8 > 4x^2 - 1$

$-0.505312 < x < 0.50533....$

Exercise 3P

1 **a** Multiplying the second equation by 2,

$14x + 6y = 10$
Therefore, subtracting this from the first equation,
$(m - 14)x = -12$
$\Rightarrow$ No solution if $m = 14$

b Multiply the first equation by $m : m(m-1)x + 2my = -m$
Multiply the second equation by 2: $6x + 2my = 22$
Subtracting these equations, $[m(m-1) - 6]x = -m - 22$
$\therefore$ No solution if $m(m-1) - 6 = m^2 - m - 6 = (m-3)(m+2) = 0$
$\Rightarrow m = 3$ or $m = -2$

2 **a** Multiplying the second equation by 2,

$6x + 2y = 8$

so by comparison with the first equation, there are infinitely many solutions if $p = 2$

b Equating ratios of coefficients and constants:

$$\therefore \frac{p-4}{p} = \frac{p-1}{-1} = \frac{-3}{p}$$

$$\Rightarrow \frac{p-4}{p} = \frac{p(1-p)}{p} = \frac{-2}{p}$$

$$\left.\begin{array}{l}(1):(p-4)(-1) = p(p-1)\\ (2):(p-1)p = -3(-1)\\ (3):(p-4)p = 3p\end{array}\right\}\text{Have only one common solution, } p = 1$$

$\Rightarrow p = 1$ gives infinitely many solutions.

3 a Multiply the first equation by 3

$6x - 3sy = 3$

Multiply the second equation by 2,

$6x - 2y = 4$

Subtracting these equations,

$(2-3s)y = -1$

Therefore for a unique solution we need

$$2 - 3s \neq 0 \Rightarrow s \neq \frac{3}{2}$$

Then, $y = -\dfrac{1}{2-3s} = \dfrac{1}{3s-2}$

and accordingly

$$x = \frac{1}{2}(1+sy) = \frac{1}{2}\left(1 + \frac{s}{3s-2}\right) = \frac{1}{2}\left(\frac{4s-2}{3s-2}\right) = \frac{2s-1}{3s-2}$$

b Rearranging the first equation,

$y = s - (s+2)x$

Substituting this into the second equation

$(5-2s)x + s[s - (s+2)x] = 4$

$\Rightarrow [(5-2s) - s(s+2)]x = 4 - s^2$

$\Rightarrow [-s^2 - 4s + 5]x = 4 - s^2$

$\therefore s \neq 1,\ s \neq -5$

$$\Rightarrow x = \frac{4-s^2}{-s^2-4s+5} = \frac{s^2-4}{s^2+4s-5}$$

$$\Rightarrow y = s - \frac{(s+2)(s^2-4)}{s^2+4s-5} = \frac{s^3+4s^2-5s-(s^3+2s^2-4s-8)}{s^2+4s-5}$$

$$= \frac{2s^2-s+8}{s^2+4s-5}$$

4 a Adding the equations,

$$(2a+2b)x = 2 \Rightarrow x = \frac{1}{a+b}$$

Subtracting the equations,

$(2a-2b)y = 0 \Rightarrow y = 0$

b The system does not have a solution when $a+b=0$

Exercise 3Q

1 $a=3, b=1+i, c=4-i, d=-(1-i), e=4+5i, f=7$

$D=ad-bc=3(i-1)-(4-i)(1+i)=-8$
$x_{num}=ed-fb=-(4+5i)(1-i)-7(1+i)=-16-8i$
$y_{num}=af-ec=3(7)-(4+5i)(4-i)=-16i$
$\begin{pmatrix} x \\ y \end{pmatrix}=-\frac{1}{8}\begin{pmatrix} -16-8i \\ -16i \end{pmatrix}=\begin{pmatrix} 2+i \\ 2i \end{pmatrix}$

2 $a=(1+4i), b=3i, c=(3-5i), d=(5+4i), e=(2+4i), f=(21-27i)$

$D=ad-bc=(1+4i)(5+4i)-3i(3-5i)=-26+12i$
$x_{num}=ed-fb=(5+4i)(2+4i)-3i(21-27i)=-87-35i$
$y_{num}=af-ec=(5i-3)(2+4i)+(1+4i)(21-27i)=103-56i$
$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{-26+12i}\begin{pmatrix} -87-35i \\ 103-56i \end{pmatrix}$

Exercise 3R

1 **a** Eliminating y from the first two equations:

$3x+(m-1)z=6 \Rightarrow x+\frac{m-1}{3}z=2$
and from the second two equations,
$5x+(2m-1)z=5 \Rightarrow x+\frac{2m-1}{5}z=1$
Eliminating x,
$\left(\frac{2m-1}{5}-\frac{m-1}{3}\right)z=-1$
$\Rightarrow \frac{6m-3-5m+5}{15}z=-1$
$\Rightarrow \frac{m+2}{15}z=-1$
$\therefore m \neq -2$

b Eliminating x from first two equations,

$(1-3m)y+(m-3)z=-5 \quad (1)$
and from the first and third:
$(2-m)y=2 \Rightarrow m \neq 2$
and substituting this into equation (1)
$\Rightarrow m \neq 3$
so $m \neq 2$ and $m \neq 3$

2 **a** Multiplying the first equation by 3,

$6x + 9y - 3z = 3$
so comparing this with the second equation,
$k \neq 6$

b $k \neq \frac{3}{2}$ but this gives no solution, not infinitely many solutions

(LHS of first and third equations become the same, RHS differ)
$k = 2$ gives infinitely many solutions as then the second
and third equations are the same

3 Take two times the third equation from the first:

$-y + (1-2m)z = m + 6 \quad (\text{Eq4})$
Take two times the second equation from the first:
$(1-2m)y - z = -m - 2 \quad (\text{Eq5})$
Multiply Eq4 by $(1-2m)$ and add to Eq5
$\therefore \left[(1-2m)^2 - 1\right]z = (m+6)(1-2m) - m - 2 = -(2m+1)(m-6)$
$\Rightarrow 4m(m-1)z = -(2m+1)(m-6)$
$\therefore$ No solution if $m = 0$ or $m = 1$ and also $m = \frac{1}{2}$ from Eq5

4 **a** $9a - 3b + c = 1$

$4a - 2b + c = -5$
$a + b + c = 4$
Gaussian elimination on this system of equations gives
$a = \frac{9}{4},\ b = \frac{21}{4},\ c = -\frac{7}{2}$

b $a - b + c = 1$

$a + b + c = -9$
$4a + 2b + c = 8$
Gaussian elimination on this system of equation gives
$a = \frac{22}{3},\ b = -5,\ c = -\frac{34}{3}$

Chapter review

1 $x(x - a - b) = 1 - ab$

$\Rightarrow x^2 - (a+b)x + ab - 1 = 0$
$\Delta = (a+b)^2 - 4(ab-1)$
$= a^2 + 2ab + b^2 - 4ab + 4$
$= a^2 - 2ab + b^2 + 4$
$= (a-b)^2 + 4 > 0$
so there are two distinct real solutions for all $a, b \in \mathbb{R}$

2 **a** $(1-i)^3 + (1+i)^3 = \left(1 - 3i + 3(-i)^2 - i^3\right) + \left(1 + 3i + 3(-i)^2 + i^3\right)$

$= (1 - 3i - 3 + i) + (1 + 3i - 3 - i)$
$= (-2 - 2i) + (2i - 2)$ □
$= -4$

b $\omega = \dfrac{i^{64} i^{63} \ldots i^{18}}{1 - 2i} = \dfrac{i^{64} i^{63} i^{62} i^{61} \ldots i^{19} i^{18}}{1 - 2i} = \dfrac{[1 \cdot i \cdot i^2 \cdot i^3]^{11} i^{19} i^{18}}{1 - 2i}$

$= \dfrac{(-1)^{11} i^3 i^2}{1 - 2i} = \dfrac{-(-i)(-1)}{1 - 2i} = \dfrac{-i}{1 - 2i} = -\dfrac{i(1 + 2i)}{1^2 + 2^2} = -\dfrac{i - 2}{5} = \dfrac{2}{5} - \dfrac{i}{5}$

c If ω is a root of $f(x)$ then so is ω^*

$\therefore f(x) = C(x + 4)\left(x - \dfrac{2 + i}{5}\right)\left(x - \dfrac{2 - i}{5}\right)$
$= C(x + 4)\left(x^2 - \dfrac{4}{5}x + \dfrac{1}{5}\right)$
$= \dfrac{C}{5}(x + 4)(5x^2 - 4x + 1)$
$= \dfrac{C}{5}(5x^3 + 16x^2 - 15x + 4)$
so set, for example, $C = 5$ to obtain integers coefficients
e.g. $f(x) = 5x^3 + 16x^2 - 15x + 4$

3 a $a(-2 - ay) + y = 2$

$(-a^2 + 1)y = 2 + 2a$
$\therefore -a^2 + 1 \neq 0 \Rightarrow a \neq \pm 1$
$x = -2 - \dfrac{2a}{1 - a} = \dfrac{-2 + 2a - 2a}{1 - a} = \dfrac{-2}{1 - a} = \dfrac{2}{a - 1}$
$y = \dfrac{2(1 + a)}{1 - a^2} = \dfrac{2(1 + a)}{(1 - a)(1 + a)} = \dfrac{2}{1 - a}$

b i $a = 1$ since then $x + y = 2$ and $x + y = -2,$ which is not possible

ii $a = -1$

$y = x + 2$

4 a $\dfrac{(1 + i)^{2019}}{(1 - i)^{2017}} = \dfrac{(1 + i)^{2019+2017}}{(1 - i)^{2017}(1 + i)^{2017}} = \dfrac{(1 + i)^{4036}}{2^{2017}} = \dfrac{\left((1 + i)^4\right)^{1009}}{2^{2017}}$

$\dfrac{(-4)^{1009}}{2^{2017}} = -\dfrac{2^{2018}}{2^{2017}} = -2$

b $\dfrac{(1 + i)^{n+2}}{(1 - i)^n} = \dfrac{(1 + i)^{2n+2}}{2^n} = \dfrac{\left((1 + i)^2\right)^{n+1}}{2^n} = \dfrac{(2i)^{n+1}}{2^n} = 2i^{n+1}$

$\therefore n + 1$ must be odd, so n must be even

5 $x^5 + 3x^4 - 2x^2 + 2x + 4 \geq x^4 + 3x^3 + 2x^2 + 3$

$x^5 + 2x^4 - 3x^3 - 4x^2 + 2x + 1 \geq 0$

Use your GDC:

$-2.45 \le x \le -1.26$,
$-0.339 \le x \le 0.715$,
$1.34 \le x$ (3s.f.)

6 $\begin{cases} 3x - 2y + z = 1 \\ 6x + 8y - 3z = 6 \\ -12x + 4y - 7z = 4 \end{cases}$

Eliminating x from the first two equations,
$12y - 5z = 4 \quad (1)$
and from the second two equations,
$20y - 13z = 16 \quad (2)$
Three times (2) minus five times (1),
$-14z = 28 \Rightarrow z = -2$
$\Rightarrow y = -\frac{1}{2}$
$\Rightarrow x = \frac{2}{3}$

7 **a** 2

b □ $(f(x))^2 = 9x^4 + 12x^3 - 26x^2 - 20x + 25$

$= (3x^2 + 2x - 5)^2 = ((x-1)(3x+5))^2$
$\therefore \alpha = 1,\ \beta = -\frac{5}{3}$
$\therefore \frac{1}{\alpha} + \frac{1}{\beta} = 1 - \frac{3}{5} = \frac{2}{5}$

8 $\begin{cases} z^* - i\omega = 3 \\ \omega^* + iz = 6 - i \end{cases} \Rightarrow \begin{cases} z + i\omega^* = 3 \\ -z + i\omega^* = 1 + 6i \end{cases}$

$\therefore 2i\omega^* = 4 + 6i$
$\Rightarrow 2i(a - bi) = 4 + 6i$
$\therefore a = 3,\ b = 2$ so $\omega = 3 + 2i$
$\Rightarrow z = 3 - i(3 - 2i) = 1 - 3i$

9 Using sum of a geometric sequence,

$1 - z + z^2 = \frac{(-z)^3 - 1}{-z - 1} = 0 \Rightarrow z^3 = -1$
$\therefore z^{2019} = (z^3)^{673} = (-1)^{673} = -1$

Exam-style questions

10 a $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (1 mark)

$x = \dfrac{6 \pm \sqrt{208}}{2}$ (1 mark)

$x = 3 \pm \sqrt{52}$ (1 mark)

$x = 3 \pm 2\sqrt{13}$ (1 mark)

b Using sketch or table

$3 - 2\sqrt{13} \le x \le 3 + 2\sqrt{13}$ (2 marks)

11 a $8x^2 + 6x - 5 = 0$

$(4x+5)(2x-1) = 0$ (2 marks)

$4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$ (1 mark)

$2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$ (1 mark)

b $8x^2 + 6x - 5 - k = 0$

No real solutions $\Rightarrow b^2 - 4ac < 0$ (1 mark)

$36 - 4 \times 8 \times (-5 - k) < 0$ (1 mark)

$36 + 32(5 + k) < 0$

$5 + k < -\dfrac{36}{32}$

$k < -\dfrac{36}{32} - 5$

$k < -\dfrac{9}{8} - \dfrac{40}{8}$

$k < -\dfrac{49}{8}$ (1 mark)

12 Two real roots implies $b^2 - 4ac > 0$ (1 mark)

$(k\sqrt{3})^2 - 4(3k)(3) > 0$ (1 mark)

$3k^2 - 36k > 0$

$k^2 - 12k > 0$

$k(k - 12) > 0$ (1 mark)

Critical values are $k = 0$ and $k = 12$ (1 mark)

Solution is $k<0$ or $k>12$ (2 marks)

13 a Using sketch of $y=(x+4)(3-x)$ (1 mark)

Correct sketch or table

Solution is $-4<x<3$ (2 marks)

b $2x^2-11x+9<0$

$(2x-9)(x-1)<0$ (1 mark)

Using sketch of $y=(2x-9)(x-1)$ (1 mark)

Correct sketch or table

Solution is $1<x<\frac{9}{2}$ (2 marks)

c Comparing answers from **a** and **b** gives

$1<x<3$ (1 mark)

14 Let $w=a+b\mathrm{i}$ where $a,b\in\mathbb{R}$

$(a+b\mathrm{i})^2=77-36\mathrm{i}$ (1 mark)

$a^2-b^2+2ab\mathrm{i}=77-36\mathrm{i}$ (1 mark)

Equating reals: $a^2-b^2=77$ (1) (1 mark)

Equating imaginary: $2ab=-36$ (2) (1 mark)

(2) gives $b=-\frac{18}{a}$

Substitute in (1): $a^2-\left(-\frac{18}{a}\right)^2=77$ (1 mark)

$a^2-\frac{324}{a^2}=77$

$a^4-77a^2-324=0$

Attempting to factorise, or using the quadratic formula:

$(a^2+4)(a^2-81)=0$ (1 mark)

Since $a\in\mathbb{R}$, $a^2=81$ (1 mark)

$a=\pm 9$

$a=9\Rightarrow b=-2$

$a=-9\Rightarrow b=2$

So $w=\pm(9-2\mathrm{i})$ (2 marks)

15 Let $p(x)=2x^3+ax^2-10x+b$

$p(1)=0\Rightarrow 2+a-10+b=0$ (2 marks)

$a+b=8$ (1) (1 mark)

$p(-2)=15 \Rightarrow -16+4a+20+b=15$

$4a+b=11$ (2) (2 marks)

Solving equations (1) and (2) simultaneously: (1 mark)

$a=1$ (1 mark)

$b=7$ (1 mark)

16 Since $3-\mathrm{i}$ is a zero, its conjugate is also a zero, i.e. $(3-\mathrm{i})^*=3+\mathrm{i}$ is a zero. (1 mark)

By the factor theorem,

$[z-(3-i)]$ is a factor of f , and $[z-(3+i)]$ is a factor of f . (1 mark)

Therefore $[z-(3-i)][z-(3+i)]=z^2-6z+10$ is also a factor of f. (2 marks)

Writing $z^4-8z^3+48z^2-176z+260=[z^2-6z+10][z^2+kz+26]$

Equating coefficients of z^2 gives $48=26-6k+10$ (2 marks)

So $k=-2$ (1 mark)

$z^2-2z+26=0$

$$z=\frac{-(-2)\pm\sqrt{(-2)^2-4\times1\times26}}{2}=\frac{2\pm\sqrt{-100}}{2}=\frac{2\pm10\mathrm{i}}{2}=1\pm5\mathrm{i}$$ (2 marks)

The zeros are therefore $(3\pm\mathrm{i})$ and $(1\pm5\mathrm{i})$.

17 a Sum of roots $=-\left(\frac{-4}{5}\right)=\frac{4}{5}$ (2 marks)

Product of roots $=(-1)^4\left(-\frac{1}{5}\right)=-\frac{1}{5}$ (2 marks)

b Sum of roots $=-\left(\frac{-4}{5}\right)=\frac{4}{5}$ (2 marks)

Product of roots $=(-1)^5\left(\frac{10}{5}\right)=-2$ (2 marks)

18 Suppose $w=a+b\mathrm{i}$ and $z=c+d\mathrm{i}$ for $a,b,c,d\in\mathbb{R}$ (1 mark)

Then $w^*=a-b\mathrm{i}$ and $z^*=c-d\mathrm{i}$ (1 mark)

So $wz^*-zw^*=(a+b\mathrm{i})(c-d\mathrm{i})-(c+d\mathrm{i})(a-b\mathrm{i})$

$=[ac+bd+\mathrm{i}(bc-ad)]-[ac+bd+\mathrm{i}(ad-bc)]$ (1 mark)

$=\mathrm{i}(bc-ad)-\mathrm{i}(ad-bc)$ (1 mark)

$=\mathrm{i}(bc-ad)+\mathrm{i}(bc-ad)$

$= 2(bc - ad)\mathrm{i}$

which is purely imaginary (1 mark)

19 At $(-1, -5)$: $a - b + c = -5$ (1 mark)

At $(3, -1)$: $9a + 3b + c = -1$ (1 mark)

At $(10, -71)$: $100a + 10b + c = -71$ (1 mark)

Solving simultaneously using GDC: (1 mark)

$a = -1$ (1 mark)

$b = 3$ (1 mark)

$c = -1$ (1 mark)

4 Measuring change: differentiation

Skills check

1 $7x^{\frac{1}{2}}; 2x^{-3}; \frac{8}{5}x^{\frac{2}{3}}$

2 Vertical asymptote: x = − 3

Horizontal asymptote: y = 0

y-intercept: $\left(0, \frac{2}{3}\right)$

3 $S_\infty = 5 \times \frac{1}{1-\frac{1}{2}} = 5 \times 2 = 10$

Exercise 4A

1 $\lim\limits_{x\to-2} \frac{x^2-4}{x+2} = \lim\limits_{x\to-2} \frac{(x-2)(x+2)}{x+2} = \lim\limits_{x\to-2}(x-2) = -4$

2 $\lim\limits_{x\to3} \frac{x^2-9}{x-3} = \lim\limits_{x\to3} \frac{(x-3)(x+3)}{x-3} = \lim\limits_{x\to3}(x+3) = 6$

3 □ $\lim\limits_{x\to2^-}(x-3) = -1$ whereas $\lim\limits_{x\to2^+}(x+1) = 3$ so the limit does not exist

4 $\lim\limits_{x\to1} \frac{x^3-1}{x-1} = \lim\limits_{x\to1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim\limits_{x\to1}(x^2+x+1) = 3$

5 $\lim\limits_{x\to1^-}(x+1) = 2$ and $\lim\limits_{x\to1^+}(x^2+1) = 2$ so $\lim\limits_{x\to1} f(x) = 2$

6 $\lim\limits_{x\to6}\left((x-6)^{\frac{2}{3}}\right) = 0$

7 $\lim\limits_{x\to2^-}[x] = 1$ whereas $\lim\limits_{x\to2^+}[x] = 2$ so the limit does not exist

Exercise 4B

1 $f(3) = \lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^+} f(x) = 7$ so the function is continuous □

2 $f(2) = \lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^+} f(x) = 3$ so the function is continuous □

3 $\lim\limits_{x\to2^+} f(x) = \frac{x-2}{|x-2|} = 1$ and $\lim\limits_{x\to2^-} f(x) = \frac{x-2}{|x-2|} = -1$ hence the function is not continuous.

4 $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = 2a$ but $f(a) = 2a$ so the function is continuous

5 $\lim\limits_{x\to1^-} f(x) = \frac{2}{3}$ and $\lim\limits_{x\to1^+} f(x) = f(1) = \frac{2}{3}$, and $f(1) = \frac{2}{3}$, hence f is continuous at $x=1$.

$\lim\limits_{x\to-2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = \infty$ but $f(-2) = 4$

so f is not continuous at $x = -2$.

6 $\lim\limits_{x\to-2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$\Rightarrow (-2)^2 - 2 = 3k(-2)$

$\Rightarrow 2 = -6k \Rightarrow k = -\dfrac{1}{3}$

7 The function is already continuous for $x \geq 3$ and for $x > 3$

Since the functions $f_1(x) = kx^2 - k$ and $f_2(x) = 4$ are both continuous on their domains, it remains to find the value of k that insures that the function is continuous at x=3

i.e. $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^-} f(x)$

$\Rightarrow k(3)^2 - k = 4 \Rightarrow k = \dfrac{1}{2}$

8 **a** $f(x) = \dfrac{x^2+9}{(x+3)(x-3)}$

$\therefore$ Discontinuous at $x = \pm 3$

b $f(x) = \dfrac{x+1}{1-x^2} = \dfrac{1}{1-x}$ so discontinuous at $x = 1$

c Continuous

d $f(x) = \dfrac{(x+2)^2+1}{(x+5)(x-1)}$

Discontinuous at $x = 1$ and $x = -5$

e $f(x) = \dfrac{x^2}{(x-1)(x^2+x+1)}$

Discontinuous at $x = 1$

f Continuous

Exercise 4C

1 **a** $\lim\limits_{x\to 3} \dfrac{x+2}{x-2} = \dfrac{3+2}{3-2} = 5$

b $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x-1} = \lim\limits_{x\to 1}(x+2) = 3$

c $\lim\limits_{x\to\sqrt[3]{3}} \dfrac{x^6-9}{x^3-3} = \lim\limits_{x\to\sqrt[3]{3}} \dfrac{(x^3-3)(x^3+3)}{(x^3-3)} = 6$

d $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-2x} = \lim\limits_{x\to 2} \dfrac{x+2}{x} = 2$

e $\lim_{x\to 0}\frac{(x+2)(x-2)}{x(x-2)}=\lim_{x\to 0}\frac{x+2}{x}$ limit does not exist

f $\lim_{x\to 2}\frac{2}{2+\frac{2}{2-x}}=\lim_{x\to 2}\frac{2(2-x)}{2(2-x)+2}=\lim_{x\to 2}\frac{2(2-x)}{2(3-x)}=0$

g $\lim_{x\to 0}\frac{(2+3x)^2-4(1+x)^2}{6x}=\lim_{x\to 0}\frac{4+12x+9x^2-4-8x-4x^2}{6x}$

$=\lim_{x\to 0}\frac{4x+5x^2}{6x}=\lim_{x\to 0}\frac{4+5x}{6}=\frac{2}{3}$

h $\lim_{x\to a}\frac{a^2x^2-b^2}{ax-b}=\lim_{x\to a}\frac{(ax+b)(ax-b)}{ax-b}=\lim_{x\to a}(ax+b)=a^2+b$

2 a $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{3x}{x+3}=\lim_{x\to\infty}\frac{3}{1+\frac{3}{x}}=3$

b $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{-2}{1-\frac{1}{x^2}}=-2$

c $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{-1+\frac{1}{x}+\frac{2}{x^2}}{3+\frac{2}{x}-\frac{1}{x^2}}=-\frac{1}{3}$

d Limit does not exist

e $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\frac{2}{x}+\frac{3}{x^2}}{5-\frac{2}{x^2}}=0$

f $\lim_{x\to\infty}f(x)=\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{3}{x}-\frac{2}{x^2}}=0$

g Limit does not exist

h $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\left(\sqrt{4-\frac{1}{x}}+2\sqrt{1+\frac{3}{x}}\right)=4$

3 a Vertical Asymptote: $6x-1=0\Rightarrow x=\frac{1}{6}$

Horizontal Asymptote:

$\lim_{x\to\infty}\frac{3x}{6x-1}=\lim_{x\to\infty}\frac{3}{6-\frac{1}{x}}=\frac{1}{2}$ so $y=\frac{1}{2}$

b Vertical Asymptote: $x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$

Horizontal Asymptote:

$$\lim_{x\to\infty} \frac{x^2+3}{3-x^2} = \lim_{x\to\infty} \frac{1+\frac{3}{x^2}}{\frac{3}{x^2}-1} = -1 \text{ so } y = -1$$

c Vertical Asymptote: $x^3 - 1 = 0 \Rightarrow (x-1)(x^2+x+1) = 0 \Rightarrow x = 1$

Horizontal Asymptote:

$$\lim_{x\to\infty} \frac{1-x-x^3}{x^3-1} = \lim_{x\to\infty} \frac{\frac{1}{x^3}-\frac{1}{x^2}-1}{1-\frac{1}{x^3}} = -1 \text{ so } y = -1$$

d Vertical Asymptote: $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$

Horizontal Asymptote:

$$\lim_{x\to\infty}\left[-\frac{5x}{x^2-2}\right] = \lim_{x\to\infty}\left[-\frac{\frac{5}{x}}{1-\frac{2}{x^2}}\right] = 0 \text{ so } y = 0$$

e Vertical Asymptote: x=0

Horizontal Asymptote: None

f $r(x) = \dfrac{x^2}{2x^2-3x+1} = \dfrac{x^2}{(2x-1)(x-1)}$

Vertical Asymptotes: $x = \frac{1}{2}$ and $x = 1$

Horizontal Asymptotes

$$\lim_{x\to\infty} \frac{x^2}{2x^2-3x+1} = \lim_{x\to\infty} \frac{1}{2-\frac{3}{x}+\frac{1}{x^2}} = \frac{1}{2}$$

Exercise 4D

1 a Divergent

b Divergent

c Convergent: $u_n = \frac{1}{n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

d Convergent: $u_n = \frac{1}{3^n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^n} = 0$

2 a Converges: $\lim_{n\to\infty} \dfrac{n+2}{n} = \lim_{n\to\infty} \dfrac{1+\frac{2}{n}}{1} = 1$

b Converges: $\lim_{x\to\infty} \dfrac{n+2}{2n+3} = \lim_{n\to\infty} \dfrac{1+\frac{2}{n}}{2+\frac{3}{n}} = \dfrac{1}{2}$

c Converges: $\lim_{n\to\infty}\frac{n^2-n}{2n^2+\sqrt{n}}=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{2+\frac{1}{n\sqrt{n}}}=\frac{1}{2}$

d Diverges

e $\lim_{n\to\infty}\frac{2n^2+1}{1-2n^3}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{1}{n^3}}{\frac{1}{n^3}-2}=0$

f $\lim_{n\to\infty}\frac{1+n^2}{1-n^3}=\lim_{n\to\infty}\frac{\frac{1}{n^3}+\frac{1}{n}}{\frac{1}{n^3}-1}=0$

3 **a** Converges since ratio of the geometric series is $\left|-\frac{1}{3}\right|<1$

$$S_\infty=1\cdot\frac{1}{1-\left(-\frac{1}{3}\right)}=\frac{3}{4}$$

b Converges since the ratio of the geometric series is $\left|\frac{1}{2}\right|<1$

$$S_\infty=\frac{3}{2}\cdot\frac{1}{1-\frac{1}{2}}=3$$

c Converges since the ratio of the geometric series is $\left|\frac{1}{10}\right|<1$

$$S_\infty=2\cdot\frac{1}{1-\frac{1}{10}}=\frac{20}{9}$$

d Converges since the ratio of both geometric series is $\left|\frac{3}{5}\right|<1$ and $\left|\frac{2}{5}\right|<1$

$$S_\infty=\frac{3}{5}\cdot\frac{1}{1-\frac{3}{5}}-\frac{2}{5}\cdot\frac{1}{1-\frac{2}{5}}=\frac{3}{2}-\frac{2}{3}=\frac{5}{6}$$

e Converges since $0<e<\pi\Rightarrow$ the ratio of the geometric series is $\left|\frac{e}{\pi}\right|<1$

$$S_\infty=\frac{1}{1-\frac{e}{\pi}}=\frac{\pi}{\pi-e}$$

f Diverges since $\pi>3.14>0\Rightarrow$ the ratio of the geometric series $\left|\frac{\pi}{3.14}\right|>1$

4 **a** $2^x>0$ for all real x so need to solve $2^x<1$

$\Rightarrow x<0$

b $S_\infty = \dfrac{42}{1-2^x} = 48$

$\Rightarrow \dfrac{7}{8} = 1 - 2^x$

$\Rightarrow 2^x = \dfrac{1}{8}$

$\Rightarrow x = -3$

5 $\left|\dfrac{3x}{x+1}\right| < 1 \Rightarrow |3x| < |x+1|$

Solve $|3x| = |x+1|$

$\therefore 3x = x+1$ or $-3x = x+1$

$\Rightarrow x = \dfrac{1}{2}$ or $x = -\dfrac{1}{4}$

By sketching graphs, deduce that

$-\dfrac{1}{4} < x < \dfrac{1}{2}$

6 Let $S_N = \displaystyle\sum_{n=0}^{N} \frac{1+2^n}{3^{n-1}}$

Then,

$S_N = 3\displaystyle\sum_{n=0}^{N}\left(\frac{1}{3}\right)^n + 3\sum_{n=0}^{N}\left(\frac{2}{3}\right)^n$

$= 3\dfrac{1-\left(\frac{1}{3}\right)^N}{1-\frac{1}{3}} + 3\dfrac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}$

$= \dfrac{9}{2}\left[1-\left(\dfrac{1}{3}\right)^{N+1}\right] + 9\left[1-\left(\dfrac{2}{3}\right)^{N+1}\right]$

$\therefore S_\infty = \displaystyle\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left\{\frac{9}{2}\left[1-\left(\frac{1}{3}\right)^{N+1}\right] + 9\left[1-\left(\frac{2}{3}\right)^{N+1}\right]\right\}$

$= \dfrac{9}{2} + 9 = \dfrac{27}{2}$

so the infinite sum converges, and is equal to $\dfrac{27}{2}$

Exercise 4E

1 a $f'(-1) = \displaystyle\lim_{h\to 0}\frac{\left[2(-1+h)^2+1\right]-\left[2(-1)^2+1\right]}{h}$

$= \displaystyle\lim_{h\to 0}\frac{2\left(h^2-2h+1\right)+1-3}{h}$

$= \displaystyle\lim_{h\to 0}\frac{2h^2-4h}{h} = \lim_{h\to 0}(2h-4) = -4$

b $f'(1) = \displaystyle\lim_{h\to 0}\frac{\left[1-3(1+h)^2\right]-\left[1-3(1)^2\right]}{h}$

$$= \lim_{h\to 0}\frac{1-3\left(h^2+2h+1\right)+2}{h}$$

$$= \lim_{h\to 0}\frac{-3h^2-6h}{h} = \lim_{h\to 0}\left(-3h-6\right) = -6$$

c $$f'(-1) = \lim_{h\to 0}\frac{\frac{2}{-1+h}-\frac{2}{-1}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{2}{h-1}+2}{h} = \lim_{h\to 0}\frac{2+2(h-1)}{h(h-1)} = \lim_{h\to 0}\frac{2h}{h^2-h}$$

$$= \lim_{h\to 0}\frac{2}{h-1} = -2$$

d $$f'(-1) = \lim_{h\to 0}\frac{-(-1+h)^2-\left[-(-1)^2\right]}{h}$$

$$= \lim_{h\to 0}\frac{-h^2+2h-1+1}{h}$$

$$= \lim_{h\to 0}\left(-h+2\right) = 2$$

e $$f'(-1) = \lim_{h\to 0}\frac{(-1+h)^3-(-1)^3}{h}$$

$$= \lim_{h\to 0}\frac{h^3-3h^2+3h-1+1}{h}$$

$$= \lim_{h\to 0}\left(h^2-3h+3\right) = 3$$

f $$f'(0) = \lim_{h\to 0}\frac{h^2+h-1-(-1)}{h}$$

$$= \lim_{h\to 0}\left(h+1\right) = 1$$

g $$f'(2) = \lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{2^2}}{h}$$

$$= \lim_{h\to 0}\frac{4-(2+h)^2}{4h(2+h)^2} = \lim_{h\to 0}\frac{-h^2-4h}{4h(2+h)^2}$$

$$= \lim_{h\to 0}\frac{-h(h+4)}{4h(2+h)^2} = -\frac{1}{4}\lim_{h\to 0}\frac{h+4}{(2+h)^2}$$

$$= -\frac{1}{4}$$

h $$f'(0) = \lim_{h\to 0}\frac{\frac{h}{h+1}-0}{h} = \lim_{h\to 0}\frac{1}{h+1} = 1$$

2 Gradient of line AB:

$$\frac{(1+h)^2+3-(1+3)}{h}=\frac{h^2+2h+1+3-4}{h}=h+2$$

This becomes the gradient of the tangent to $f(x)$ i.e. $f'(x)$ in the limit $h \to 0$

$\therefore f'(1)=\lim_{h\to 0}(h+2)=2$

3 **a** $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{\left[3(x+h)^2+2(x+h)-1\right]-\left[3x^2+2x-1\right]}{h}$$
$$=\lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h-1-3x^2-2x+1}{h}$$
$$=\lim_{h\to 0}\frac{6xh+3h^2+2h}{h}$$
$$=\lim_{h\to 0}(6x+2+3h)=6x+2$$

b $f'(x)=-4 \Rightarrow 6x+2=-4 \Rightarrow x=-1$

$\therefore \left(-1, 3(-1)^2+2(-1)-1\right)=(-1,0)$

4 $f'(x)=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

$$=\lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}$$
$$=\lim_{h\to 0}\frac{-2x-h}{x^4+2x^3h+h^2x^2}$$
$$=\lim_{h\to 0}\frac{-2x}{x^4}=-\frac{2}{x^3}$$

$\therefore$ If the gradient is $-\frac{1}{4}$,

$-\frac{2}{x^3}=-\frac{1}{4} \Rightarrow x^3=8 \Rightarrow x=2$

$\therefore \left(2, \frac{1}{4}\right)$

Exercise 4F

1 a $f'(x) = \lim_{h\to 0}\frac{(x+h)^2 + (x+h) - 2 - x^2 - x + 2}{h}$

$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + x + h - 2 - x^2 - x + 2}{h}$

$= \lim_{h\to 0}\frac{2xh + h^2 + h}{h}$

$= \lim_{h\to 0}(2x + h + 1) = 2x + 1$

$\therefore f'(0) = 1$

b $f'(x) = \lim_{h\to 0}\frac{2 - (x+h) + 3(x+h)^2 - 2 + x - 3x^2}{h}$

$= \lim_{h\to 0}\frac{2 - x - h + 3x^2 + 6xh + 3h^2 - 2 + x - 3x^2}{h}$

$= \lim_{h\to 0}\frac{(-1+6x)h + 3h^2}{h}$

$= \lim_{h\to 0}(-1 + 6x + 3h) = -1 + 6x$

$\therefore f'(-1) = -7$

c $f'(x) = \lim_{h\to 0}\frac{-\frac{2}{x+h} + \frac{2}{x}}{h}$

$= \lim_{h\to 0}\frac{-2x + 2(x+h)}{hx(x+h)} = \lim_{h\to 0}\frac{-2x + 2x + 2h}{hx^2 + h^2x}$

$= \lim_{h\to 0}\frac{2}{x^2 + hx} = \frac{2}{x^2}$

$\therefore f'(1) = 2$

d $f'(x) = \lim_{h\to 0}\frac{\sqrt{x+1+h} - \sqrt{x+1}}{h}$

$= \lim_{h\to 0}\frac{\sqrt{x+1+h} - \sqrt{x+1}}{h}\cdot\frac{\sqrt{x+1+h} + \sqrt{x+1}}{\sqrt{x+1+h} + \sqrt{x+1}}$

$= \lim_{h\to 0}\frac{x+1+h-(x+1)}{h\sqrt{x+1+h} + \sqrt{x+1}} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+1+h} + \sqrt{x+1}\right)}$

$= \lim_{h\to 0}\frac{1}{\left(\sqrt{x+1+h} + \sqrt{x+1}\right)} = \frac{1}{2\sqrt{x+1}}$

$\therefore f'(3) = \frac{1}{4}$

e $f'(x) = \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$

$$= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} = \lim_{h\to 0}\frac{\left(\sqrt{x}-\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}\left(\sqrt{x}+\sqrt{x}\right)} = -\frac{1}{2x\sqrt{x}}$$

$$\therefore f'(9) = -\frac{1}{2\cdot 9\cdot 3} = -\frac{1}{54}$$

f $f'(x) = \lim_{h\to 0}\frac{(x+h)^3 - 1 - x^3 + 1}{h}$

$$= \lim_{h\to 0}\frac{3hx^2 + 3h^2x + h^3}{h}$$

$$= \lim_{h\to 0}\left(3x^2 + 3hx + h^2\right)$$

$$= 3x^2$$

$$\therefore f'(1) = 3$$

2 **a** $v_{avg} = \frac{8 + 2(a+h)^2 - 8 - 2a^2}{h} = \frac{4ah + 2h^2}{h} = 4a + 2h$

b In the limit $h \to 0$, $v_{avg} \to v_A = \lim_{h\to 0}(4a + 2h) = 4a$

3 **a** $v = \lim_{h\to 0}\frac{10(t+h)^2 - (t+h)^3 - 10t^2 - t^3}{h}$

$$= \lim_{h\to 0}\frac{10t^2 + 20th + 10h^2 - t^3 - 3ht^2 - 3h^2t - h^3 - 10t^2 + t^3}{h}$$

$$= \lim_{h\to 0}\frac{20th + 10h^2 - 3ht^2 - 3h^2t - h^3}{h}$$

$$= \lim_{h\to 0}\left(20t - 3t^2 + 1 - h - h^2 - 3ht\right)$$

$$= 20t - 3t^2$$

b $v(1) = 17$, $v(10) = -100$

The sign indicates the direction the particle moves in. At $t = 1$, the particle is moving in the positive direction and in the opposite direction at $t = 10$

Exercise 4G

1 **a** $f'(1) = \lim_{h\to 0}\frac{2(1+h)^2 - (1+h) + 1 - 2 + 1 - 1}{h}$

$$= \lim_{h\to 0}\frac{2\left(h^2 + 2h + 1\right) - (h+1) - 1}{h} = \lim_{h\to 0}\frac{2h^2 + 3h}{h}$$

$$= \lim_{h\to 0}(2h + 3) = 3$$

b $y = f(1) = 2$

$y - 2 = 3(x-1) \Rightarrow y = 3x - 1$

c The normal has gradient $-\dfrac{1}{3}$ and also passes through $(1,2)$

$\therefore y - 2 = -\dfrac{1}{3}(x-1) \Rightarrow y = -\dfrac{1}{3}x + \dfrac{7}{3}$

2 $f'(x) = \lim\limits_{h\to 0} \dfrac{\dfrac{1}{2-(x+h)} - \dfrac{1}{2-x}}{h} = \lim\limits_{h\to 0} \dfrac{2-x-(2-x-h)}{h(2-x)(2-x-h)}$

$= \lim\limits_{h\to 0} \dfrac{h}{h(2-x)(2-x-h)}$

$= \lim\limits_{h\to 0} \dfrac{1}{(2-x)(2-x-h)}$

$= \dfrac{1}{(2-x)^2}$

$\therefore f'(x) = 1 \Rightarrow \dfrac{1}{(2-x)^2} = 1 \Rightarrow (2-x)^2 = 1$

$\Rightarrow x = 1$ or $x = 3$

At $x = 1$, $y = \dfrac{1}{2-1} = 1$ so the tangent here is

$y - 1 = x - 1 \Rightarrow y = x$

At $x = 3$, $y = \dfrac{1}{2-3} = -1$ so the tangent here is

$y - (-1) = x - 3 \Rightarrow y = x - 4$

3 **a** $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{2(x+h)^2 - 1 - 2x^2 + 1}{h} = \lim\limits_{h\to 0} \dfrac{4xh + 2h^2}{h}$

$= \lim\limits_{h\to 0}(4x + 2h) = 4x$

so there exists a horzontal tangent at $x = 0 \Rightarrow (0,-1)$

b $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{2 - 3(x+h) - (x+h)^2 - 2 + 3x + x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{2 - 3x - 3h - x^2 - 2xh - h^2 - 2 + 3x + x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{(-3-2x)h - h^2}{h}$

$= \lim\limits_{h\to 0}[-3 - 2x - h] = -3 - 2x$

so there is a horizontal tangent at $x = -\dfrac{3}{2}$

i.e. at the point $\left(-\dfrac{3}{2}, \dfrac{17}{4}\right)$

c $\dfrac{dy}{dx} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - 1 - x^3 + 1}{h}$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-1-x^3+1}{h}$$
$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$
$$= \lim_{h\to 0}\left(3x^2+3xh+h^2\right)=3x^2$$

so there is a horizontal tangent when $x = 0$

i.e. at the point $(0,-1)$

d $$\frac{dy}{dx}=\lim_{h\to 0}\frac{(x+h)^3-3(x+h)-x^3+3x}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$$
$$= \lim_{h\to 0}\left(3x^2+3xh+h^2-3\right)$$
$$= 3x^2-3$$

so there is a horizontal tangent at $x = \pm 1$

i.e. at the points $(\pm 1, \mp 2)$

4 $$f'(1)=\lim_{h\to 0}\frac{1+h+\frac{1}{1+h}-1-1}{h}=\lim_{h\to 0}\frac{h-1+\frac{1}{1+h}}{h}$$

$$= \lim_{h\to 0}\frac{h^2-1+1}{h}$$
$$= \lim_{h\to 0} h = 0$$

At $x = 1,\ y = 1+\frac{1}{1}=2$

$\therefore$ Tangent: $y = 2$

Normal: $x = 1$

Exercise 4H

1 **a** $y=(3x-1)^2=9x^2-6x+1$

$$\frac{dy}{dx}=2\cdot 9x-6=18x-6$$

b $$\frac{dy}{dx}=5\cdot 3x^4-2\cdot 4x+2=15x^4-8x+2$$

c $$\frac{dy}{dx}=2\cdot\frac{1}{4}x+\frac{2}{3}=\frac{1}{2}x+\frac{2}{3}$$

d $y=5x^5-4x^3+x-\frac{1}{4}x^{-3}-\frac{1}{5}x^{-5}$

$$\frac{dy}{dx}=25x^4-12x^2+1+\frac{3}{4}x^{-4}+x^{-6}$$
$$=25x^4-12x^2+1+\frac{3}{4x^4}+\frac{1}{x^6}$$

e $$y=\frac{3-2x^3+x^4}{x}=3x^{-1}-2x^2+x^3$$

$$\frac{dy}{dx} = -3x^{-2} - 4x + 3x^2$$
$$= -\frac{3}{x^2} - 4x + 3x^2$$

f $y = \sqrt{x} = x^{\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

g $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2x\sqrt{x}}$$

h $y = \sqrt[5]{x^2} = x^{\frac{2}{5}}$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{5}x^{-\frac{3}{5}} = \frac{2}{5\sqrt[5]{x^3}}$$

i $y = \frac{2}{\sqrt[3]{x}} - \frac{3}{\sqrt{x^5}} = 2x^{-\frac{1}{3}} - 3x^{-\frac{5}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{3}x^{-\frac{4}{3}} + \frac{15}{2}x^{-\frac{7}{2}} = -\frac{2}{3\sqrt[3]{x^4}} + \frac{15}{2\sqrt{x^7}}$$

j $y = \left(1+\sqrt{x}\right)\left(3-\sqrt[3]{x}\right) = \left(1+x^{\frac{1}{2}}\right)\left(3-x^{\frac{1}{3}}\right)$

$$= 3 - x^{\frac{1}{3}} + 3x^{\frac{1}{2}} - x^{\frac{5}{6}}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2}x^{-\frac{1}{2}} - \frac{5}{6}x^{-\frac{1}{6}}$$

2 $y = -2\left(x^2+3x\right) = -2x^2 - 6x$

$\frac{dy}{dx} = -4x - 6$

$\therefore$ At $x = 1$,

$y = -8$ and $\frac{dy}{dx} = -10$

So the equation of the tangent at $(1,-8)$ is

$y - (-8) = -10(x-1)$

$\Rightarrow y = -10x + 2$

3 $y = \frac{x-3}{x} = 1 - 3x^{-1}$

$\therefore \frac{dy}{dx} = 3x^{-2}$

Therefore at $x = -1$,

$y = 4$ and $\frac{dy}{dx} = 3$

$\Rightarrow$ The gradient of the normal is $-\frac{1}{3}$

So the equation of the normal at $(-1,4)$ is

$$y-4=-\frac{1}{3}(x-(-1))$$

$$\Rightarrow y=-\frac{1}{3}x+\frac{11}{3}$$

$$\Rightarrow x+3y-11=0$$

4 $f'(x)=15x^2+24x-7$

Therefore at $x=1$,

$$y=f(1)=10 \text{ and } \frac{dy}{dx}=f'(1)=32$$

So the tangent at $(1,10)$ is

$$y-10=32(x-1)$$

$$\Rightarrow y=32x-22$$

At $x=-1$,

$$y=f(-1)=14 \text{ and } \frac{dy}{dx}=f'(-1)=-16$$

$$y-14=-16(x-(-1))$$

$$\Rightarrow y=-16x-2$$

5 $f'(x)=3x^2-10x+5$

$$\therefore f'(x)=2$$

$$\Rightarrow 3x^2-10x+5=2$$

$$\Rightarrow 3x^2-10x+3=0$$

$$\Rightarrow (3x-1)(x-3)=0$$

so $x=\frac{1}{3}$ or $x=3$

If $x=\frac{1}{3}$, $y=f\left(\frac{1}{3}\right)=-\frac{77}{27}$

so $y+\frac{77}{27}=2\left(x-\frac{1}{3}\right)$

$$\Rightarrow y=2x-\frac{77}{27}-\frac{18}{27}=2x-\frac{95}{27}$$

If $x=3$, $y=-7$

$$\therefore y-(-7)=2(x-3)$$

$$\Rightarrow y=2x-13$$

6 $f'(x)=2x-3$

$\therefore$ The normal at $x=1$ has gradient $-\frac{1}{2-3}=1$

$$y=f(1)=-1$$

$$\therefore y-(-1)=1(x-1)\Rightarrow y=x-2$$

$$\Rightarrow x^2-3x+1=x-2$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-3)(x-1)=0$$

$$\Rightarrow x=1 \text{ or } x=3$$

$\therefore$ The other point is $(3,f(3))=(3,1)$

7 $f'(x) = 3x^2 + 2x + 1$

The line has gradient $-\frac{1}{2}$

So, set $f'(x) = 2$

$\therefore 3x^2 + 2x + 1 = 2$

$\Rightarrow 3x^2 + 2x - 1 = 0$

$\Rightarrow (3x - 1)(x + 1) = 0$

$\Rightarrow \ x = -1$ or $x = \frac{1}{3}$

If $x = -1$,

$y = f(-1) = -2$

$\therefore y - (-2) = 2(x - (-1))$

$\Rightarrow y = 2x$

If $x = \frac{1}{3}$,

$y = f\left(\frac{1}{3}\right) = \frac{1}{27} + \frac{1}{9} + \frac{1}{3} - 1 = \frac{1 + 3 + 9 - 27}{27} = -\frac{14}{27}$

$\therefore y - \left(-\frac{14}{27}\right) = 2\left(x - \frac{1}{3}\right)$

$\Rightarrow 27y + 14 = 54x - 18$

$\Rightarrow -54x + 27y + 32 = 0$

Exercise 4I

1 **a** Let $u = 4x - 3$, then $y = u^5$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (5u^4)(4) = 20u^4 = 20(4x - 3)^4$$

b Let $u = 1 - 4x$, then $y = u^{\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(-4) = -2u^{-\frac{1}{2}} = -2(1 - 4x)^{-\frac{1}{2}}$$

c $y = \frac{2 - x^2 - 3x^5}{x} = 2x^{-1} - x - 3x^4$

$$\therefore \frac{dy}{dx} = -2x^{-2} - 1 - 12x^3 = -\frac{2 + x^2 + 12x^5}{x^2}$$

d Let $u = 1 - 3x^2$, then $y = -2u^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = u^{-\frac{3}{2}}(-6x) = -\frac{6x}{(1 - 3x^2)^{\frac{3}{2}}}$$

e $y = \left(\frac{1 - \sqrt{x}}{x}\right)^3 = \left(x^{-1} - x^{-\frac{1}{2}}\right)^3$

Let $u = x^{-1} - x^{-\frac{1}{2}}$, then $y = u^3$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2\left(-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}\right) = 3\left(x^{-1} - x^{-\frac{1}{2}}\right)^2\left(-x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}\right)$$

$$= 3\left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)^2\left(-\frac{1}{x^2} + \frac{1}{2x\sqrt{x}}\right)$$

$$= 3\left(\frac{\sqrt{x} - 1}{x}\right)^2\left(\frac{-2 + \sqrt{x}}{2x^2}\right)$$

$$= \frac{3\left(\sqrt{x} - 1\right)^2\left(\sqrt{x} - 2\right)}{2x^4}$$

f Let $u = 2x^2 - 4$, then $y = u^{\frac{1}{3}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{3}u^{-\frac{2}{3}}\right)(4x) = \frac{4x}{3\left(2x^2 - 4\right)^{\frac{2}{3}}} = \frac{2^{\frac{4}{3}}x}{3\left(x^2 - 2\right)^{\frac{2}{3}}}$$

2 $\dfrac{dy}{dx} = 6x - 12x^2$

Therefore the gradient at $x = 1$ is -6

$y = f(1) = -1$

$\therefore y - (-1) = -6(x - 1)$

$\Rightarrow y = -6x + 5$

3 $y = 1 - \dfrac{2}{x} = 1 - 2x^{-1}$

$\therefore \dfrac{dy}{dx} = 2x^{-2}$

The gradient at $x = -1 \Rightarrow$ the gradient of the normal to the curve at this point is $-\dfrac{1}{2}$

$y = f(-1) = \dfrac{-1 - 2}{-1} = 3$

$\therefore y - 3 = -\dfrac{1}{2}(x - (-1))$

$\Rightarrow y = -\dfrac{1}{2}x + \dfrac{5}{2}$

$\Rightarrow x + 2y - 5 = 0$

4 $y = \sqrt{2 - \sqrt{x}}$

Let $u = 2 - \sqrt{x}$, then $y = u^{\frac{1}{2}}$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{4\sqrt{x}\sqrt{2 - \sqrt{x}}}$$

5 $f'(x) = \dfrac{20}{x^2}$ and $g'(x) = 5$

$\therefore f'(x) = g'(x) \Rightarrow \frac{20}{x^2} = 5$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

6 **a** $f'(x) = 15ax^2 - 4bx + 4c$

b $f'(x) \geq 0 \Rightarrow \Delta \leq 0$

$\therefore (-4b)^2 - 4(15a)(4c) \leq 0$

$\Rightarrow 16b^2 \leq 240ac$

$\Rightarrow b^2 \leq 15ac$

7 **i** Let $f(x)$ be an even function i.e. $f(-x) = f(x)$

Let $u = -x$, then $f(-x) = f(u)$

$\therefore \frac{d}{dx}[f(x)] = f'(x)$

and since $f(x)$ is an even function,

$\frac{d}{dx}[f(x)] = \frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(u)] = \frac{du}{dx}\frac{d}{du}[f(u)] = -f'(u) = -f'(-x)$

$\therefore f'(x) = -f'(-x) \Rightarrow f'(-x) = -f(x)$

i.e. the derivative of $f(x)$ is an odd function

ii Let $g(x)$ be an odd function i.e. $g(-x) = -g(x)$

Let $u = -x$, then $g(-x) = g(u)$

Then,

$\frac{d}{dx}[g(-x)] = \frac{d}{dx}[-g(x)] = -\frac{d}{dx}[g(x)] = -g'(x)$

and also

$\frac{d}{dx}[g(-x)] = \frac{d}{dx}[g(u)] = \frac{du}{dx}\frac{d}{du}[g(u)] = -g'(u) = -g'(-x)$

$\therefore -g'(x) = -g'(-x)$

$\Rightarrow g'(-x) = g'(x)$

i.e. the derivative of $g(x)$ is an even function

Exercise 4J

1 Let $u = 2x - 3$ and $v = (x+3)^3$

Then,

$\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3(x+3)^2$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = 2(x+3)^3 + (2x-3)\cdot 3(x+3)^2$

$= (x+3)^2[2(x+3) + 3(2x-3)]$

$= (x+3)^2(8x-3)$

2 Let $u = (2x+3)^2$ and $v = (3-x)^3$

Then,

$\frac{du}{dx} = 4(2x+3)$ and $\frac{dv}{dx} = -3(3-x)^2$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = 4(2x+3)(3-x)^3 + (2x+3)^2 \cdot \left[-3(3-x)^2\right]$

$= (2x+3)(3-x)^2\left[4(3-x) - 3(2x+3)\right]$

$= (2x+3)(3-x)^2(3-10x)$

3 $y = \frac{x-1}{x+1} = (x-1)(x+1)^{-1}$

Let $u = x-1$ and $v = (x+1)^{-1}$

Then,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -(x+1)^{-2}$

$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x-1)\left[-(x+1)^{-2}\right] + (x+1)^{-1}$

$= \frac{1}{x+1} - \frac{x-1}{(x+1)^2}$

$= \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$

4 $y = x(2-3x)^{\frac{1}{2}}$

Let $u = x$ and $v = (2-3x)^{\frac{1}{2}}$

Then,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\frac{3}{2}(2-3x)^{-\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= (2-3x)^{\frac{1}{2}} - \frac{3}{2}(2-3x)^{-\frac{1}{2}}$

$= \frac{2(2-3x)-3}{2\sqrt{2-3x}} = \frac{4-9x}{2\sqrt{2-3x}}$

5 $y = \frac{1}{x^3 - 2x^2 + 3x + 1} = \left(x^3 - 2x^2 + 3x + 1\right)^{-1}$

$\Rightarrow \frac{dy}{dx} = -\left(3x^2 - 4x + 3\right)\left(x^3 - 2x^2 + 3x + 1\right)^{-2}$

$= -\frac{3x^2 - 4x + 3}{\left(x^3 - 2x^2 + 3x + 1\right)^2}$

6 $y = (x+1)^4(2-3x)^{\frac{2}{3}}$

Let $u=(x+1)^4$ and $v=(2-3x)^{\frac{2}{3}}$

Then,

$\frac{du}{dx}=4(x+1)^3$ and $\frac{dv}{dx}=-2(2-3x)^{-\frac{1}{3}}$

$\therefore \frac{dy}{dx}=u\frac{dv}{dx}+\frac{du}{dx}v$

$=4(x+1)^3(2-3x)^{-\frac{1}{3}}\left[2(2-3x)-(x+1)\right]$

$=2(x+1)^3(2-3x)^{-\frac{1}{3}}(3-7x)$

$=\frac{2(x+1)^3(3-7x)}{\sqrt[3]{2-3x}}$

7 $y=(2x-1)^3(4-x)^{-2}$

Let $u=(2x-1)^3$ and $v=(4-x)^{-2}$

Then,

$\frac{du}{dx}=6(2x-1)^2$ and $\frac{dv}{dx}=2(4-x)^{-3}$

$\therefore \frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}=6(2x-1)^2(4-x)^{-2}+2(2x-1)^3(4-x)^{-3}$

$=2(2x-1)^2(4-x)^{-3}\left[3(4-x)+(2x-1)\right]$

$=2(2x-1)^2(4-x)^{-3}(11-x)$

8 $y=\frac{1-2x}{\sqrt{3x^2+2}}=(1-2x)(3x^2+2)^{-\frac{1}{2}}$

Let $u=1-2x$ and $v=(3x^2+2)^{-\frac{1}{2}}$

Then,

$\frac{du}{dx}=-2$ and $\frac{dv}{dx}=-\frac{3x}{2}(3x^2+2)^{-\frac{3}{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}$

$=-2(3x^2+2)^{-\frac{1}{2}}-3x(1-2x)(3x^2+2)^{-\frac{3}{2}}$

$=-(3x^2+2)^{-\frac{3}{2}}\left[2(3x^2+2)+3x(1-2x)\right]$

$=-(3x^2+2)^{-\frac{3}{2}}(3x+4)$

9 $y=\frac{x^2+1}{x^2-3}=(x^2+1)(x^2-3)^{-1}$

Let $u=x^2+1$ and $v=(x^2-3)^{-1}$

Then,

$\frac{du}{dx}=2x$ and $\frac{dv}{dx}=-2x(x^2-3)^{-2}$

$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}$

$= 2x\left(x^2-3\right)^{-1} - 4x^2\left(x^2-3\right)^{-2}$

$= 2x\left(x^2-3\right)^{-2}\left[x^2-3-2x\right]$

$= 2x\left(x^2-3\right)^{-2}(x-3)(x+1)$

At $x = 1$,

$\frac{dy}{dx} = 2(-2)^{-2}(-2)(2) = -2$

Therefore the tangent has gradient -2 and the normal has gradient $\frac{1}{2}$

Tangent: $y - 0 = -2(x-1) \Rightarrow y = -2x + 2$

Normal: $y - 0 = \frac{1}{2}(x-1) \Rightarrow y = \frac{1}{2}x - \frac{1}{2}$

10 a $y = (x+1)^{\frac{1}{2}}(3-x)^2$

Let $u = (x+1)^{\frac{1}{2}}$ and $v = (3-x)^2$

Then,

$\frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ and $\frac{dv}{dx} = -2(3-x)$

$\therefore \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)^2 - 2(x+1)^{\frac{1}{2}}(3-x)$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)\left[(3-x) - 4(x+1)\right]$

$= \frac{1}{2}(x+1)^{-\frac{1}{2}}(3-x)(-1-5x)$

$= \frac{-(3-x)(5x+1)}{2\sqrt{x+1}}$

b $\frac{dy}{dx} = 0 \Rightarrow x = 3$ or $x = -\frac{1}{5}$

Exercise 4K

1 a $\frac{dy}{dx} = \frac{(5-x)(3)-(1+3x)(-1)}{(5-x)^2} = \frac{16}{(5-x)^2}$

b $\frac{dy}{dx} = \frac{(2-x)\left(\frac{1}{2\sqrt{x}}\right)-\sqrt{x}(-1)}{(2-x)^2} = \frac{(2-x)+\sqrt{x}\left(2\sqrt{x}\right)}{2\sqrt{x}(2-x)^2} = \frac{2+x}{2\sqrt{x}(2-x)^2}$

c $\frac{dy}{dx} = \frac{\sqrt{1-x^2}(2)-(1+2x)\frac{-x}{\sqrt{1-x^2}}}{1-x^2} = \frac{2\sqrt{1-x^2}+\frac{x(1+2x)}{\sqrt{1-x^2}}}{1-x^2}$

$$= \frac{2\left(1-x^2\right)+x\left(1+2x\right)}{\left(1-x^2\right)^{\frac{3}{2}}}$$

$$= \frac{2+x}{\left(1-x^2\right)^{\frac{3}{2}}}$$

d $\frac{dy}{dx} = \frac{\left(x^2+1\right)(3)-(1+3x)(2x)}{\left(x^2+1\right)^2} = \frac{3-2x-3x^2}{\left(x^2+1\right)^2}$

2 **a** $u = x^2-2; \frac{du}{dx} = 2x; v = x^3-1; \frac{dv}{dx} = 3x^2$

$$\frac{dy}{dx} = \frac{\left(x^3-1\right)(2x)-\left(x^2-2\right)\left(3x^2\right)}{\left(x^3-1\right)^2} = \frac{2x^4-2x-3x^4+6x^2}{\left(x^3-1\right)^2}$$

$$= \frac{-2x+6x^2-x^4}{\left(x^3-1\right)^2}$$

b $u = x^2+2x; \frac{du}{dx} = 2x+2; v = (x^3+1)^{\frac{1}{2}}; \frac{dv}{dx} = \frac{3x^2}{2}(x^3+1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{2(x+1)(x^3+1)^{\frac{1}{2}} - \frac{3x^2}{2}(x^3+1)^{-\frac{1}{2}}\left(x^2+2x\right)}{x^3+1}$$

$$\frac{dy}{dx} = \frac{(x^3+1)^{-\frac{1}{2}}\left[2(x+1)(x^3+1) - \frac{3x^2(x^2+2x}{2}\right]}{x^3+1}$$

$$\frac{dy}{dx} = \frac{2(x^3+1)^{-\frac{1}{2}}\left[(x+1)(x^3+1) - \frac{3x^2(x^2+2x}{4}\right]}{x^3+1}$$

$$\frac{dy}{dx} = \frac{x^4-2x^3+4x+4}{2\left(x^3+1\right)^{\frac{3}{2}}}$$

c $y = \frac{1}{\sqrt{x^4-x^5+2x^6}} = \left(x^4-x^5+2x^6\right)^{-\frac{1}{2}}$

$$u = 1; \frac{du}{dx} = 0; v = \left(x^4-x^5+2x^6\right)^{-\frac{1}{2}}; \frac{dv}{dx} = -\frac{1}{2}\left(x^4-x^5+2x^6\right)^{-\frac{3}{2}}\left(4x^3-5x^4+12x^5\right)$$

$$\therefore \frac{dy}{dx} = \frac{0-\left[-\frac{1}{2}\left(x^4-x^5+2x^6\right)^{-\frac{2}{3}}(4x^3-5x^4+12x^5)\right]}{\left(x^4-x^5+2x^6\right)}$$

$$= \frac{-\left(12x^2-5x+4\right)}{2x^3\left(2x^2-x+1\right)^{\frac{3}{2}}}$$

d $\frac{dy}{dx} = \frac{\left(1+\sqrt{x}\right)\left(-\frac{1}{2\sqrt{x}}\right)-\left(1-\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right)}{\left(1+\sqrt{x}\right)^2}$

$$= \frac{-\left(1+\sqrt{x}\right)-\left(1-\sqrt{x}\right)}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} = \frac{-2}{2\sqrt{x}\left(1+\sqrt{x}\right)^2} = -\frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$$

e $\dfrac{dy}{dx} = \dfrac{3\sqrt{\sqrt{x}-1} - (1+3x)\left(\dfrac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}\right)}{\sqrt{x}-1}$

$= \dfrac{12\sqrt{x}\left(\sqrt{x}-1\right) - (1+3x)}{4\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{3}{2}}}$

$= \dfrac{9x - 12\sqrt{x} - 1}{4\sqrt{x}\left(\sqrt{x}-1\right)^{\frac{3}{2}}}$

f $\dfrac{dy}{dx} = \dfrac{2}{3}\dfrac{1}{(x+2)^2}\left(1 - \dfrac{1}{2+x}\right)^{-\frac{1}{3}} = \dfrac{2}{3(x+1)^{\frac{1}{3}}(x+2)^{\frac{5}{3}}}$

3 $\dfrac{dy}{dx} = \dfrac{(x^2+1)(4) - (4x)(2x)}{(x^2+1)^2} = \dfrac{-4x^2+4}{(x^2+1)^2}$

At $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 4$ so the normal has gradient $-\dfrac{1}{4}$

$\therefore$ The normal has equation $y = -\dfrac{1}{4}x$

4 $y = 8\left(4+x^2\right)^{-1}$

$\Rightarrow \dfrac{dy}{dx} = -8(2x)\left(4+x^2\right)^{-2} = -\dfrac{16x}{\left(4+x^2\right)^2}$

At $x = 1$, $y = \dfrac{8}{5}$ and $\dfrac{dy}{dx} = -\dfrac{16}{25}$

$\therefore$ The normal has gradient $\dfrac{25}{16}$ and its equation is

$y - \dfrac{8}{5} = \dfrac{25}{16}(x-1) \Rightarrow y = \dfrac{25}{16}x - \dfrac{17}{80}$

5 $y = x^2 + x + 1 + \dfrac{1}{x}$

$\dfrac{dy}{dx} = 2x + 1 - \dfrac{1}{x^2} = \dfrac{2x^3 + x^2 - 1}{x^2}$

$\dfrac{dy}{dx} = 2$

$\Rightarrow 2x^3 + x^2 - 1 = 2x^2$

$\Rightarrow 2x^3 - x^2 - 1 = (x-1)\left(2x^2 + x + 1\right) = 0$

$x = 1$ is the only real solution and the corresponding coordinate is $(1, 4)$

Exercise 4L

1 $f(x) = 1 - 4x - x^{-1}$

$f'(x) = -4 + x^{-2}$

$f''(x) = -2x^{-3} = -\frac{2}{x^3}$

2 $f(x) = 3x^5 - 2x^2 + 1$

$f'(x) = 15x^4 - 4x$

$f''(x) = 60x^3 - 4$

$f'''(x) = 180x^2$

$f^{(4)}(x) = 360x$

$\therefore f^{(4)}(1) = 360$

3 $y = 1 - 3ax + 3a^2x^2 - a^3x^3$

$\frac{d^3y}{dx^3} = -6a^3 = 162 \Rightarrow a^3 = -27$

$\therefore a = -3$

4 $f'(x) = \frac{x^3}{3} - 4x + 5$

$f''(x) = x^2 - 4 = 0 \Rightarrow x = \pm 2$

5 $\frac{dy}{dx} = 4x^3 - 12x^2 + 16$

$\frac{d^2y}{dx^2} = 12x^2 - 24x$

$\frac{d^3y}{dx^3} = 24x - 24$

$\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} \Rightarrow 12x^2 - 24x = 24x - 24$

$\Rightarrow x^2 - 4x + 2 = 0$

$\Rightarrow (x-2)^2 = 2 \Rightarrow x = 2 \pm \sqrt{2}$

6 $f(x) = x^4 + px^2 + qx + r$

$f'(x) = 4x^3 + 2px + q$

$f''(x) = 12x^2 + 2p$

$f''(-1) = 16 \Rightarrow 12 + 2p = 16 \Rightarrow p = 2$

$f'(-1) = -16 \Rightarrow -4 - 2p + q = 16 \Rightarrow q = 16 + 4 + 2p = 24 \Rightarrow q = 24$

$f(-1) = 1 + p - q + r = 16 \Rightarrow r = 16 - 1 - p + q = 37 \Rightarrow r = 37$

7 **a** $s(t) = 4(t-2)^5$

$v(t) = 20(t-2)^4$

$\therefore v(3) = 20$

b $a(t) = 80(t-2)^3$

$\therefore a(3) = 80$

c $a'(t) = 240(t-2)^2$

$a'(1) = 240$ □

8 $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

$f'''(x) = -\frac{6}{x^4}$

$f^{(4)}(x) = \frac{24}{x^5}$

$f^{(5)}(x) = -\frac{120}{x^6}$

Conjecture that $P(n): f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$

The statement $P(1)$ is true (see above)

Assume that $P(k)$ is true for some $k \in \mathbb{Z}^+$

i.e. that $f^{(k)}(x) = (-1)^k \frac{k!}{x^{k+1}}$

Then,

$f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x)$

$= \frac{d}{dx}(-1)^k \frac{k!}{x^{k+1}}$

$= (-1)^k k! \frac{d}{dx}\left[x^{-(k+1)}\right]$

$= -(-1)^k k!(k+1)x^{-k-2}$

$= (-1)^{k+1} \frac{(k+1)!}{x^{k+1+1}}$

so $P(k) \Rightarrow P(k+1)$

Therefore, $P(1)$ has been shown to be true and if $P(k)$ is true for some $k \in \mathbb{Z}^+$ then so is $P(k+1)$. Hence, the statement is true for all positive integers by the principle of mathematical induction

9 $f(x) = (x-1)^4\left[(x+1)^4(1-2x^3)^3\right]$

Note that all derivatives up to and including the third derivative of $(x-1)^4$ will include a factor of $(x-1)$ and therefore disappear upon being evaluated at $x = 1$. Therefore, we only need to include the first term in Leibniz:

$$f'(1) = \frac{d^4\left[(x-1)^4\right]}{dx^4} \frac{d^0}{dx^0}\left[(x+1)^4(1-2x^3)^3\right]\Bigg|_{x=1}$$

$$= 24\left[(x+1)^4(1-2x^3)^3\right]\Big|_{x=1} = -384$$

Exercise 4M

1 **a** $f'(x) = 0 \Rightarrow f$ has a stationary point at x. So, $x = -3, -1, 2, 4$

b $f'(x) > 0 \Rightarrow f$ is increasing, so $x < -3, -1 < x < 2, x > 4$

c $f'(x) < 0 \Rightarrow f$ is decreasing, so $-3 < x < -1, 2 < x < 4$

d By inspection of the graph of $f, x \le -3, -1 \le x \le 2, x > 4$

e By inspection of the graph of f, f' is decreasing at $x < -2, -0.5 < x < 3$

2 **a** **i** By inspection of the graph of f, $x < -3, x > 4$

ii By inspection of the graph of f, $-3 < x < 4$

iii By inspection of the graph of f, $x = -3, 4$

b **i** By inspection of the graph of f, $x < -1, 0 < x < 1, 2 < x$

ii By inspection of the graph of f, $-1 < x < 0, 1 < x < 2$

iii By inspection of the graph of f, $x = -1, 0, 1, 2$

3 **a** $y = 2x^3 - 6x^2 + 3$

i $\frac{dy}{dx} = 6x^2 - 12x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$

$f'(-0.5) < 0, f'(0.5) > 0 \Rightarrow (0, 3)$ is a maximum.
$f'(1.5) > 0, f'(2.5) < 0 \Rightarrow (2, -5)$ is a minimum.

ii $6x^2 - 12 > 0 \quad \Rightarrow x < 0$ or $x > 2$

iii $6x^2 - 12 < 0 \Rightarrow 0 < x < 2$

iv Both turning points are local.

b $y = -3x^4 + 2x^3 + 3x^2 - 4$

i □ $\frac{dy}{dx} = -12x^3 + 6x^2 + 6x = 6x(-2x^2 + x + 1) = 6x(-2x - 1)(x - 1)$

$\therefore \frac{dy}{dx} = 0$ when $x = -\frac{1}{2}$, $x = 0$ or $x = 1$

$f'(-1) > 0; f'(-\tfrac{1}{4}) < 0 \Rightarrow \left(-\frac{1}{2}, -\frac{59}{16}\right)$ is a maximum

$f'(-0.5) < 0; f'(0.5) > 0 \Rightarrow (0, -4)$ is a minimum

$f'(0.5) > 0; f'(1.5) < 0 \Rightarrow (1, -2)$ is a maximum

ii $\frac{dy}{dx} > 0$ when $x < -\frac{1}{2}$ or $0 < x < 1$

iii $\frac{dy}{dx} < 0$ when $-\frac{1}{2} < x < 0$ or $x > 1$

iv $\left(-\frac{1}{2}, -\frac{59}{16}\right)$ and (0,-4) are local turning points, and (1,-3) is a global maximum.

c $y = 2\left(3 - 2x - x^2\right)^{-1}$

i $\therefore \frac{dy}{dx} = -2(-2 - 2x)\left(3 - 2x - x^2\right)^{-2} = 4(1 + x)\left(3 - 2x - x^2\right)^{-2}$

$\frac{dy}{dx} = 0$ when $x = -1$

By testing points either side of $x = -1$ in $\frac{dy}{dx}$, $\left(-1, \frac{1}{2}\right)$ is a minimum

ii $\frac{dy}{dx} > 0$ when $x > -1$ $(x \neq 1)$

iii $\frac{dy}{dx} < 0$ when $x < -1$ $(x \neq -3)$

iv $\left(-1, \frac{1}{2}\right)$ is a local minimum.

d $y = \frac{3x + 3}{3x - x^2}$

i □ $\frac{dy}{dx} = \frac{\left(3x - x^2\right)(3) - (3x + 3)(3 - 2x)}{\left(3x - x^2\right)^2}$

$= \frac{3x^2 + 6x - 9}{\left(3x - x^2\right)^2} = \frac{3(x - 1)(x + 3)}{\left(3x - x^2\right)^2}$

$\therefore \frac{dy}{dx} = 0$ when $x = -3$ or $x = 1$

By testing points to the left and right of these x-values in f',

$\left(-3, \frac{1}{3}\right)$ is a maximum and

$(1, 3)$ is a minimum

ii $\frac{dy}{dx} > 0$ when $x < -3$ or $x > 1$ $(x \neq 3)$

iii $\frac{dy}{dx} < 0$ when $-3 < x < 1$ $(x \neq 0)$

iv Both turning points are local.

e $y = x + (x - 1)^{-1}$

i $\frac{dy}{dx} = 1 - (x - 1)^{-2} = 1 - \frac{1}{(x - 1)^2} = \frac{(x - 1)^2 - 1}{(x - 1)^2}$

$= \frac{x^2 - 2x}{(x - 1)^2} = \frac{x(x - 2)}{(x - 1)^2}$

so $\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$

By testing points to the left and to the right of these values in f',

$(0, -1)$ is a maximum and

$(2, 3)$ is a minimum

ii $\frac{dy}{dx} > 0$ when $x < 0$ or $x > 2$

iii $\frac{dy}{dx} < 0$ when $0 < x < 2$ $(x \neq 1)$

iv The turning points are local.

4 $f(x) = 4 + 5x - x^2 - x^3$

$f'(x) = 5 - 2x - 3x^2 = 0$
$\Rightarrow 3x^2 + 2x - 5 = (3x + 5)(x - 1) = 0$
$\Rightarrow x = -\frac{5}{3}$ or $x = 1$

$\therefore$ By testing points to the right and left of these values in f' $\left(-\frac{5}{3}, -\frac{67}{27}\right)$
is a local minimum and $(1, 7)$ is a local maximum.
but $f(-3) = 7$ and $f(3) = -17$ so the local minimum is not
the least value of $f(x)$ in the given interval:
$\therefore f_{\text{min}} = -17$
$f_{\text{max}} = 7$

5 $f(x) = x^2 + \frac{4}{x}$

As $x = 0$ is in the interval, the global maximum and minimum are $\pm\infty$.

6 $\frac{dy}{dx} = 3x^2 + 2ax = 0 \Rightarrow x(3x + 2a) = 0$
$\Rightarrow x = 0; x = \frac{-2a}{3}$

So the turning points are at x=0 and x=4.

$x = 4 \Rightarrow 4 = -\frac{2a}{3} \Rightarrow a = -6$

$\therefore 48 + 8a = 0 \Rightarrow a = -6$
$\Rightarrow f(4) = -11 \Rightarrow 4^3 - 6(4^2) + b = -11$
$\Rightarrow b = 21$
$x = 0 \Rightarrow (0, b) = (0, 21)$

7 The graph passes through $(1, 1)$

$\therefore 1 + a + b + c = 1 \Rightarrow a + b + c = 0 \quad (1)$
The graph has turning points at $x = -1$ and $x = 3$
$\frac{dy}{dx} = 3x^2 + 2ax + b$
$\therefore 3 - 2a + b = 0 \Rightarrow 2a - b = 3 \quad (2)$
and

$27 + 6a + b = 0 \Rightarrow 6a + b = -27 \quad (3)$

Solving (2) and (3) simultaneously,

$a = -3$ and $b = -9$ and therefore

$c = -(a + b) = 12$

Exercise 4N

1 a $\frac{dy}{dx} = 7x^6 - 7 = 0 \Rightarrow x = \pm 1$

$\frac{d^2y}{dx^2} = 42x^5$

$42(1)^5 > 0 \Rightarrow (1, -6)$ is a local minimum

$42(-1)^5 < 0 \Rightarrow (-1, 6)$ is a local maximum

b $\frac{dy}{dx} = 20x^3 - 5x^4 = 0 \Rightarrow x^3(20 - 5x) = 0 \Rightarrow x = 0$ or $x = 4$

$\frac{d^2y}{dx^2} = 60x^2 - 20x^3$

$60(4)^2 - 10(4^3) < 0 \Rightarrow (4, 256)$ is a local maximum

At $x = 0$, the 2nd derivative test is inconclusive, so resort to 1st derivative test by testing points to the left and right of $x = 0$, in the first derivative.

$f'(-0.5) < 0; f'(0.5) > 0 \Rightarrow (0,0)$, hence $(0,0)$ is a minimum.

c $\frac{dy}{dx} = 4 - \frac{1}{x^2} = 0 \Rightarrow x = \pm\frac{1}{2}$

$\frac{d^2y}{dx^2} = \frac{2}{x^3}$

$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} > 0 \Rightarrow \left(\frac{1}{2}, 5\right)$ is local minimum by second derivative test

$\left.\frac{d^2y}{dx^2}\right|_{x=-\frac{1}{2}} < 0 \Rightarrow \left(-\frac{1}{2}, -3\right)$ is a local maximum by second derivative test

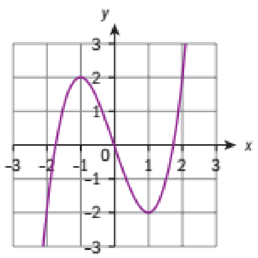
2 a $f'(x) = 3x^2 - b$

$f'(1) = 3 - b = 0 \Rightarrow b = 3$

b $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

$\therefore (1, -2)$ and $(-1, 4)$

c



3 a $f'(x) = 2ax + b = 0$

$\Rightarrow x = -\dfrac{b}{2a}$

b Local minimum if $a > 0$, local maximum if $a < 0$

Justified as a direct consequence of the second derivative test since $f''(x) = 2a$

4 a $f'(x) = 3x^2 + b$

$f'(1) = 3 + b = 0 \Rightarrow b = -3$
$f(1) = 1 + b + c = c - 2 = 4 \Rightarrow c = 6$

b $f(x) = x^3 - 3x + 6$
$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$
so there is another turning point at
$(-1, 8)$

Exercise 4O

1 a $f'(x) = 3x^2 - 1$

$f''(x) = 6x$
$f''(0) = 0$ and $y = f(x)$ changes concavity through this point so
$(0,0)$ is a point of inflexion

b $x > 0$

c $x < 0$

2 a $f'(x) = 4x^3 - 3$

$f''(x) = 12x^2$
$f''(0) = 0$ but $y = f(x)$ does not change concavity through this point so no points of inflexion

b $x \in \mathbb{R} \setminus \{0\}$

c $\varnothing$

3 a $f'(x) = 3x^2 - 12x - 12$

$f''(x) = 6x - 12 = 0 \Rightarrow x = 2$
$y = f(x)$ changes concavity through this point so
$(2, -38)$ is a point of inflexion

b $x > 2$

c $x < 2$

4 **a** $f'(x) = 3x^2 + 2x$

$f''(x) = 6x + 2 = 0 \Rightarrow x = -\frac{1}{3}$

$y = f(x)$ changes concavity through this point so

$\left(-\frac{1}{3}, -\frac{25}{27}\right)$ is a point of inflexion

b $x > -\frac{1}{3}$

c $x < -\frac{1}{3}$

5 **a** $f'(x) = 12x^2 - 4x^3$

$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$

The concavity of $y = f(x)$ changes through both of these points so

$(0,0)$ and $(2,16)$ are both points of inflexion

b $0 < x < 2$

c $x < 0$ or $x > 2$

6 **a** $f'(x) = 3x^2 - 6x + 3$

$f''(x) = 6x - 6 = 0 \Rightarrow x = 1$

$y = f(x)$ changes concavity through this point so

$(1,0)$ is a point of inflexion

b $x > 1$

c $x < 1$

7 **a** $f'(x) = 8x^3 + 3x^2$

$f''(x) = 24x^2 + 6x = 6x(4x + 1) = 0$

$\Rightarrow x = -\frac{1}{4}$ or $x = 0$

The concavity of $y = f(x)$ changes through both of these points

so $\left(-\frac{1}{4}, \frac{127}{128}\right)$ and $(0,1)$ are both points of inflexion

b $x < -\frac{1}{4}$ or $x > 0$

c $-\frac{1}{4} < x < 0$

8 **a** $f'(x) = 4x^3 - 12x^2 + 16$

$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$

$\Rightarrow x = 0$ or $x = 2$

The concavity of $y = f(x)$ changes through both of these points

so $(0,-16)$ and $(2,0)$ are points of inflexion

b $x < 0$ or $x > 2$

c $0 < x < 2$

9 a $f'(x) = 3ax^2 + 2bx + c$

since there are two distinct stationary points, the discriminant of this quadratic is positive

$\therefore \Delta = (2b)^2 - 4(3a)(c) > 0$

$\Rightarrow 4b^2 - 12ac > 0$

$\Rightarrow b^2 > 3ac$

b $f''(x) = 6ax + 2b = 0 \Rightarrow x = -\dfrac{b}{3a}$

$(a \neq 0$ by construction since $f(x)$ is a cubic$)$

$$f'(x) = 0 \Rightarrow x = \frac{-2b \pm \sqrt{4b^2 - 4(3a)(c)}}{2(3a)} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

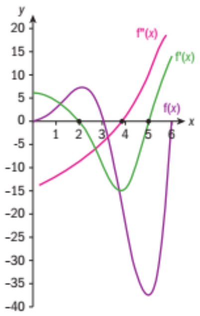
$$\therefore \text{Let } x_1 = \frac{-b + \sqrt{b^2 - 3ac}}{3a} \text{ and } x_2 = \frac{-b - \sqrt{b^2 - 3ac}}{3a}$$

$$\Rightarrow \frac{x_1 + x_2}{2} = \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 3ac} - b - \sqrt{b^2 - 3ac}}{3a}\right) = -\frac{b}{3a}$$

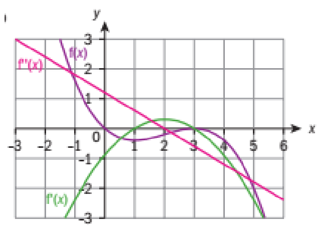
as required

Exercise 4P

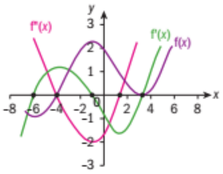
1 a



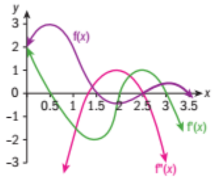
b



c

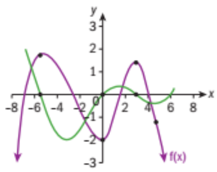


d

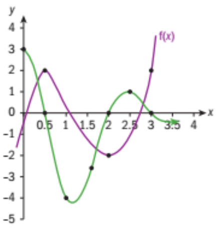


Exercise 4Q

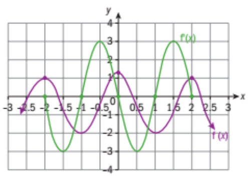
1 a



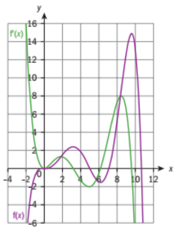
b



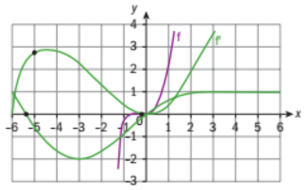
c



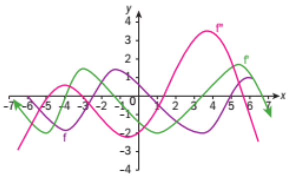
d



2 **a**



b



Exercise 4R

1 Let the length of the fence opposite the wall be denoted by x

Since there is 800m of fencing available, the remaining sides must each measure $\frac{800-x}{2}$

Therefore the area of the enclosed rectangular plot is

$A(x) = x\left(\frac{800-x}{2}\right) = 400x - \frac{x^2}{2}$

$\therefore A'(x) = 400 - x = 0 \Rightarrow x = 400$

This is a minimum, which can be justified by the shape of the graph or by showing $A''(400) > 0$

$\therefore A_{\text{min}} = A(400) = 400(400) - \frac{400^2}{2} = 80000$ so $80{,}000 \text{ m}^2$

2 Let the two side lengths be denoted by x and y, side x being opposite the river

Since the area enclosed is 200m^2, $xy = 200 \Rightarrow y = \frac{200}{x}$

and the total length of fencing required is

$L(x) = x + 2y = x + \frac{400}{x}$

$\therefore L'(x) = 1 - \frac{400}{x^2} = 0 \Rightarrow x^2 = 400 \Rightarrow x = 20 \ (x > 0)$

$L''(20) > 0$ so this is a minimum

$\therefore L_{\text{min}} = L(20) = 20 + \frac{400}{20} = 40$ so 40m

3 Let the radius of the semicircle be denoted by r

Then the horizontal edge of the window measures $2r$ and since the perimeter of the window is 12m, if the vertical edges of the window are length x,

$2r + 2x + \pi r = 12 \Rightarrow x = \frac{12-(2+\pi)r}{2}$

$\therefore$ The area of the window is

$A(r) = 2xr + \frac{1}{2}\pi r^2 = [12 - (2+\pi)r]r + \frac{1}{2}\pi r^2$

$= 12r + \left[\frac{\pi}{2} - (2+\pi)\right]r^2 = 12r - \left(\frac{\pi}{2} + 2\right)r^2$

$\therefore A'(r) = 12 - (\pi + 4)r = 0 \Rightarrow r = \frac{12}{4+\pi}$

This is a minimum, which can be justified either

by considering the shape of the graph or by showing

$A''\left(\frac{12}{4+\pi}\right) > 0$

$\therefore r = \frac{12}{4+\pi} = 1.68\ldots$ so radius is 1.68m (3s.f.)

and the dimensions of the rectangle are

3.36 m $\times$ 1.68 m (3s.f.)

4 Let L be the length of the wire, then:

$L = \sqrt{36 + x^2} + \sqrt{196 + (20 - x)^2}$

$\frac{dL}{dx} = \frac{x}{\sqrt{36+x^2}} - \frac{20-x}{\sqrt{196-(20-x^2)}}$

To minimize the length of L, set the first derivative equal to 0, i.e.,

$\frac{x}{\sqrt{36+x^2}} - \frac{20-x}{\sqrt{196-(20-x^2)}} = 0$

$\Rightarrow x = 6$

5 Let the side length of the congruent squares be denoted by x

After cutting out the congruent squares, the sides of the open rectangular box

will measure $(24-2x)$cm by $(45-2x)$cm by x cm

Therefore the volume of the box is

$V(x) = x(24-2x)(45-2x) = -4x^3 - 138x^2 + 1080x$

$\therefore V'(x) = -12x^2 - 276x + 1080 = 0$

$\Rightarrow x^2 + 23x - 90 = 0$

$\Rightarrow (x-18)(x-5) = 0$

$\Rightarrow x = 18$ or $x = 5$

$x = 18$ is not feasible and $V''(5) < 0$ so

$V_{\text{min}} = V(5) = 5(24-10)(45-10) = 5(14)(35) = 2450$

6 Let the height and radius of the cylindrical can respectively be h and r.

Then, since the surface area is 3π m^2,

$2\pi rh + \pi r^2 = 3\pi \Rightarrow r^2 + 2rh - 3 = 0 \Rightarrow h = \frac{3-r^2}{2r}$

Therefore the volume of the can is

$V(r) = \pi r^2 h = \pi r^2 \dfrac{3-r^2}{2r} = \dfrac{\pi}{2}\left(3r - r^3\right)$

$\therefore V'(r) = \dfrac{\pi}{2}\left(3 - 3r^2\right) = 0 \Rightarrow r = 1 \ (r > 0)$

$\therefore r = 1, \ h = \dfrac{3-1}{2} = 1$ so the radius and height are both 1m and

$V_{\min} = V(1) = \pi$ so the volume is π m^3

7 $1l = \dfrac{1}{1000} m^3$

Let the radius of the cylinder be r and the height be h
Then,

$\pi r^2 h = \dfrac{1}{1000} \Rightarrow h = \dfrac{1}{1000\pi r^2}$

$S(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{1}{500r}$

$S'(r) = 4\pi r - \dfrac{1}{500r^2} = 0 \Rightarrow r^3 = \dfrac{1}{2000\pi} \Rightarrow r = \left(\dfrac{1}{2000\pi}\right)^{\frac{1}{3}}$

$\therefore S_{\min} = S\left(\left(\dfrac{1}{2000\pi}\right)^{\frac{1}{3}}\right) = 0.055358\ldots$

Therefore the minimum surface area is 0.055358...m^3 or 554cm^3 to 3s.f.

8 Let the radius of the right-circular cone be r. Accordingly, by Pythagoras',

the height of the cone is $h = 10 + \sqrt{100 - r^2}$

$\Rightarrow r^2 = 100 - (h-10)^2 = 20h - h^2$

$\therefore V(h) = \dfrac{\pi r^2 h}{3} = \dfrac{\pi h}{3}\left(20h - h^2\right) = \dfrac{\pi}{3}\left(20h^2 - h^3\right)$

$\Rightarrow V'(h) = \dfrac{\pi}{3}\left(40h - 3h^2\right) = \dfrac{\pi h}{3}(40 - 3h) = 0$

$h \neq 0$ so $h = \dfrac{40}{3}$ and $V''\left(\dfrac{40}{3}\right) < 0$

$\therefore V_{\max} = V\left(\dfrac{40}{3}\right) = 1241.12\ldots.$

so 1240cm^3 (3s.f.)

9 A general point on the curve has coordinates $\left(x, \sqrt{x}\right)$

By Pythagoras', the distance from $(1.5, 0)$ is

$D(x) = \sqrt{\left(x - \dfrac{3}{2}\right)^2 + \left(\sqrt{x}\right)^2} = \sqrt{x^2 - 2x + \dfrac{9}{4}}$

$= \sqrt{(x-1)^2 + \dfrac{5}{4}} \geq \sqrt{\dfrac{5}{4}} = \dfrac{\sqrt{5}}{2}$

so $D_{\min} = \dfrac{\sqrt{5}}{2}$

10 Let the radii of the circles be r

Then the pieces of wire for each circle measure $2\pi r$ and the piece of wire for the square measures $80-4\pi r$ so the square has side length $20-\pi r$
Therefore the area of each circle is πr^2 and the area of the square is $(20-\pi r)^2$

So the combined area is

$A(r)=2\pi r^2+(20-\pi r)^2=(2\pi+\pi^2)r^2-40\pi r+400$

$\therefore A'(r)=(4\pi+2\pi^2)r-40\pi=0\Rightarrow r=\dfrac{20}{2+\pi}$

and $A''\left(\dfrac{20}{2+\pi}\right)>0$ so this is a minimum

11 Let the poster have height h and width w

Then, $hw=320$
and the total area is

$A(h)=(h+10)(w+8)=hw+8h+10w+80=400+8h+10w$

$=400+8h+\dfrac{3200}{h}$

$\therefore A'(h)=8-\dfrac{3200}{h^2}=0\Rightarrow h^2=400\Rightarrow h=20\ \ (h>0)$

$\therefore$ 16cm by 20cm

12 Suppose I land my boat x km along the shore from the point on the coastline

directly opposite my initial position. Then, by Pythagoras', I row $\sqrt{x^2+4}$ km and jog $(6-x)$ km so the total time taken is

$T(x)=\dfrac{\sqrt{x^2+4}}{2}+\dfrac{6-x}{5}$

$\therefore T'(x)=\dfrac{x}{2\sqrt{x^2+4}}-\dfrac{1}{5}=0$

$\Rightarrow 5x=2\sqrt{x^2+4}$

$\Rightarrow 25x^2=4(x^2+4)$

$\Rightarrow 21x^2=16\Rightarrow x^2=\dfrac{16}{21}\Rightarrow x=\dfrac{4}{\sqrt{21}}=\dfrac{4\sqrt{21}}{21}\ \ (x>0)$

$\therefore$ I should land my boat 0.873km (to 3s.f.) along the coastline from the point directly opposite my initial position

Exercise 4S

1 $C(x)=0.01x^3-10x+150$

$\therefore C'(x)=0.03x^2-10=0\Rightarrow x^2=\dfrac{1000}{3}$

$\Rightarrow x=\sqrt{\dfrac{1000}{3}}\ \ (x>0)$

$C''\left(\sqrt{\dfrac{1000}{3}}\right)>0$ so this is a minimum and a global minimum

by consideration of the graph $\sqrt{\dfrac{1000}{3}}=18.257\ldots$ so 18

2 $f(x)=\frac{3}{4}x^2-\frac{1}{8}x^3$

The largest drop in systolic pressure is the maximum of $f(x)$

$\therefore f'(x)=\frac{3}{2}x-\frac{3}{8}x^2=\frac{3x}{8}(4-x)=0 \Rightarrow x=4cm^3$

$\therefore f(4)=\frac{3}{4}(4^2)-\frac{1}{8}(4^3)=4mmHg$

3 **a** $7\div 0.002=3500,$ so the domain of $p(x)$ is $0\le x\le 3500$

b $x(7-0.002x)-(500+3x)=0$

$\Rightarrow 0.002x^2-4x+500=0$
$\Rightarrow x^2-2000x+250000=0$
$x=1000\pm 500\sqrt{3}=1866\ldots$ or $133.97\ldots$

c From part **b**, $1000-500\sqrt{3}<x<1000+500\sqrt{3}$ or
$134<x<1870$

d $3500^2-2000(3500)+250000=5500000$
$=5.5$ million

4 **a** $c'(x)=20-0.4x+0.0012x^2=0$

$\Rightarrow x=61.25741133$ or $x=272.075922$
The latter is the minimum so 272

b $p(x)=r(x)-c(x)=15x-403+0.2x^2-0.0004x^3$

$p'(x)=15+0.4x-0.0012x^2=0$
$\Rightarrow x=367.3599\ldots$ so 367

c Individual Response

5 Let the plot measure x m by y m and suppose that

the side with fencing is of length x
$\therefore xy=1000$
and the total cost is

$T(x)=(15+3)x+(15+15)y=18x+\dfrac{30000}{x}$

$\therefore T'(x)=18-\dfrac{30000}{x^2}=0$

$\Rightarrow x^2=\dfrac{30000}{18}\Rightarrow x=\dfrac{50\sqrt{6}}{3}=40.8248\ldots$

$\Rightarrow y=10\sqrt{6}=24.4949\ldots$
so the plot is 24.5m by 40.8m (to 3s.f.)
and the minimised cost is

$T\left(\dfrac{50\sqrt{6}}{3}\right)=1470$ (to 3s.f.)

6 Suppose the airline reduces the price by €10 n times

Then the price of an individual ticket is $500 - 10n$ and the number of passengers is $180 + 2n$, and therefore the revenue is

$r(n) = (500 - 10n)(180 + 2n) = 90000 - 800n - 20n^2$

$\therefore r'(n) = -800 - 40n = 0 \Rightarrow n = 20$

so the optimal price is \$300 and the corresponding number of passengers is 220

Exercise 4T

1 **a** $v(t) = h'(t) = 96 - 32t \Rightarrow v(0) = 96$

b $h'(t) = 0 \Rightarrow t = 3$

$h_{max} = h(3) = 256$

c $h(t) = 16(7 + 6t - t^2) = 16(1+t)(7-t) = 0$

$t > 0 \Rightarrow t = 7$ so $v(7) = -128$

2 **a** $s(0) = 10$

b $s(t) = 5(2 + t - t^2) = 5(1+t)(2-t) = 0$

$t > 0 \Rightarrow t = 2$

c $v(t) = s'(t) = 5 - 10t$ so $v(2) = -15$

$a(t) = v'(t) = s''(t) = -10$

The diver hits the water with a velocity of 15 m s^{-1}, and a constant vertical acceleration of -10 m s^{-2}, which is approximately the force of gravity. Since both velocity and acceleration are negative, the diver is speeding up as he/she approaches the water.

3 Take the ground to have height 0, so that $h_0 = 0$

$\therefore h(t) = 50t - 4.9t^2$

$h'(t) = 50 - 9.8t = 0 \Rightarrow t = \dfrac{50}{9.8}$

$\therefore h_{max} = h\left(\dfrac{50}{9.8}\right) = 127.55\ldots = 128$ to 3s.f.

$h(t) = 0$ when $t = 0$ or $t = \dfrac{50}{4.9} = 10.2041\ldots$

so the rocket is at ground level again after 10.2s (to 3s.f.)

4 **a** $t = 0,\ 3,\ 6,\ 11$

b **i** $0 < t < 3,\ 6 < t < 11$

ii $3 < t < 6$

c **i** $t = 1.5$

ii $t = 4.5$

d $t = 1.5,\ 4.5$

e **i** $0 < t < 1.5,\ 3 < t < 4.5,\ 6 < t < 9$

ii $1.5 < t < 3,\ 4.5 < t < 6,\ 9 < t < 11$

5 $v(t) = s'(t) = -3t^2 - 6t + 4$

$a(t) = v'(t) = s''(t) = -6t - 6$

6 **a** $v_{avg} = \dfrac{s(3) - s(0)}{3} = \dfrac{63}{3} = 21$

b $v(t) = s'(t) = 20t - 3t^2$

$\Rightarrow v(3) = 33$
$a(t) = v'(t) = s''(t) = 20 - 6t$
$\Rightarrow a(3) = 2$

c Both the velocity and accleration are positive, so the particle is speeding up

d $v(t) = 0 \Rightarrow t = 0 \text{ or } t = \dfrac{20}{3}$

7 **a** $v(t) = s'(t) = 3t^2 - 14t + 11$

$a(t) = v'(t) = s''(t) = 6t - 14$

b $v(t) = 0 \Rightarrow 3t^2 - 14t + 11 = (3t - 11)(t - 1) = 0$

$\therefore t = 1 \text{ and } t = \dfrac{11}{3}$

c **i** □ Require $v(t)$ and $a(t)$ to have the same sign

$a(t) > 0$ when $t > \dfrac{14}{6} = \dfrac{7}{3}$ and $a(t) < 0$ when $t < \dfrac{7}{3}$
$v(t) > 0$ when $0 < t < 1$ or $t > \dfrac{11}{3}$ and $v(t) < 0$ when $1 < t < \dfrac{11}{3}$
Therefore the particle is speeding up when
$1 < t < \dfrac{7}{3}$ or $t > \dfrac{11}{3}$

ii The particle is slowing down when $a(t)$ and $v(t)$ have different signs

Using the working in part i, this is when
$0 < t < 1$ or $\dfrac{7}{3} < t < \dfrac{11}{3}$

d $v(t) = 0 \Rightarrow t = 1 \text{ or } t = \dfrac{11}{3}$

e $D = |s(1) - s(0)| + |s(3) - s(1)| = 5 + 8 = 13$

Exercise 4U

1 **a** $4y\dfrac{dy}{dx}-6x=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{3x}{2y}$

b $3y^2\dfrac{dy}{dx}=4x^3 \Rightarrow \dfrac{dy}{dx}=\dfrac{4x^3}{3y^2}$

c $4x+8y\dfrac{dy}{dx}-3+4\dfrac{dy}{dx}=0$

$\Rightarrow (8y+4)\dfrac{dy}{dx}=3-4x$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{3-4x}{8y+4}$

d $2x-3y-3x\dfrac{dy}{dx}+4y\dfrac{dy}{dx}=0$

$\Rightarrow (4y-3x)\dfrac{dy}{dx}=3y-2x$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{3y-2x}{4y-3x}$

e $2(x+y)\left(1+\dfrac{dy}{dx}\right)=-3\dfrac{dy}{dx}$

$\left[2(x+y)+3\right]\dfrac{dy}{dx}=-2(x+y)$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{2(x+y)}{2(x+y)+3}$

f $2x^2=\dfrac{x+y}{x-y}$

$$4x=\frac{(x-y)\left(1+\dfrac{dy}{dx}\right)-(x+y)\left(1-\dfrac{dy}{dx}\right)}{(x-y)^2}=\frac{2x\dfrac{dy}{dx}-2y}{(x-y)^2}$$

$\Rightarrow 4x(x-y)^2+2y=2x\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{4x(x-y)^2+2y}{2x}=\dfrac{2x(x-y)^2+y}{x}$

g $1=\dfrac{1}{2}\left(4x-18y^2\dfrac{dy}{dx}\right)\left(2x^2-6y^3\right)^{-\frac{1}{2}}$

$\Rightarrow 4x-18y^2\dfrac{dy}{dx}=2\sqrt{2x^2-6y^3}$

$\Rightarrow \dfrac{dy}{dx}=-\dfrac{1}{18y^2}\left[2\sqrt{2x^2-6y^3}-4x\right]$

2 $-\dfrac{3}{x^4}-\dfrac{3}{y^4}\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{y^4}{x^4}$

$\therefore$ At $(1,1)$, the gradient of the tangent is -1 and the gradient of the normal is 1

Tangent: $y-1=-1(x-1) \Rightarrow y=-x+2$

Normal: $y-1=1(x-1) \Rightarrow y=x$

3 $$-\frac{1}{(x+1)^2}-\frac{1}{(y+1)^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{(y+1)^2}{(x+1)^2}$$

$\therefore$ At $(1,1)$, the gradient of the tangent is -1 and the gradient of the normal is 1

Tangent: $y-1=-(x-1) \Rightarrow y=-x+2$

Normal: $y-1=x-1 \Rightarrow y=x$

4 $x^2+y^2=6x+8y$

$2x+2y\frac{dy}{dx}=6+8\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2y-8)=6-2x$

$\Rightarrow \frac{dy}{dx}=\frac{3-x}{y-4}$

$\therefore \frac{dy}{dx}=0$ when $x=3$

$\Rightarrow 9+y^2=18+8y$

$\Rightarrow y^2-8y-9=(y-9)(y+1)=0$

Therefore $(3,9)$ and $(3,-1)$

5 **a** $x+y=x^2-2xy+y^2$

$1+\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+2y\frac{dy}{dx}$

$\Rightarrow (1+2x-2y)\frac{dy}{dx}=2x-2y-1$

$\Rightarrow \frac{dy}{dx}=\frac{2x-2y-1}{2x-2y+1}$

b $$1-\frac{dy}{dx}=1-\frac{2x-2y-1}{1+2x-2y}=\frac{1+2x-2y-(2x-2y-1)}{2x-2y+1}$$

$$=\frac{1+2x-2y-(2x-2y-1)}{1+2x-2y}$$

$$=\frac{2}{2x-2y+1}$$

c Differentiating the result in part *b*,

$$-\frac{d^2y}{dx^2}=\frac{-2\left(2-2\frac{dy}{dx}\right)}{(2x-2y+1)^2}=-\frac{4\left(1-\frac{dy}{dx}\right)}{(2x-2y+1)^2}$$

$$=-\frac{4\left(\frac{2}{2x-2y+1}\right)}{(2x-2y+1)^2}=-\frac{8}{(2x-2y+1)^3}$$

$$\therefore \frac{d^2y}{dx^2}=\left(\frac{2}{2x-2y+1}\right)^3=\left(1-\frac{dy}{dx}\right)^3$$

Exercise 4V

1 **a** $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

b $\frac{dA}{dt} = 2\pi\frac{dr}{dt} + 2\pi\frac{dh}{dt}$

c $\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{\pi r}{3}\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$

d $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

2 Let the diagonal distance be D. Then,

$D^2 = l^2 + w^2 + h^2$

$\therefore 2D\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt}$

$\Rightarrow \frac{dD}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{D}$

$= \frac{l\frac{dl}{dt} + w\frac{dw}{dt} + h\frac{dh}{dt}}{\sqrt{l^2 + w^2 + h^2}}$

3 $\frac{dl}{dt} = 3, \quad \frac{dw}{dt} = -3$

a $A = lw \Rightarrow \frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt} = 3(w - l)$

$\therefore$ Initially, $\frac{dA}{dt} = 3(7 - 24) = -51$

b $P = 2(l + w) \Rightarrow \frac{dP}{dt} = 2\left(\frac{dl}{dt} + \frac{dw}{dt}\right) = 0$

c Let D denote the diagonal distance. Then,

$D^2 = l^2 + w^2 \Rightarrow 2D\frac{dD}{dt} = 2l\frac{dl}{dt} + 2w\frac{dw}{dt}$

$\Rightarrow \frac{dD}{dt} = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2 + w^2}} = \frac{3l - 3w}{\sqrt{l^2 + w^2}}$

$\therefore$ Initially,

$\frac{dD}{dt} = \frac{3(24 - 7)}{\sqrt{24^2 + 7^2}} = \frac{51}{25}$

4 Let the distance of the base of the ladder from the wall be x

and the vertical height of the ladder up the wall be y

Then, by Pythagoras,

$y = \sqrt{100 - x^2}$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{100-x^2}}\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{\sqrt{100-x^2}}{x}\frac{dy}{dx} = \frac{\sqrt{100-x^2}}{2x}$$

$\therefore$ when $x = 6$,

$$\frac{dx}{dt} = \frac{\sqrt{100-36}}{12} = \frac{2}{3}$$

5 Let the cube have side length l

$$V = l^3 \Rightarrow \frac{dV}{dt} = 3l^2\frac{dl}{dt} = \frac{3}{2}$$

$$S = 6l^2$$

$$\therefore \frac{dS}{dt} = 12l\frac{dl}{dt}$$

When $V = 27$, $l = 3$ and $\frac{dl}{dt} = \frac{1}{18}$

so $\frac{dS}{dt} = 12(3)\left(\frac{1}{18}\right) = 2$

6 Let the radius of the cylinder be r and the height h

$$\frac{dr}{dt} = -3 \text{ and } \frac{dh}{dt} = 6$$

$$S = 2\pi r^2 + 2\pi rh = 2\pi r(r+h)$$

$$\therefore \frac{dS}{dt} = 2\pi\left[\frac{dr}{dt}(r+h) + r\left(\frac{dr}{dt}+\frac{dh}{dt}\right)\right]$$

$$= 2\pi\left[-3(r+h)+3r\right]$$

so when $r = 12,\ h = 10$

$$\frac{dS}{dt} = 2\pi\left[-3(22)+36\right] = -60\pi$$

7 $h^2 = a^2 + b^2$

$$2h\frac{dh}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{\sqrt{a^2+b^2}} = \frac{5a-4b}{\sqrt{a^2+b^2}}$$

$\therefore$ When $a = 15$ and $b = 20$,

$$\frac{dh}{dt} = \frac{75-80}{\sqrt{225+400}} = -\frac{5}{25} = -\frac{1}{5}$$

8 $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$

$$\therefore -7 = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = -\frac{7}{4\pi r^2}$$

When $V = 36\pi$,

$$\frac{4}{3}\pi r^3 = 36\pi \Rightarrow r^3 = 27 \Rightarrow r = 3$$

so $\frac{dr}{dt} = -\frac{7}{36\pi}$

9 $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{\pi r}{3}\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = 3$

Let θ denote the angle between the axis of symmetry through the centre of the cone and the curved surface of the cone. Then,

$\tan\theta = \frac{r}{h} = \frac{1.5}{2} = \frac{3}{4} \Rightarrow \frac{3}{4}h = r \Rightarrow \frac{dr}{dt} = \frac{3}{4}\frac{dh}{dt}$

$\therefore \frac{\pi r}{3}\left(\frac{3h}{2}\frac{dh}{dt} + r\frac{dh}{dt}\right) = 3$

$\Rightarrow \frac{dh}{dt} = \frac{18}{\pi r(3h + 2r)}$

so when $h = 2$ and $r = 1.5$,

$\frac{dh}{dt} = \frac{4}{3\pi}$

10 $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r}\frac{dA}{dt} = \frac{1}{\pi r}$

$\therefore$ When $r = 5$,

$\frac{dr}{dt} = \frac{1}{5\pi}$

Chapter review

1 **a** Limit does not exist since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

b $\lim_{x\to -1} f(x) = 2$

c $\lim_{x\to 0} f(x) = 2.25$

d no limit as there is a cusp at x= 3.

2 **a** $\lim_{x\to 2}\left(-x^2 + 5x - 2\right) = 4$

b $\lim_{x\to 2}\frac{x+3}{x+6} = \frac{5}{8}$

c $\lim_{x\to 0}(8 - 2x)^{\frac{1}{3}} = 2$

d $\lim_{x\to 3}\frac{x^2 - 4x + 3}{x - 3} = \lim_{x\to 3}\frac{(x-3)(x-1)}{x-3} = \lim_{x\to 3}(x-1) = 2$

e $\lim_{x\to\infty}\frac{\sqrt{x^2 - 2}}{x} = \lim_{x\to\infty}\sqrt{1 - \frac{2}{x^2}} = 1$

f $\lim_{x\to -\infty}\frac{2}{x^3 + 1} = \lim_{x\to -\infty}\frac{\frac{2}{x^3}}{1 + \frac{1}{x^3}} = 0$

3 $\lim_{x\to -2^-} f(x) = (-2)^2 - 4 = 0$

$\lim_{x\to -2^+} f(x) = (-2)^3 - 6(-2) = 4$

so not continuous at $x = -2$

4 **a** $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\Rightarrow 2 + a = 1 \Rightarrow a = -1$

b $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

$\Rightarrow 6 = a - 1 \Rightarrow a = 7$

5 **a** $\lim_{n\to\infty} \frac{n+3}{3n-4} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{3-\frac{4}{n}} = \frac{1}{3}$

b $\lim_{n\to\infty} \frac{4-2n^2}{n^3-1} = \lim_{n\to\infty} \frac{\frac{4}{n^3}-\frac{2}{n}}{1-\frac{1}{n^3}} = 0$

c No limit

6 $\left|-\frac{1}{3}\right| < 1$ so the series does converge

$S_\infty = 2\cdot\frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{3}{2}$

7 The common ratio is $\frac{1}{n^2+1}$, so the condition is

$\frac{1}{n^2+1} < 1 \Rightarrow n^2 + 1 > 1 \Rightarrow n^2 > 0$

i.e. the series converges for all non-zero n

$S_\infty = \frac{n^2}{1-\frac{1}{n^2+1}} = \frac{n^2\left(n^2+1\right)}{n^2+1-1} = \frac{n^2\left(n^2+1\right)}{n^2} = n^2 + 1$

8 **a** Vertical: $x^2 - 9 = 0 \Rightarrow x = \pm 3$

Horizontal:

$\lim_{x\to\infty} f(x) = 6$ so $y = 6$

b Vertical: $3x^3 + 81 = 0 \Rightarrow x^3 = -27 \Rightarrow x = -3$

Horizontal:

$\lim_{x\to\infty} f(x) = 0$ so $y = 0$

c Vertical: $x = 0$

9 **a** $y = (1-2x)^5(3x-2)^6$

$$\frac{dy}{dx} = -10(1-2x)^4(3x-2)^6 + (1-2x)^5 \cdot 18(3x-2)^5$$

$$= (1-2x)^4(3x-2)^5\left[-10(3x-2)+18(1-2x)\right]$$
$$= (1-2x)^4(3x-2)^5(6x-2)$$
$$= -2(33x-19)(2x-1)^4(3x-2)^5$$

b $$\frac{dy}{dx} = \frac{\sqrt{x-1}(2x)-(x^2-1)\frac{1}{2\sqrt{x-1}}}{x-1}$$

$$= \frac{4x(x-1)-(x^2-1)}{2(x-1)^{\frac{3}{2}}} = \frac{3x^2-4x+1}{2(x-1)^{\frac{3}{2}}}$$

c $$\frac{dy}{dx} = \frac{1}{2}\left(2+\frac{3x^2}{2\sqrt{x^3+1}}\right)\frac{1}{\sqrt{2x+\sqrt{x^3+1}}}$$

10 a Vertical: $x^2 - 1 = 0 \Rightarrow x = \pm 1$

Horizontal:

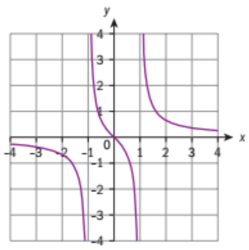
$$\lim_{x\to\infty}\frac{x}{x^2-1} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1-\frac{1}{x^2}} = 0 \quad \text{so} \quad y = 0$$

b $$f(x) = \frac{x}{x^2-1}$$

$$f(-x) = \frac{-x}{(-x)^2-1} = -\frac{x}{x^2-1} = -f(x)$$

c $$\frac{dy}{dx} = \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} = \frac{-1-x^2}{(x^2-1)^2} = -\frac{1+x^2}{(x^2-1)^2} < 0$$

d



11 a $$|OP| = \sqrt{a^2+(3-a^2)^2} = \sqrt{a^4-5a^2+9}$$

b $$|OP| = \sqrt{\left(a^2-\frac{5}{2}\right)^2+9-\frac{25}{4}} = \sqrt{\left(a^2-\frac{5}{2}\right)^2+\frac{11}{4}}$$

$\therefore$ Closest when $a^2 - \frac{5}{2} = 0 \Rightarrow a = \pm\sqrt{\frac{5}{2}}$

12 $f(x)=\dfrac{x^2-3x+2}{x^2+3x+2}=\dfrac{(x-2)(x-1)}{(x+2)(x+1)}$

a Vertical: $x=-2$ and $x=-1$

Horizontal:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{3}{x}+\frac{2}{x^2}}=1 \quad \text{so} \quad y=1$$

b $(x-2)(x-1)=0 \Rightarrow x=2$ or $x=1$

so $(2,0)$, $(1,0)$, $(0,1)$

c $f'(x)=\dfrac{(x^2+3x+2)(2x-3)-(x^2-3x+2)(2x+3)}{(x^2+3x+2)^2}$

$=\dfrac{2x^3+3x^2-5x-6-2x^3+3x^2+5x-6}{(x^2+3x+2)^2}$

$=\dfrac{6x^2-12}{(x^2+3x+2)^2}=0 \Rightarrow x=\pm\sqrt{2}$

13 a $f(x)=0 \Rightarrow x=-1$

b Horizontal Asymptote: $y=0$

Vertical Asymptote: $x=0$

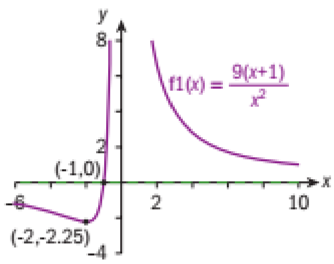
c $f'(x)=0 \Rightarrow x=-2$

When $x=-2$, $y=f(-2)=-\dfrac{9}{4}$

$f''(-2)=\dfrac{9}{8}>0$ so $\left(-2,-\dfrac{9}{4}\right)$ is a minimum

d $f''(x)>0 \Rightarrow x>-3$

e



14 a $f(x)=0 \Rightarrow \sqrt{x}\left(\sqrt{x}-b\right)=0 \Rightarrow x=0$ or $x=b^2$

b $f'(x)=1-\dfrac{b}{2\sqrt{x}}=\dfrac{2\sqrt{x}-b}{2\sqrt{x}}$

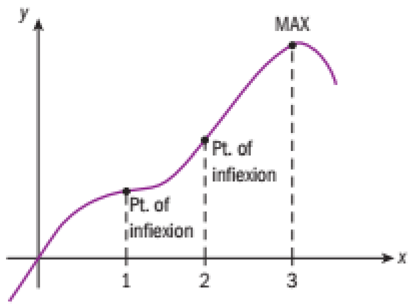
i $f'(x) > 0$ when $x > \dfrac{b^2}{4}$

ii $f'(x) < 0$ when $0 \le x < \dfrac{b^2}{4}$

c $f(x) \ge f\left(\dfrac{b^2}{4}\right) = -\dfrac{b^2}{4}$

d $f''(x) = \dfrac{b}{4x\sqrt{x}} > 0$ so concave up

15



16 $8x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{4x}{y}$

$\therefore$ At $\left(\dfrac{2}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$,

$\dfrac{dy}{dx} = -4$

17 $4xy + 2x^2\dfrac{dy}{dx} + 3y^2 + 6xy\dfrac{dy}{dx} = 0$

$\Rightarrow (2x^2 + 6xy)\dfrac{dy}{dx} = -(3y^2 + 4xy)$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{3y^2 + 4xy}{2x^2 + 6xy}$

$\therefore$ At $(1,1)$,

$\dfrac{dy}{dx} = -\dfrac{3+4}{2+6} = -\dfrac{7}{8}$

so the normal to the curve at this point has gradient $\dfrac{8}{7}$

$\therefore y - 1 = \dfrac{8}{7}(x - 1)$

$\Rightarrow y = \dfrac{8}{7}x - \dfrac{1}{7}$

18 $\sqrt{a}+\sqrt{b}=\sqrt{p}$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

$\therefore$ The gradient at (a,b) is $-\sqrt{\frac{b}{a}}$

and the equation of the tangent is

$$y-b=-\sqrt{\frac{b}{a}}(x-a)$$

$$\Rightarrow y=-\sqrt{\frac{b}{a}}x+\sqrt{ab}+b$$

The tangent passes through $(0,r)$:

$r=\sqrt{ab}+b$

and $(s,0)$:

$$0=-\sqrt{\frac{b}{a}}s+\sqrt{ab}+b \Rightarrow s=a+\sqrt{ab}$$

$$\therefore r+s=\sqrt{ab}+b+a+\sqrt{ab}=a+2\sqrt{ab}+b$$

$=\left(\sqrt{a}+\sqrt{b}\right)^2=\left(\sqrt{p}\right)^2=p$ as required

19 Let D denote the distance from the observer to the drone and x the height of the drone. Then,

$$D^2=36+x^2 \Rightarrow 2D\frac{dD}{dt}=2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dD}{dt}=\frac{x}{\sqrt{36+x^2}}\frac{dx}{dt}$$

$\therefore$ When $\frac{dx}{dt}=3$ and $x=8$,

$$\frac{dD}{dt}=\frac{8}{\sqrt{36+64}}(3)=2.4$$

20 a $v(0)=-2$

b $v(t)=0$

$\Rightarrow 1+t=\sqrt{4t+9}$

$\Rightarrow (1+t)^2=1+2t+t^2=4t+9$

$\Rightarrow t^2-2t-8=0$

$\Rightarrow (t-4)(t+2)=0$

$t>0$ so $t=4$

c $a(t)=v'(t)=1-\frac{2}{\sqrt{4t+9}}$

$a(4)=1-\frac{2}{\sqrt{25}}=\frac{3}{5}$

d **i** $v(t) < 0$ for $t < 4$ and $v(t) > 0$ for $t > 4$

$a(t) > 0$ for all $t > 0$

$\therefore$ slowing down when $a(t)$ and $v(t)$ have different signs

$\Rightarrow t < 4$

ii $v(t)$ and $a(t)$ have the same sign when $t > 4 \Rightarrow t > 4$

21 Let the (initially) inner circle have radius r_1 and the (initially) outer circle have radius r_2

Then, $\frac{dr_1}{dt} = 1.2$ and $\frac{dr_2}{dt} = 0.8$

The area between the circles is given by

$A = \pi\left(r_2^2 - r_1^2\right)$ so at time t,

$$\Rightarrow \frac{dA}{dt} = 2\pi\left(r_2\frac{dr_2}{dt} - r_1\frac{dr_1}{dt}\right) = 2\pi\left(0.8\times3 - 1\times1.2\right) = \frac{12\pi}{5}$$

22 $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 3 \Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r\frac{3}{4\pi r^2} = \frac{6}{r}$$

$\therefore$ When $r = 1$,

$$\frac{dS}{dt} = 6$$

23 $|PQ| = x \Rightarrow |QT| = |RS| = x$ since PQT is an equilateral triangle and QRST is a rectangle. Let $|ST| = |QR| = y$, then

$p = 3x + 2y$ and $A = xy + \frac{\sqrt{3}}{4}x^2$ (area of rectangle plus eq. triangle)

$$\therefore y = \frac{p - 3x}{2}$$

$$\Rightarrow A = \frac{x(p - 3x)}{2} + \frac{\sqrt{3}}{4}x^2$$

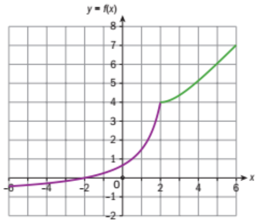
$$\frac{dA}{dx} = \frac{p}{2} - 3x + \frac{\sqrt{3}}{2}x = 0$$

$$\Rightarrow \left(3 - \frac{\sqrt{3}}{2}\right)x = \frac{p}{2}$$

$$\Rightarrow \frac{p}{x} = 2\left(3 - \frac{\sqrt{3}}{2}\right) = 6 - \sqrt{3}$$

Exam-style questions

24 a Graphical approach:



Attempt to draw graph (1 mark)

Each branch correct (2 marks)

Then $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 4$

Hence $\lim\limits_{x\to 2} f(x)$ exists and is equal to 4. (1 mark)

OR: Algebraic approach (note that an algebraic approach will be accepted, but not expected, in examinations)

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} -\dfrac{x^2-4}{x^2-5x+6} = \lim\limits_{x\to 2^-} -\dfrac{(x-2)(x+2)}{(x-2)(x-3)}$ (1 mark)

$= \lim\limits_{x\to 2} \dfrac{-(x+2)}{x-3} = 4$ (1 mark)

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} e^{2-x} + x + 1 = 4$ (1 mark)

Then $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 4$

Hence $\lim\limits_{x\to 2} f(x)$ exists and is equal to 4. (1 mark)

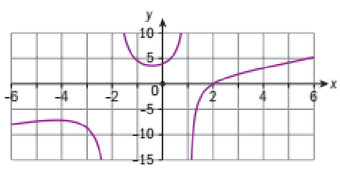
b $a^2 + 3a + 6 = 4$ (1 mark)

$(a+1)(a+2) = 0$ (1 mark)

$a = -1, a = -2$ (1 mark)

25 a $x^2 + x - 2 \neq 0 \Rightarrow x \neq 1, x \neq -2$ (2 marks)

b



(Shape of each branch correct gains 1 mark) (3 marks)

c i Vertical asymptotes: $x = 1$ and $x = -2$ (2 marks)

ii Using long division, $f(x) = x - 1 + \dfrac{3x-10}{x^2+x-2}$ (1 mark)

As $x \to \infty$, $f(x) \to x - 1$ which is a slant asymptote. (1 mark)

26 a $g'(x) = \dfrac{\frac{1}{2\sqrt{x}}(x^2+1) - \sqrt{x} \cdot 2x}{(x^2+1)^2}$ (2 marks)

$= \dfrac{1 - 3x^2}{2\sqrt{x}(x^2+1)^2}$ (1 mark)

b $g(1) = \dfrac{1}{2}$ (1 mark)

$g'(1) = -\dfrac{1}{4}$ (1 mark)

Equation of normal: $y - \dfrac{1}{2} = 4(x-1)$ or $y = 4x - 3\dfrac{1}{2}$ (1 mark)

c g' is not defined at $x = 0$ because a derivative is not defined at the end point of a closed interval. (1 mark)

Therefore, there is no tangent to the graph of g at $x = 0$. (1 mark)

27 a $2x = \dfrac{\left(1 + \frac{dy}{dx}\right)(x-y) - \left(1 - \frac{dy}{dx}\right)(x+y)}{(x-y)^2}$ (3 marks)

Make $\dfrac{dy}{dx}$ the subject (1 mark)

$2x(x-y)^2 = 2x\dfrac{dy}{dx} - 2y$

$\dfrac{dy}{dx} = (x-y)^2 + \dfrac{y}{x}$ (1 mark)

b $y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ (1 mark)

$\dfrac{dy}{dx} = 1$ (1 mark)

$y = x \mp 1$ (2 marks)

28 a i $\dfrac{101.1 - 98.5}{2002 - 2000} = 1.3$ (2 marks)

ii $\dfrac{102.3 - 101.1}{2004 - 2002} = 0.6$ (1 mark)

b The average annual profit between 2000 and 2002 was almost double the average annual profit between 2002 and 2004. (2 marks)

29 a $f'(x) = \lim\limits_{h \to 0} \dfrac{\left(2(x+h)^2 + 3(x+h) - 4\right) - \left(2x^2 + 3x - 4\right)}{h}$ (2 marks)

$= \lim_{h \to 0} \frac{(2x^2 + 4hx + 2h^2 + 3x + 3h - 4) - (2x^2 + 3x - 4)}{h}$ (1 mark)

$= \lim_{h \to 0} \frac{4hx + 2h^2 + 3h}{h}$ (1 mark)

$= \lim_{h \to 0} (4x + 2h + 3) = 4x + 3$ (1 mark)

b $f(1) = 2 + 3 - 4 = 1$ (1 mark)

$f'(1) = 4 + 3 = 7$ (1 mark)

Equation of tangent: $y - 1 = 7(x - 1)$ (or $y = 7x - 6$) (1 mark)

30 a $h(4) = 370$ and $h(5) = 438$ (3 s.f.) (2 marks)

b $v(t) = h'(t) = 112 - 9.8t$ (2 marks)

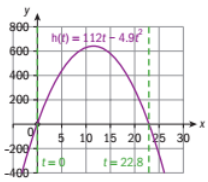
c $v(t) = 0 \Rightarrow 112 - 9.8t = 0$ (1 mark)

$t = 11.4$ (3 s.f.) (1 mark)

d double x-coordinate of maximum, or determine zero (1 mark)

22.8 (3 s.f.) (1 mark)

e



(Shape: 1 mark; Domain: 1 mark; Maximum: 1 mark)

f $v(22.8571...) = -112$ ms^{-1} (2 marks)

g $a(t) = v'(t) = -9.8$ which is constant (2 marks)

31 a i $\left(\frac{f}{g}\right)'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2}$ (1 mark)

$= \frac{10 \times 4 - 9 \times \left(-\frac{4}{3}\right)}{4^2}$ (1

mark)

$= \frac{52}{16}\left(= \frac{13}{4} = 3.25\right)$ (1 mark)

ii $(g \circ f)'(1) = g'(f(1))f'(1) = g'(2)f'(1)$ (1 mark)

$= -\frac{4}{3} \times 4 = -\frac{16}{3}$ (1 mark)

b **i** False (1 mark)

as derivative changes sign. (1 mark)

ii False (1 mark)

as the derivatives at these points are not negative reciprocals. (1 mark)

32 a Let $d(x)$ be the total length of the pipeline.

$d(x) = \sqrt{x^2 + 75^2} + 100 - x$ (2 marks)

b Let $c(x)$ be proportional to the construction costs of the pipeline.

$c(x) = 3\sqrt{x^2 + 75^2} + 100 - x$ (1 mark)

$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{3x}{\sqrt{x^2 + 75^2}} - 1$ (2 marks)

$\frac{\mathrm{d}c}{\mathrm{d}x} = 0 \Rightarrow \frac{3x}{\sqrt{x^2 + 75^2}} = 1$ (1 mark)

Solve equation

$9x^2 = x^2 + 75^2$ (2 marks)

$x = \frac{75}{4}\sqrt{2}$ (1 mark)

c $d(x) = \sqrt{\left(\frac{75}{4}\sqrt{2}\right)^2 + 75^2} + 100 - \frac{75}{4}\sqrt{2}$ (2 marks)

$= \sqrt{75^2\left(\frac{2}{16} + 1\right)} + 100 - \frac{75}{4}\sqrt{2}$

$= 75\sqrt{\frac{9}{8}} + 100 - \frac{75}{4}\sqrt{2}$ (1 mark)

$= 3 \times \frac{75}{4}\sqrt{2} + 100 - \frac{75}{4}\sqrt{2}$

$= \frac{75}{2}\sqrt{2} + 100$ (1 mark)

33 a Vertical Asymptote: $x = a$ (1 mark)

Horizontal Asymptote: $y = a$ (1 mark)

b $f'(x) = \frac{a \cdot (x - a) - 1 \cdot (ax - 4)}{(x - a)^2} = \frac{4 - a^2}{(x - a)^2}$. (3 marks)

c $f'(x) = 0$ for turning points (1 mark)

$\frac{4 - a^2}{(x - a)^2} = 0 \Rightarrow a^2 = 4 \Rightarrow a = 2$ (1 mark)

For $a = 2$, $f(x) = \frac{2x - 4}{x - 2} = \frac{2(x - 2)}{x - 2} = 2$ so the function is constant, and there are no turning points. (1 mark)

For $a \neq 2$, $f'(x) \neq 0$, so the function has no max/min. (1 mark)

d $f'(1) = \dfrac{4 - a^2}{(1-a)^2}$ (1 mark)

gradient of normal is $m = \dfrac{(1-a)^2}{a^2 - 4}$ (1 mark)

$f(1) = \dfrac{a-4}{1-a}$ (1 mark)

$y - \dfrac{a-4}{1-a} = \dfrac{(1-a)^2}{a^2-4}(x-1)$ (or equivalent) (1 mark)

e Asymptotes intersect at (a, a). Substitute (a, a) into normal equation. (1 mark)

$a - \dfrac{a-4}{1-a} = \dfrac{(1-a)^2}{a^2-4}(a-1)$ (or equivalent) (1 mark)

Simplify (1 mark)

$(a^2 - 4)^2 = (a-1)^4$

$4a^3 - 14a^2 + 4a + 15 = 0$ (1 mark)

f From GDC (1 mark)

$a = 2.5$ or $a = 1.82$ (2 marks)

(For part **f**, award 2 marks only if negative root $a = -0.823$ is included)